AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **43** (2009), Pages 103–113

A new expression for the adjoint polynomial of a path*

Jianfeng Wang†

College of Mathematics and System Science
Xinjiang University
Urumqi 830046
P. R. China
jfwang4@yahoo.com.cn

K.L. Teo

Institute of Fundamental Sciences
Massey University, Palmerston North
New Zealand
K.L.Teo@massey.ac.nz

Qiongxiang Huang

College of Mathematics and System Science
Xinjiang University, Urumqi, Xinjiang 830046
P. R. China
huangqx@xju.edu.cn

Chengfu Ye   Ruying Liu

Department of Mathematics and Information Science
Qinghai Normal University, Xining, Qinghai 810008
P. R. China
yechf@qhnu.edu.cn   liury@qhnu.edu.cn

Abstract

For $k \geq 3$, let $G = T(l_1, l_2, \ldots, l_k)$ be the tree with exactly one vertex v of degree k, and $G - v = P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_k}$, where P_n is the path of order n.

* Supported by the National Science Foundation of China (No. 10761008).

† Corresponding author. Second address: Department of Mathematics and Information Science, Qinghai Normal University, Xining, Qinghai 810008, P. R. China.

A graph is adjointly integral if all the roots of its adjoint polynomial are integers. In this paper, we derive a new expression (different from the one given by Dong, Teo, Little and Hendy in *Australas. J. Combin.* 25 (2002), 167–174) for the adjoint polynomial of a path to provide a new method for finding the roots of the adjoint polynomials of paths and cycles. Moreover, we establish the following results:

1. $\frac{k^2}{1-k} < \beta(G) \leq -k$, where $\beta(G)$ is the minimum real root of the adjoint polynomial of G.
2. The tree $T(l, l, \ldots, l)$ is adjointly integral if and only if $l \in \{1, 2\}$.
3. If $l_1 \geq 12$ or $1 \leq l_i \leq 2$ and $(l_1, l_2, \ldots, l_k) \notin \{(1, 1, \ldots, 1), (2, 2, \ldots, 2)\}$, then $T(l_1, l_2, \ldots, l_k)$ is not adjointly integral for $l_1 \geq l_2 \geq \cdots \geq l_k \geq 1$.

1 Introduction

All graphs considered here are finite and simple. For a graph G, let $V(G)$, $E(G)$, and $\overline{G}$, respectively, be the set of vertices, the set of edges, and the complement of G. Notation and terminology not defined here will conform to those in [1]. Let G be a graph with n vertices. A partition $\{A_1, A_2, \ldots, A_k\}$ of the vertex set of G, where k is a positive integer, is called an *k-independent partition* if every A_i is a nonempty independent set of G. We denote by $\alpha(G, k)$ the number of k-independent partitions of G. Then *the chromatic polynomial* of G is given by

$$P(G, \lambda) = \sum_{r \geq 1} \alpha(G, r)(\lambda)_r,$$

where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-r+1)$ for all $r \geq 1$. The readers can turn to [4, 5] for details on chromatic polynomials.

An *ideal subgraph* of a graph G is a spanning subgraph of G whose components are all complete graphs([10]). Let $N(G, k)$ denote the number of an ideal subgraphs with $p-k$ components. For a graph G with n vertices and q edges, it is obvious that $N(G, k) = \alpha(\overline{G}, n-k)$ for $k \geq 0$.

Definition 1.1.([10]) Let graph G with n vertices, the polynomial

$$h(G, x) = \sum_{k=0}^{n-1} N(G, k)x^{n-k}$$

is called the *adjoint polynomial* of G.

Remark 1.1. Another polynomial that is related to the adjoint polynomial of a graph is the σ–polynomial ([6]). This polynomial is the same as $h(\overline{G}, x)/x^{\chi(G)}$, where $\chi(G)$ is the chromatic number of G.

Definition 1.2.([10]) Let G be a graph and $h_1(G, x)$ the polynomial with a nonzero constant term such that $h(G, x) = x^{\rho(G)}h_1(G, x)$. If $h_1(G, x)$ is an irreducible polynomial over the rational number field, then G is called *an irreducible graph.*

Remark 1.2. It follows, from Remark 1.1 and the above definition, that $\rho(G) = \chi(\overline{G})$, where $\chi(\overline{G})$ is the chromatic number of $\overline{G}$.

Definition 1.3. The adjoint roots of a graph G are the zeros of the adjoint polynomial of G.

