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The integrity of small cage graphs

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Abstract

Integrity, a measure of network reliability, is defined as
$I(G) = \min_{S\subset V} \{|S| + m(G-S)\}$,
where G is a graph with vertex set V and $m(G-S)$ denotes the order of the largest component of $G-S$. In this article the integrity of the cubic cage graphs $(3, g)$ up to $g = 10$ and the integrity of the known small cages with less than or equal to 60 vertices is computed.

1 Introduction

The integrity of a graph measures the reliability of a communication network. If the network is modeled by a graph, then the integrity measures how easy it is to cut the graph (or the network) into several small pieces by deleting as few vertices as possible. Formally, the *integrity* of a graph G with vertex set V is defined as

$$I(G) = \min_{S\subset V} \{|S| + m(G-S)\},$$

where $m(G-S)$ denotes the order of the largest component of $G-S$ [6]. This elegant and simple concept was introduced by Barefoot, Entringer and Swart in 1987 [3]. The goal in constructing networks resilient to disruptions is to find infinite families of graphs which have large integrity. Complete graphs have a large integrity, but they are "expensive" in the sense that the number of edges is very large. For

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this reason, it is natural to consider the integrity of k-regular graphs, for a fixed integer k.

There is a substantial literature on integrity. For example, an easy result cited in the survey article on integrity by Bagga, Beineke, Goddard, Lipman and Pippert [2] gives the integrity of a path P_n with n vertices as $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$; the integrity of a cycle C_n with n vertices as $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$; the integrity of the complete graph K_n as n; and the integrity of the complete bipartite graph $K_{n,m}$ as $\min(n, m) + 1$. For most graphs that do not belong to the previously mentioned families the integrity is not known because it is difficult to compute even for small graphs [9]. However, there are results that give lower and upper bounds on the integrity [1, 7, 11].

In this paper we study the integrity of a small class of regular graphs called cages. The length of the shortest cycle in a graph G that contains a cycle, is called the *girth* of G. An (r, g)-*cage* or *cage* is an r-regular graph with girth g, that has the smallest number of vertices among all r-regular graphs with girth g. For some values of r and g there is a unique (r, g)-cage (up to isomorphism), while for other values of r and g there are several (r, g)-cages. If $r = 3$, then an (r, g)-cage is simply referred to as a g-cage. Cages are known only for small values of g and r, see [10]. In this article the integrity of the cages in Table A is computed. These include all the known cages with less or equal to 60 vertices. One of the questions that motivated this study was the following: If for some fixed pair (r, g), there are several nonisomorphic cages, do these cages have the same integrity? We will show that all of the 18 different 9-cages have the same integrity, as do the 3 different 10-cages. On the other hand the 4 different $(5, 5)$-cages have two different integrities.

Table A

vertices	(r, g)-cage	number	integrity
10	$(3, 5)$	1	6
14	$(3, 6)$	1	8
19	$(4, 5)$	1	11
24	$(3, 7)$	1	12
26	$(4, 6)$	1	14
30	$(3, 8)$	1	14
30	$(5, 5)$	4	18 or 19
40	$(6, 5)$	1	25
42	$(5, 6)$	1	22
50	$(7, 5)$	1	30
58	$(3, 9)$	18	24
70	$(3, 10)$	3	28

For graphs with at most 30 vertices the integrity can be computed directly by a program that simply searches through all possible subsets $S \subset V(G)$. However, once the vertex set grows to 40 and more vertices this exhaustive procedure becomes impractical to carry out on a regular desktop computer and one needs to look for theoretical bounds on the integrity to restrict the search time for the computer program. This paper is organized as follows: In Section 2 some upper bounds on the integrity of cages are established, in Section 3 these bounds are used to establish the exact integrity of 3-cages. In Section 4 we establish the integrity of some higher cages and Section 5 contains some general observations about the integrity of cages.

2 General bounds on the integrity of cages

In the following G is a connected, r-regular graph with n vertices, $r \geq 3$, and girth $g \geq 5$. The *cycle rank* $\beta(H)$ of a connected graph H is $|E(H) - V(H) + 1|$.

Lemma 1 *If the integrity of G can be achieved by removing a set S of $s = |S|$ vertices from G, such that all the components of $G - S$ are trees, then*

$$I(G) \geq \frac{nr + 2\sqrt{2nr} - 2(n+1)}{2(r-1)}$$

Proof: The graph G has $rn/2$ edges. Remove a set S of $s = |S|$ vertices from G and let $z = m(G - S)$. The integrity bound obtained from this is $I(G) \leq z + s$. The $rn/2$ edges of G are either removed together with the set S or reside in $G - S$. Thus we have $rn/2 =$ edges removed + edges remaining. At most rs edges are removed. The number of edges remaining in $G - S$ is $n - s - t$ where t is the number of tree components in $G - S$. Thus we have the following: $rn/2 \leq rs + n - s - t$. However there are at least $(n - s)/z$ components. Our inequality now becomes $rn/2 \leq rs + n - s - (n - s)/z$. Solving this for s results in

$$s \geq \frac{n(rz - 2z + 2)}{2(rz - z + 1)}.$$

Thus

$$I(G) \geq \min_z\left(z + \frac{n(rz - 2z + 2)}{2(rz - z + 1)}\right). \tag{1}$$

Differentiating this with respect to z results in a minimum at

$$z = \frac{2 - \sqrt{2nr}}{2(1 - r)}. \tag{2}$$

Substituting (2) into (1) results in the statement of the lemma. □

Lemma 2 *If the integrity of G can be achieved by removing a set S of $s = |S|$ vertices from G, such that not all the components of $G - S$ are trees and no component of $G - S$ contains more than one cycle, then*

$$I(G) \geq \max\left\{2g, g + \frac{n}{2}\left(\frac{r-2}{r-1}\right)\right\}.$$

Proof: The proof is similar to the previous lemma. Again we have $rn/2 =$ edges removed + edges remaining.

If all vertices in S are nonadjacent to each other, then there were rs edges removed. Since each component is acyclic or unicyclic, the number of remaining edges in $G-S$ is at most $n-s$. Thus we have the following: $rn/2 \leq rs + n - s$. Solving this for s results in

$$s \geq \frac{n}{2}(\frac{r-2}{r-1}).$$

Thus

$$I(G) \geq \min_z (z + \frac{n}{2}(\frac{r-2}{r-1})),$$

where $z = m(G-S)$ and $z \geq g$ by our assumption. Since a cycle C of order at least g is in one of the components of $G-S$, $|S| + m(G-S) \geq |N_G[C]| \geq 2g$ where $N_G[C]$ is the closed neighborhood of the cycle C in G. □

Lemma 3 *If the integrity of G can be achieved by removing a set S of $s = |S|$ vertices from G, such that at least one component, say G_1, of $G-S$ contains more than one cycle ($\beta(G_1) > 1$) and there is no component G_2 of $G-S$ with $\beta(G_2) > 3$ then*

$$I(G) \geq \min\{g + \lceil (g-2)/2 \rceil + \frac{n}{2}(\frac{r-2}{r-1}) - \frac{n}{(r-1)(g + \lceil (g-2)/2 \rceil)},$$

$$2g - 2 + \frac{n}{2}(\frac{r-2}{r-1}) - \frac{n}{(r-1)(g-1)}\}.$$

Proof: We first prove the following claim.

Claim: Let G be a connected graph with girth g. Then

$$|V(G)| \geq \begin{cases} g + \lceil \frac{g-2}{2} \rceil & \text{if } \beta(G) = 2 \\ 2g - 2 & \text{if } \beta(G) = 3 \end{cases}$$

Proof of Claim:

Let G be a graph with $\beta(G) = 2$ and with a minimal number of vertices. Let T be a spanning tree of G. Adding two edges of G to T creates two cycles C_1 and C_2 of order at least g. If C_1 and C_2 are edge disjoint then it is easy to see that G cannot be minimal. Thus G has the shape of a letter θ. The girth of G is g, so G must contain a cycle C of order g to which a path P is attached at both endpoints of P. To minimize the length of P, the endpoints of P must be as far apart on C as possible. This is achieved as follows: If g is even (or odd) split C at two vertices v_1 and v_2 into two components of order $(g-2)/2$ each (or of orders $(g-3)/2$ and $(g-1)/2$). Now attach P at v_1 and v_2. In order that $C \cup P$ still has girth g, P needs to have a length of $(g-2)/2 + 2$ (or $(g-1)/2 + 2$) (the $+2$ accounts for the end vertices of P). Now if g is even, $C \cup P$ has $g + (g-2)/2$ vertices and if g is odd, there are $g + (g-1)/2$ vertices. Thus $C \cup P$ has order $g + \lceil (g-2)/2 \rceil$.

If $\beta(G) = 3$, then we will show that a minimal graph G has $2g-2$ vertices. If this is possible, then G cannot contain two disjoint cycles. This implies that G has the shape of the θ-graph discussed above with an additional arc added that connects two of the arcs of the theta. In order to construct the smallest graph G with $\beta(G) = 3$, we begin with the above described graph $C \cup P$ to which we add a path P'. If g is even then there are three cycles of order g in $C \cup P$. We simply add a path P' of length $(g-2)/2 + 2$ to two vertices that are distance $g/2$ apart on the θ-graph. $C \cup P \cup P'$ has now an order of $2g - 2$. If g is odd then there are two cycles of length g, and a cycle C' of length $g+1$ in $C \cup P$. It is now to our advantage to add P' to the cycle C'. Split C' at two vertices v_1 and v_2 into two components of order $(g-1)/2$ each and add a path P' of length $(g-3)/2 + 2$ at the vertices v_1 and v_2. Now $C \cup P \cup P'$ has $g + \lceil (g-2)/2 \rceil + (g-3)/2 = 2g - 2$ vertices.