Now, we introduce some graphs and notation used in this paper. By P_n and C_n we denote the path and the cycle with orders n, respectively. The T-shaped tree $T(l_1, l_2, l_3)$, which was first defined and studied by R.Liu ([9]), is a tree with exactly one vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$. We generalize this by defining the graph $T_{l_1,l_2,\ldots,l_k}$ to be the graph obtained by replacing each the edges of the star $K_{1,k}$ by paths of lengths $l_1, l_2, \ldots, l_k$. We call this a *star-tree* with k branches. In what follows, it will be assumed that $l_1 \geq l_2 \geq \cdots \geq l_k \geq 1$ and $k \geq 3$. We write $k \cdot l = (l_1, l_2, \ldots, l_k)$ if $l_1 = l_2 = \cdots = l_k = l$. By $\beta(G)$ we denote the minimum adjoint root of graph G. For graphs G_1 and G_2, $G_1 \subset G_2$(resp. $G_1 \subseteq G_2$) denote that G_1 is a proper subgraph of G_2(resp. G_1 is a subgraph of G_2).

It is obvious that the adjoint polynomial of a graph is a polynomial in variable x. In this paper, we give new expressions of adjoint polynomials of paths and cycles by means of trigonometric functions. As an application, we find the adjoint roots of all the paths and cycles, which were first obtained by Dong et al ([3]). We also obtain the lower and upper bounds for the minimum adjoint root of the star-tree $T(l_1, l_2, \ldots, l_k)$. The notion of adjointly integral graph whose adjoint roots are all integers is first introduced and the conditions for the star-tree $T(l_1, l_2, \ldots, l_k)$ to be adjointly integral is investigated.

2 A new method for finding the adjoint roots of paths and cycles

Lemma 2.1.([10],[13])

(1) *For* $n \geq 2$, $h(P_n, x) = \sum_{k \leq n} \binom{k}{n-k} x^k$; $\beta(P_n) > 4$.

(2) *For* $n \geq 3$, $h(P_n, x) = x(h(P_{n-1}, x) + h(P_{n-2}, x))$.

(3) *For* $n \geq 4$, $h(C_n, x) = \sum_{k \leq n} \frac{n}{k} \binom{k}{n-k} x^k$, $\beta(C_n) > -4$.

(4) *For* $n \geq 6$, $h(C_n, x) = x(h(C_{n-1}, x) + h(C_{n-2}, x))$.

Lemma 2.2.([3]) (**1**) *For* $uv \in E(G)$ *not contained in any triangle of* G, *we have*

$$h(G, x) = h(G - uv, x) + xh(G - \{u, v\}, x).$$

(**2**) *For any positive integer* n, $h(P_n, x)$ *has the following zeros:*

$$\underbrace{0, 0, \ldots, 0}_{\lceil \frac{n}{2} \rceil}, \quad -2 - 2\cos\frac{2s\pi}{n+1}, \quad s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

(**3**) *For any positive integer* $n \geq 4$, $h(C_n, x)$ *has the following zeros:*

$$\underbrace{0, 0, \ldots, 0}_{\lceil \frac{n}{2} \rceil}, \quad -2 - 2\cos\frac{(2s-1)\pi}{n}, \ s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

In this section, we shall find an explicit expression for $h(P_n, -4\cos^2\theta)$ and $h(C_n, -4\cos^2\theta)$, from which we find a new proof of (2) and (3) of Lemma 2.2.

Lemma 2.3. **(1)** *For* $n \geq 4$, $h_1(C_n, x) = h_1(P_n, x) + h_1(P_{n-2}, x)$.

(**2**) *For* $n \geq 2$, $h_1(P_n, x) = \begin{cases} xh_1(P_{n-1}, x) + h_1(P_{n-2}, x), & \text{if } n = 2k; \\ h_1(P_{n-1}, x) + h_1(P_{n-2}, x), & \text{if } n = 2k+1. \end{cases}$

Proof. **(1)** We distinguish the following two cases:

Case 1. $n = 2k$.

From (1) of Lemma 2.2, we have that

$$h(C_n, x) = h(P_n, x) + xh(P_{n-2}, x). \tag{2.1}$$

Noting that both n and $n-2$ are even, we obtain, from (1) and (3) of Lemma 2.1, that $\rho(C_n) = \frac{n}{2}$, $\rho(P_n) = \frac{n}{2}$ and $\rho(P_{n-2}) = \frac{n-2}{2}$. Eliminating the common factor $x^{\frac{n}{2}}$ from both sides of Equation (2.1), we get the result.

Case 2. $n = 2k+1$.

Remark that both n and $n-2$ are all odd, it follows, from (1) and (3) of Lemma 2.1, that $\rho(C_n) = \frac{n+1}{2}$, $\rho(P_n) = \frac{n+1}{2}$ and $\rho(P_{n-2}) = \frac{n-2}{2}$. Eliminating the common factor $x^{\frac{n+1}{2}}$ from both sides of Equation (2.1), we arrive at the result.

Part (2) of the lemma can be proved in a similar way. □

Theorem 2.1. *For* $n \geq 2$,

$$h_1(P_n, -4\cos^2\theta) = \begin{cases} (-1)^k \dfrac{\sin(n+1)\theta}{\sin\theta}, & \text{if } n = 2k; \\ (-1)^k \dfrac{\sin(n+1)\theta}{\sin 2\theta}, & \text{if } n = 2k+1. \end{cases}$$

Proof. We will prove the theorem by mathematical induction on n.

By (1) of Lemma 2.1, we see that $h_1(P_2, x) = x+1$. By substituting $x = -4\cos^2\theta$ in $h_1(P_2, x)$, we obtain that

$$h_1(P_2) = 1 - 4\cos^2\theta = -2\cos 2\theta - 1 = \frac{-2\sin\theta\cos 2\theta - \sin\theta}{\sin\theta} = -\frac{\sin 3\theta}{\sin\theta}.$$

Thus the result holds for $n = 2$.

Similarly, $h_1(P_3) = x + 2$. By substituting $x = -4\cos^2\theta$ in $h_1(P_3, x)$, we obtain that

$$h_1(P_3) = 2 - 4\cos^2\theta = -2\cos 2\theta = -\frac{2\sin 2\theta\cos 2\theta}{\sin 2\theta} = -\frac{\sin 4\theta}{\sin 2\theta}.$$

Thus the result holds for $n = 3$.

Assume that the theorem holds when the path has less than n vertices, where $n \geq 4$. We distinguish the following cases:

Case 1. $n = 2k$.

In the light of (2) of Lemma 2.3 and by substituting $x = -4\cos^2\theta$, we have, by the inductive assumption, that

$$\begin{aligned} h_1(P_n, x) &= xh_1(P_{n-1}, x) + h_1(P_{n-2}, x) \\ &= (-1)^k \frac{4\cos^2\theta \sin n\theta}{\sin 2\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin\theta} \\ &= (-1)^k \frac{2\sin n\theta \cos\theta}{\sin\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin\theta} \\ &= (-1)^k \frac{\sin(n+1)\theta + \sin(n-1)\theta}{\sin\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin\theta} \\ &= (-1)^k \frac{\sin(n+1)\theta}{\sin\theta}. \end{aligned}$$

Thus the theorem holds in this case.

Case 2. $n = 2k + 1$.

In terms of (2) of Lemma 2.3 and by substituting $x = -4\cos^2\theta$, we obtain, by the inductive assumption, that

$$\begin{aligned} h_1(P_n, x) &= h_1(P_{n-1}, x) + h_1(P_{n-2}, x) \\ &= (-1)^k \frac{\sin n\theta}{\sin\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin 2\theta} \\ &= (-1)^k \frac{2\cos\theta \sin n\theta}{\sin 2\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin 2\theta} \\ &= (-1)^k \frac{\sin(n+1)\theta + \sin(n-1)\theta}{\sin 2\theta} + (-1)^{k-1} \frac{\sin(n-1)\theta}{\sin 2\theta} \\ &= (-1)^k \frac{\sin(n+1)\theta}{\sin 2\theta}. \end{aligned}$$

Thus the theorem holds in this case. □

Theorem 2.2. *Let C_n be the cycle with order $n \geq 4$. Then*

$$h_1(C_n, -4\cos^2\theta) = \begin{cases} (-1)^k 2\cos n\theta, & \text{if } n = 2k; \\ (-1)^k \dfrac{\cos n\theta}{\cos\theta}, & \text{if } n = 2k+1. \end{cases}$$

Proof. Case 1. $n = 2k$.