Now the proof of the lemma is similar to the previous lemmas. Again we have $rn/2 =$ edges removed + edges remaining. If all vertices in S are nonadjacent to each other, then there were rs edges removed. The number of remaining edges in $G - S$ is $n - s + \sum (i-1)t_i$ where t_i equals the number of components in $G - S$ with cycle rank i. Then we have

$rn/2 \leq rs + n - s + 2t_3 + t_2 - t_0 \leq rs + n - s + 2t_3 + t_2$.

If $G - S$ contains a component with cycle rank 2 and no component with cycle rank greater than 2, then $t_3 = 0$ and t_2 is as large as possible, if there are as many components with two cycles as possible. That is $t_2 \leq n/(g + \lceil (g-2)/2 \rceil)$. Solving this for s results in the statement:

$$I(G) \geq \min_z \left(z + \frac{n}{2}\left(\frac{r-2}{r-1}\right) - \frac{n}{(r-1)(g + \lceil (g-2)/2 \rceil)}\right).$$

This minimum is achieved if z is as small as possible, that is $z = g + \lceil (g-2)/2 \rceil$.

If in $G - S$ there is a component with cycle rank 3 then $2t_3 + t_2$ is maximized if t_3 is as large as possible, that is we can bound $2t_3 + t_2$ by $2t_3 + t_2 \leq 2n/(2g-2)$. Solving this for s results in the statement:

$$I(G) \geq \min_z \left(z + \frac{n}{2}\left(\frac{r-2}{r-1}\right) - \frac{2n}{(r-1)(2g-2)}\right).$$

This minimum is achieved if z is as small as possible, that is $z = 2g - 2$. □

The three lemmas above can be summarized in the following theorem.

Theorem 4 *The integrity of G is bounded below by*

$$I(G) \geq \min\Big\{\frac{nr + 2\sqrt{2nr} - 2(n+1)}{2(r-1)}, \max\{2g, g + \frac{n}{2}\left(\frac{r-2}{r-1}\right)\},$$
$$g + \lceil (g-2)/2 \rceil + \frac{n}{2}\left(\frac{r-2}{r-1}\right) - \frac{n}{(r-1)(g + \lceil (g-2)/2 \rceil}, $$
$$2g - 2 + \frac{n}{2}\left(\frac{r-2}{r-1}\right) - \frac{2n}{(r-1)(2g-2)}\Big\}$$

or the integrity is achieved with a component in $G - S$ of size greater than or equal to $2g - 1$.

3 The integrity of 3-regular cages

Theorem 4 of the previous section can now be used to prove bounds on the integrity for many of the known 3-regular cages. The results are summarized in Table B.

Table B

r-cage	$\|G\|$	Theorem 4
5-cage	10	6
6-cage	14	8
7-cage	24	12
8-cage	30	14
9-cage	58	24
10-cage	70	28

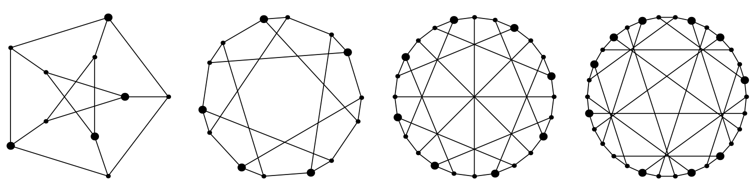

Figure 1: Shown are the 5-cage with $S = 4$ and $m(G-S) = 2$; the 6-cage with $S = 5$ and $m(G - S) = 3$; the 7-cage with $S = 8$ and $m(G - S) = 4$ and the 8-cage with $S = 10$ and $m(G - S) = 4$.

Theorem 5 *The integrity bound given by Theorem 4 (third column of Table B) is the actual integrity.*

Proof: It suffices to find a vertex set that achieves the integrity given by Theorem 4; see also Theorem 6. Observe that in the case of the 9- and 10-cages, it is not possible to have a component of order greater than or equal to 17 or 19 vertices and still achieve a good integrity bound. □

Figures 1, 2, 3 and 4 show these cages together with a (fat) vertex set S, that can be used to achieve the integrity. Figure 1 shows this for the 5-, 6-, 7- and 8-cages. There are 18 9-cages each with 58 vertices; see [4] and [5]. Figures 2 and 3 shows these graphs together with I-sets. There are three 10-cages each with 70 vertices; see [12]. Figure 4 shows these graphs together with I-sets. (An I-set S is a subset of the vertices of G such that $I(G) = |S| + m(G - S)$.)
This settles the question of whether all 18 9-cages have the same integrity, which motivated this study. We summarize this in a theorem below:

Theorem 6 *All* 18 9*-cages have integrity* 24 *and all three* 10*-cages have integrity* 28*.*

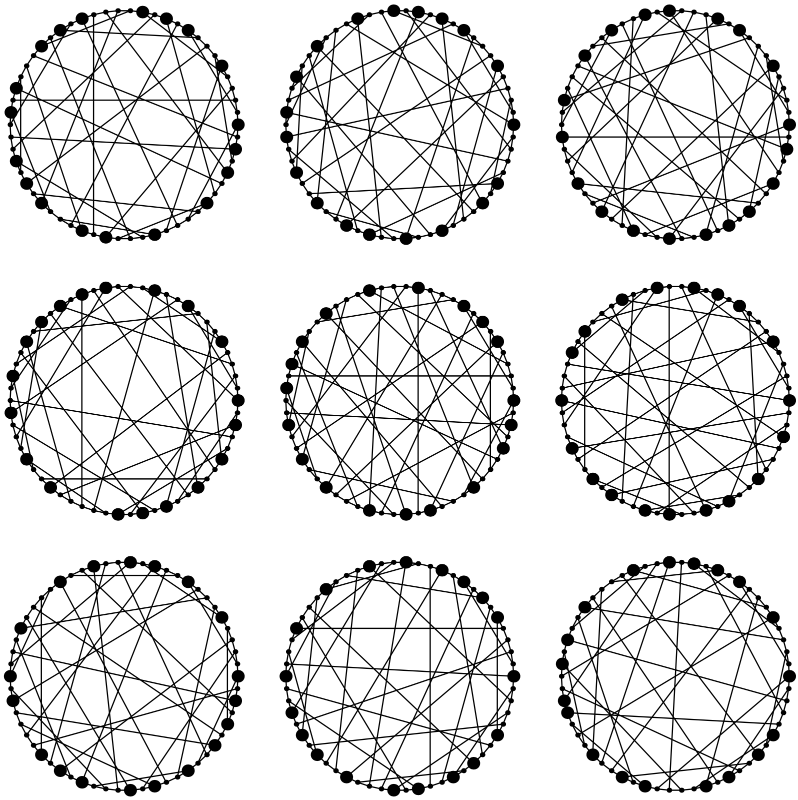

Figure 2: Shown are the first nine 9-cages. For each graph $S = 19$ and $m(G-S) = 5$ resulting in an integrity of 24.

Even with the knowledge that the integrity of a 9- or 10-cage is at least 24 or 28 by Theorem 4, it is nontrivial to search for a vertex set that achieves this integrity. We explain how the set S was found using an example of a 10-cage. In the proof of Lemma 1 we know that the integrity of 28 could be achieved with a value of $z = (2-\sqrt{2nr})/(2(1-r)) = (-1+\sqrt{105})/2 \approx 4.6$. Thus we cut a 5 vertex component from G by deleting its 7 neighbors. The remaining graph has $70 - 12 = 58$ vertices. To obtain an integrity of 28, another 16 vertices could be deleted. The computer program employed went through an exhaustive search deleting all possible 16 vertices sets. If this failed to find a vertex set that achieved the required integrity a different 5 vertex set was selected for the initial cut. Using a standard PC such a search could be completed within a couple of days and at most 4 to 5 tries of this procedure were needed for each of the graphs. Two other 3-regular cages are known, namely

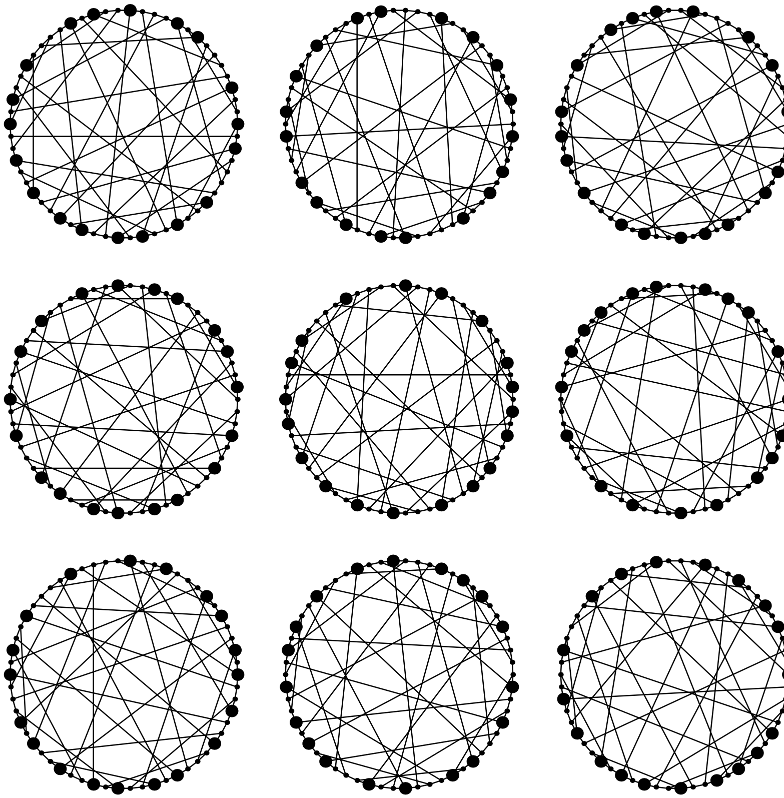

Figure 3: Shown are the second nine 9-cages. For each graph $S = 19$ and $m(G-S) = 5$ resulting in an integrity of 24.