We obtain, from (1) of Lemma 2.3 and Theorem 2.1, that

$$\begin{aligned} h_1(C_n,-4\cos^2\theta) &= h_1(P_n,-4\cos^2\theta)+h_1(P_{n-2},-4\cos^2\theta) \\ &= (-1)^k\frac{\sin(n+1)\theta}{\sin\theta}+(-1)^{k-1}\frac{\sin(n-1)\theta}{\sin\theta} \\ &= (-1)^{k-1}\frac{\sin(n-1)\theta-\sin(n+1)\theta}{\sin\theta} \\ &= (-1)^{k-1}\frac{2\cos n\theta\sin(-\theta)}{\sin\theta} \\ &= (-1)^k 2\cos n\theta. \end{aligned}$$

Case 2. $n=2k+1$.
We obtain, from (1) of Lemma 2.3 and Theorem 2.1, that

$$\begin{aligned} h_1(C_n,-4\cos^2\theta) &= h_1(P_n,-4\cos^2\theta)+h_1(P_{n-2},-4\cos^2\theta) \\ &= (-1)^k\frac{\sin(n+1)\theta}{\sin 2\theta}+(-1)^{k-1}\frac{\sin(n-1)\theta}{\sin 2\theta} \\ &= (-1)^{k-1}\frac{\sin(n-1)\theta-\sin(n+1)\theta}{\sin 2\theta} \\ &= (-1)^{k-1}\frac{2\cos n\theta\sin(-\theta)}{2\sin\theta\cos\theta} \\ &= (-1)^k\frac{\cos n\theta}{\cos\theta}. \end{aligned}$$

□

Theorem 2.3. **(1)** *For* $n\ge 1$,

$$h(P_n,-4\cos^2\theta)=(-1)^n\frac{2^n\sin(n+1)\theta\cos^n\theta}{\sin\theta}.$$

(2) *For* $n\ge 4$,

$$h(C_n,-4\cos^2\theta)=(-1)^n 2^{n+1}\cos n\theta\cos^n\theta.$$

Proof. **(1)** By (1) of Lemma 2.1, we obtain that $h(P_1,x)=x=-4\cos^2\theta$. Hence the formula holds for $n=1$. If $n\ge 2$, from (1) of Lemma 2.1, we arrive at $\rho(P_n)=\lceil\frac{n}{2}\rceil$. By $h(P_n,x)=x^{\rho(P_n)}h_1(P_n,x)$, we arrive at

$$h(P_n,-4\cos^2\theta)=(-4\cos^2\theta)^{\rho(P_n)}h_1(P_n,-4\cos^2\theta).$$

The result follows from Theorem 2.1.

(2) Result (2) can be proved in a similar way. □

Corollary 2.1. (**1**) *For any positive integer* n, *the set of adjoint roots of* P_n *is*

$$ARS(P_n)=\left\{\left\lceil\frac{n}{2}\right\rceil\cdot 0,\ -2-2\cos\frac{2k\pi}{n+1}\ \middle|\ 1\le k\le\left\lfloor\frac{n}{2}\right\rfloor\right\}.$$

(**2**) *For any positive integer* $n \geq 4$*, the set of adjoint roots of* C_n *is*

$$ARS(C_n) = \left\{ \left\lceil \frac{n}{2} \right\rceil \cdot 0, \ -2 - 2\cos\frac{(2k-1)\pi}{n} \ \Big| \ 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Proof. **(1)** According to (1) of Theorem 2.3, we have

$$h(P_n, -4\cos^2\theta) = 0 \text{ if and only if } \sin{(n+1)\theta}\cos^n\theta = 0,$$

which leads to

$$\theta = \frac{k\pi}{n+1} \text{ or } \theta = k\pi + \frac{\pi}{2}, \text{ where } k = 0, \pm 1, \pm 2, \ldots.$$

By substituting the values of θ in $x = -4\cos^2\theta = -2(1+\cos 2\theta)$, we obtain the following adjoint roots of P_n

$$x = -2 - 2\cos\frac{2k\pi}{n+1} \text{ or } x = 0, \text{ where } k = 0, \pm 1, \pm 2, \ldots.$$

From (1) of Lemma 2.1, we arrive at $\rho(G) = \lceil \frac{n}{2} \rceil$, which implies that the multiplicity of the adjoint root 0 is $\lceil \frac{n}{2} \rceil$. It is obvious that

$$-2 - 2\cos\frac{-2k\pi}{n+1} = -2 - 2\cos\frac{2k\pi}{n+1}, \text{ where } k \geq 0.$$

If $k = 0$, then $x = -4$, which contradicts $\beta(P_n) > -4$. Thus $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