the 11- and 12-cages, see [10]. We have been unable to establish the integrity of either one, due to the increase in complexity in computing a set S that realizes the given integrity bound. An example will illustrate this. The 11-cage has 112 vertices. Lemma 1 gives a bound of 41 on the integrity with $m(G-S)$ having a size of 6 vertices and $|S| = 35$. Lemma 2 gives a bound of 39 that occurs when $|S| = 28$ and $m(G-S) = 11$. However $112-28$ is not divisible by 11 and thus not all components in $G-S$ can be unicyclic components of size 11. Next, Lemma 3 gives a bound of 41 that occurs when $|S| = 28$ and $m(G-S) = 12$. Now $112 - 28 = 84$ which is divisible by 12, so this is theoretically possible. A brute force search for either $|S| = 28$ or 35 (out of 112 vertices) has been beyond our computational capabilities (that is within a reasonable run time on a standard PC). However we can conclude that the integrity of the 11-cage will be at least 40 and the integrity of the 12-cage (which has 126 vertices) will be at least 44.

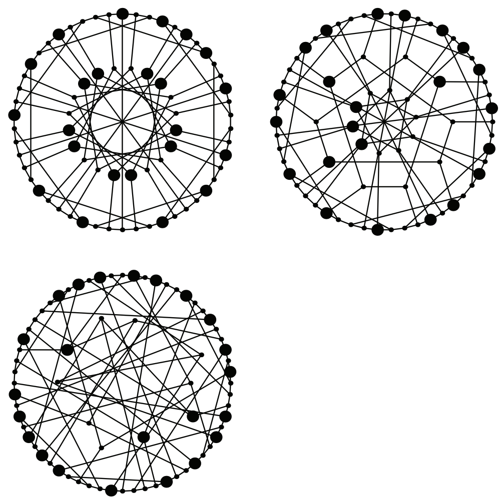

Figure 4: Shown are the 3 10-cages. For each graph $S = 23$ and $m(G - S) = 5$ resulting in an integrity of 28.

4 The integrity of other cages

The integrity of some small other known cages can be computed directly. For not quite so small higher cages, the techniques from the previous section do not work well since many of these cages are highly connected. As a result the vertices in the set S used to achieve the actual integrity may no longer be nonadjacent. The Table C summaries the situation.

Table C

(r, g)-cage	$\|G\|$	Theorem 4
$(4, 5)$-cage	19	11
$(4, 6)$-cage	26	14
$(5, 5)$-cage	30	16
$(6, 5)$-cage	40	21
$(5, 6)$-cage	42	21
$(7, 5)$-cage	50	26

For some of these graphs, we can see that the predicted bound is $n/2 + 1$. This bound occurs when we delete $n/2$ vertices and all components in $G - S$ are single vertices, that is $z = 1$. However for these higher cages this bound will not be sharp since there is no set of size $n/2$ of nonadjacent vertices. In following we list results for higher cages.

1. Complete graphs. The $(n,3)$-cage is the complete graph K_{n+1} and therefore has integrity $n+1$.

2. Complete bipartite graphs. The $(n,4)$-cage is the complete bipartite graph $K_{n,n}$ and therefore has integrity $n+1$.

3. Other 4-regular cages.

Lemma 7 $I((4,5)-cage)=11$ *and* $I((4,6)-cage)=14$

Proof: This is done by a direct computation. See Figure 5 for the I-sets. □.

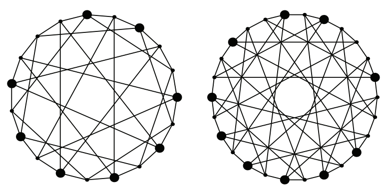

Figure 5: Shown are the $(4,5)$-cage with $S=8$ and $m(G-S)=3$ and the $(4,6)$-cage with $S=10$ and $m(G-S)=4$.

4. Other 5-regular cages.

Lemma 8 *There are four* $(5,5)$*-cages. Three have integrity* 18 *and one has integrity* 19*.*

Proof: There are four $(5,5)$-cages each containing 30 vertices, which is still small enough to allow a direct computation. See Figure 6 for the I-sets. □.

For the $(5,5)$-cages the formula of Lemma 1 yields a bound on the integrity of 16. However this is not a sharp bound since the integrity will be achieved by removing a set S of vertices that contains adjacent vertices. Interestingly the integrity of these four graphs is not the same for all four. In Figure 6 the $(5,5)$-cage on the top right has integrity 19 and the other three have integrity 18.

This was a surprise since the examples of the 3-cages led us to believe that for a fixed g and r different (r,g)-cages have the same integrity. We summarize this in a theorem below:

Theorem 9 *There exist values of* g *and* r *such that nonisomorphic* (r,g)*-cages have different integrities.*

Lemma 10 *The integrity of the* $(5,6)$*-cage is* 22*.*

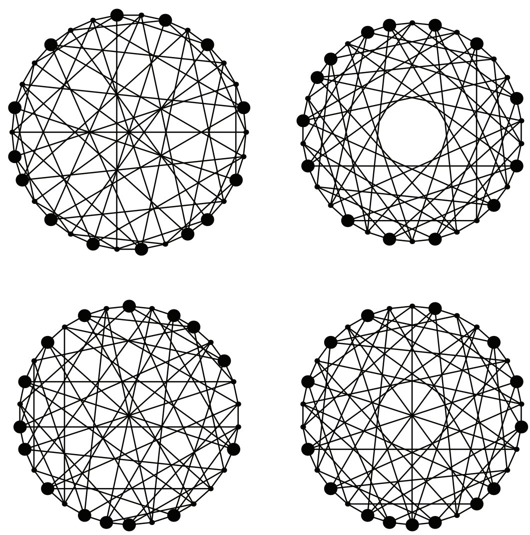

Figure 6: Shown are the four $(5,5)$-cages. For the top left $S = 14$ and $m(G-S) = 4$; top right $S = 14$ and $m(G-S) = 5$ and for two cages in the bottom row $S = 15$ and $m(G-S) = 3$.

Proof: The $(5,6)$-cage has 42 vertices and 105 edges. This is too big to do a brute force search for any possible set S. Our proof will use a mixture of analytic argument and computer search. We want to show that $I(G) > 21$. Assume that $I(G) \leq 21$ and let S be an I-set. Assume that a component in $m(G-S) \geq 11$. By searching through all possible vertex sets S up to order 10, we can rule out that $I(G) \leq 21$ can be achieved with $m(G-S) \geq 11$. For example, if $|S| = 10$, then $m(G-S) \geq 30$. If $m(G-S) < 11$, then the formula of Theorem 4 yields a bound on the integrity of 21. Moreover, from Equation (2), $m(G-S) = 2$ or 3 and therefore $|S| = 19$ or 18.

Consider the case when 19 vertices are deleted and $m(G-S) = 2$. If there is more than one pair of adjacent vertices in S then at most 93 edges are deleted, leaving at least 12 edges. However the remaining $42 - 19 = 23$ vertices can not accommodate 12 edges if $m(G-S) = 2$. Thus at most one pair of the 19 vertices can be adjacent. Similarly, we can show that in the case of $|S| = 18$ and $m(G-S) = 3$ at most one pair of the 18 vertices can be adjacent. If one removes 18 or 19 vertices with at most one adjacent pair from the $(5,6)$-cage then at least two of the vertices removed are such that they have exactly one vertex between them on the large 42-cycle shown in Figure 7. From Figure 7, we see that $(5,6)$-cage has a rotational symmetry of order 21. This means that there exactly two nonequivalent ways to choose two vertices on the 42-cyle that have exactly one vertex between them. For both choices of two such vertices, an exhaustive search was conducted on graphs of 40 vertices (obtained by deleting the two vertices) using $|S| = 16$ and $|S| = 17$. An integrity less than or equal to 21 could not be obtained this way. Therefore $I(G) > 21$. On the other hand an integrity of 22 is possible, see Figure 7. □.

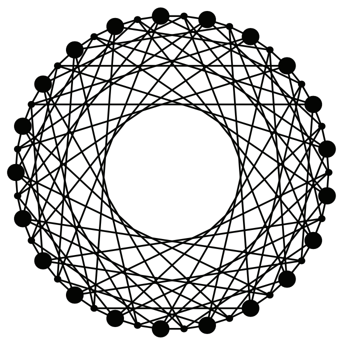

Figure 7: Shown is the $(5,6)$-cage together with one possible choice of a set S (the fat vertices). Notice that the thin vertices are another choice for S.

5. Other small cages.

Lemma 11 *The* $(7,5)$*-cage has an integrity of* 30.

Proof: There is one unique $(7,5)$-cage containing 50 vertices and 175 edges [8]. This cage can be constructed by starting with 5 pentagons $P_1, \ldots, P_5$ and 5 pentagrams $Q_1, \ldots, Q_5$. All the remaining edges are edges that connect each vertex contained in a pentagon with five vertices each belonging to one of the 5 different pentagrams [8]. Thus if one deletes the 5 pentagrams (i.e. 25 vertices), then there will be five disjoint pentagons remaining. This shows that $I((7,5) - cage) \leq 30$.

To show that $I((7,5) - cage) = 30$, we first observe that if a component of $G - S$ has 6 vertices, then more than 25 vertices have to be deleted from G. Thus to achieve the integrity, no component can have a order greater than 5. The formula of Lemma 1 yields only a bound on the integrity of 26 and is not helpful. However, the large group of automorphisms [8] can be used to exhaust all possibilities where $G - S$ contains a component of order less than 5. For example the group of automorphisms is edge transitive. Thus if there is a component of order 2 in $G - S$ we can simply cut out a single edge (any edge will do) by deleting its 12 neighbors. The remaining graph has 36 vertices and its integrity can be obtained by an exhaustive search on a computer. The result shows that no integrity less or equal to 30 can be achieved, if $m(G - S) = 2$. The cases where there is a component of orders 1, 3 or 4 are dealt with similarly and the details are left to the reader. For an example of a vertex set S achieving the integrity see Figure 8. Since there are 175 edges in the figure, it is very crowded. We give an explicit description of the $(7,5)$-cage together with the set S in Appendix A. □.

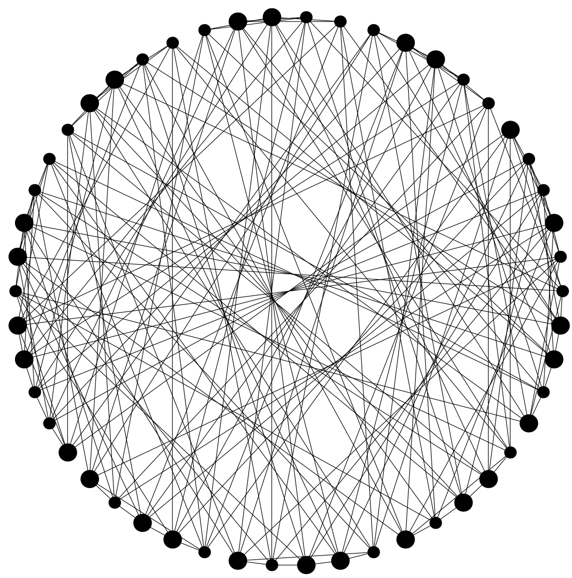

Figure 8: Shown is the $(7,5)$-cage together with one possible choice of the sets S (the fat vertices) achieving the minimal integrity.

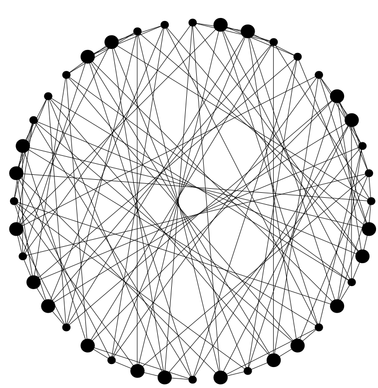

Figure 9: Shown is the $(6,5)$-cage together with one possible choice of the sets S (the fat vertices) achieving the minimal integrity.

Lemma 12 *The* $(6,5)$*-cage has an integrity of* 25.

Proof: The $(6,5)$-cage can be obtained from the $(7,5)$-cage by deleting one pentagon and one pentagram. If one deletes the four remaining pentagrams (20 vertices), then the remaining graph will have four components each of which is a pentagon. Thus the integrity is at most 25. To show that this is the actual integrity, computations similar to the previous lemma are needed and the details will be left to the reader. One possible set S is show in Figure 9 and a description of the $(6,5)$-cage is given in Appendix A. □

The above analysis shows the limitations of the Lemmas 1 and 2 in finding bounds for the integrity of these cages. Most of the other known higher cages have a much larger number of vertices than the ones discussed here and thus the computation of the exact integrity remains elusive.

5 Observations and Conclusions

The following points are worthwhile, because they refute conjectures that suggested themselves to the authors at some point during this investigation.

1. If for a pair of values (r,g), there are several nonisomorphic cages, then these do not have to have identical integrities. (See the $(5,5)$-cage example discussed in the previous section.)

2. Among all graphs G which are r-regular and of order n, one can ask which graph G has the largest integrity? If there exists an (r,g)-cage G_0 of order n, does G_0 have the maximal integrity among all graphs G? This is not true; see the example in Figure 10. The graph on the left is the $(3,4)$-cage graph with integrity 4 but the one on the right is 3-regular with integrity 5.

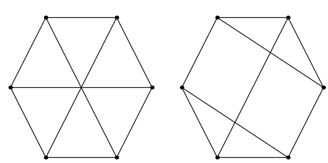

Figure 10:

3. We define the *k-integrity* $I_k(G)$ of G to be the lowest upper bound one can achieve on the integrity be deleting k vertices, that is $I_k(G) = k + \min_{|S|=k} m(G-S)$. Then one could conjecture that as k increases the values of $I_k(G)$ are at first monotonically decreasing and then change to monotonically increasing. That is, the values of k for which $I_k(G)$ is minimal are a set of consecutive integers. Note, this is clearly not true in general, but it could have been true for cages. However, even if one restricts attention to cages, this is not true. Computer calculations have shown that there are examples with two distinct minima. For example for the $(5,5)$-cage: $I_{14}((5,5)-\text{cage}) = 18$, $I_{15}((5,5)-\text{cage}) = 19$ and $I_{16}((5,5)-\text{cage}) = 18$.

Appendix A

The first column in the tables give the vertex number. The other columns list the vertex numbers of its neighbors. The set S for the $(7,5)$-cage in Figure 8 is

$S = \{26, 31, 36, 41, 46, 2, 34, 39, 44, 49, 9, 14, 19, 24, 30, 5, 33, 38, 43, 48, 8, 13, 18, 23, 27\}$.

The set S for the $(6,5)$-cage in Figure 9 is

$S = \{6, 16, 26, 36, 18, 28, 38, 40, 2, 19, 29, 39, 5, 14, 24, 34, 7, 13, 23, 33\}$

$(7,5)$-cage:

1	3	4	26	31	36	41	46	26	1	6	11	16	21	27	30
2	4	5	27	32	37	42	47	27	2	7	12	17	19	22	28
3	1	5	28	33	38	43	48	28	3	8	13	18	23	27	29
4	1	2	29	34	39	44	49	29	4	9	14	19	24	28	30
5	2	3	30	35	40	45	50	30	5	10	15	20	25	26	29
6	8	9	26	32	38	44	50	31	1	10	14	18	22	32	35
7	9	10	27	33	39	45	46	32	2	6	15	19	23	31	33
8	6	10	28	34	40	41	47	33	3	7	11	20	24	32	34
9	6	7	29	35	36	42	48	34	4	8	12	16	25	33	35
10	7	8	30	31	37	43	49	35	5	9	13	17	21	31	34
11	13	14	26	33	40	42	49	36	1	9	12	20	23	37	40
12	14	15	27	34	36	43	50	37	2	10	13	16	24	36	38
13	11	15	28	35	37	44	46	38	3	6	14	17	25	37	39
14	11	13	29	31	38	45	47	39	4	7	15	18	21	38	40
15	12	13	30	32	39	41	48	40	5	8	11	19	22	36	39
16	18	19	26	34	37	45	48	41	1	8	15	17	24	42	45
17	19	20	27	35	38	41	49	42	2	9	11	18	25	41	43
18	16	20	28	31	39	42	50	43	3	10	12	19	21	42	44
19	16	17	29	32	40	43	46	44	4	6	13	20	22	43	45
20	17	18	30	33	36	44	47	45	5	7	14	16	23	41	44
21	23	24	26	35	39	43	47	46	1	7	13	19	25	47	50
22	24	25	27	31	40	44	48	47	2	8	14	20	21	46	48
23	21	25	28	32	36	45	49	48	3	9	15	16	22	47	49
24	21	22	29	33	37	41	50	49	4	10	11	17	23	48	50
25	22	23	30	34	38	42	46	50	5	6	12	18	24	46	49

(6, 5)-cage:

1	3	4	22	28	34	40	21	5	9	13	17	22	25
2	4	5	23	29	35	36	22	1	10	14	18	21	23
3	1	5	24	30	31	37	23	2	6	15	19	22	24
4	1	2	25	26	32	38	24	3	7	11	20	23	25
5	2	3	21	27	33	39	25	4	8	12	16	21	24
6	8	9	23	30	32	39	26	4	7	15	18	27	30
7	9	10	24	26	33	40	27	5	8	11	19	26	28
8	6	10	25	27	34	36	28	1	9	12	20	27	29
9	6	7	21	28	35	37	29	2	10	13	16	28	30
10	7	8	22	29	31	38	30	3	6	14	17	26	29
11	13	14	24	27	35	38	31	3	10	12	19	32	35
12	14	15	25	28	31	39	32	4	6	13	20	31	33
13	11	15	21	29	32	40	33	5	7	14	16	32	34
14	11	12	22	30	33	36	34	1	8	15	17	33	35
15	12	13	23	26	34	37	35	2	9	11	18	31	34
16	18	19	25	29	33	37	36	2	8	14	20	37	40
17	19	20	21	30	34	38	37	3	9	15	16	36	38
18	16	20	22	26	35	39	38	4	10	11	17	37	39
19	16	17	23	27	31	40	39	5	6	12	18	38	40
20	17	18	24	28	32	36	40	1	7	13	19	36	39

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