(2) With a similar method, we can prove that (2) of the theorem holds. □

3 The minimum adjoint root of a star-tree and adjointly integral star-trees

A polynomial is integral if all its zeros are integers. In the theory of graph spectra, a graph is called integral if its characteristic polynomial is integral. In [7] Lepovic and Gutman studied some spectral properties of star-trees. In particular, they provided a new way of characterizing integral star-trees. The main aim of this section is to investigate which star-trees possess integral adjoint polynomials. A graph is said to be adjointly integral if its adjoint polynomial is integral. We will see that if a tree is integral, then it is necessarily adjointly integral. However, the converse is not true. For example, the path P_3 is adjointly integral but is not integral.

By Corollary 2.1 and $h(C_3, x) = x(x^2 + 3x + 1)$, we have the following result:

Corollary 3.1. (**1**) *The path* P_n *is adjointly integral if and only if* $n \in \{1, 2, 3, 5\}$ *for any positive integer* n*.*

(2) *The cycle* C_n *is not adjointly integral for* $n \geq 3$*.*

Lemma 3.1.([12]) *Let T be a tree and $f(T,\lambda)=\lambda^{\theta(T)}f_1(T,\lambda)$ be its characteristic polynomial, where $\theta(T)$ denotes the degree of the lowest degree term of $f(T,\lambda)$. If $x=-\lambda^2$, then $h_1(T,x)=(-1)^k f_1(T,\lambda)$, where k is the number of edges of the maximum matching of T.*

As a consequence of Lemma 3.1 we have:

Corollary 3.2. *Let T be a tree of order n. If the eigenvalues of T are $\lambda_1,\lambda_2,\dots,\lambda_n$, then the adjoint roots of T are $-\lambda_1^2,-\lambda_2^2,\dots,-\lambda_n^2$.*

Lemma 3.2.([7]) *If λ_1 is the largest eigenvalue of the star-tree $T(l_1,l_2,\dots,l_k)$, then $\sqrt{k}\le\lambda_1<k/\sqrt{k-1}$ holds for any positive integers $l_1\ge l_2\ge\cdots\ge l_k\ge 1$.*

It is well-known that the eigenvalues in Corollary 3.2 satisfy the property that $\lambda_i=-\lambda_{n-i+1}$, for $i=1,2,\dots,\lfloor\frac{n}{2}\rfloor$. Consequently, $\beta(T)=-\lambda_1^2=-\lambda_n^2$. By using this together with Lemma 3.2 we obtain the following upper and lower bounds for the minimum adjont root of a star-tree.

Theorem 3.1. *For any positive integers $l_1\ge l_2\ge\cdots\ge l_k\ge 1$ and $k\ge 3$, the minimum adjoint root of $T(l_1,l_2,\dots,l_k)$ satisfies the following inequality*

$$\frac{k^2}{1-k}<\beta(T(l_1,l_2,\dots,l_k))\le -k.$$

From the above theorem, the distribution of the minimum adjoint roots of star-trees with three branches is completely determined.

Corollary 3.3. *Let $\mathcal{S}=\{T(l_1,l_2,l_3)\mid l_1\ge l_2\ge l_3\ge 1\}$, then*

(1)([13]) $\beta(T(l_1,l_2,l_3))=-3$ *if and only if*

$$(l_1,l_2,l_3)\in\mathcal{T}_1=\{(1,1,1)\}.$$

(2)([13]) $-4<\beta(T(l_1,l_2,l_3))<-3$ *if and only if*

$$(l_1,l_2,l_3)\in\mathcal{T}_2=\{(l_1,1,1,),(4,2,1),(3,2,1),(2,2,1)\mid l_1\ge 2\}.$$

(3)([13]) $\beta(T(l_1,l_2,l_3))=-4$ *if and only if*

$$(l_1,l_2,l_3)\in\mathcal{T}_3=\{(5,2,1),(2,2,2),(3,3,1)\}.$$

(4)([13]) $-2-\sqrt{5}\le\beta(T(l_1,l_2,l_3))<-4$ *if and only if* $(l_1,l_2,l_3)\in\mathcal{T}_4=$

$$\{(3,3,2),(l_1,2,1)\mid l_1>5\}\cup\{(l_1,l_2,1)\mid l_1>3,l_2>2\}\cup\{(l_1,2,2)\mid l_1>2\}.$$

(5) $-4.5<\beta(T(l_1,l_2,l_3))<-2-\sqrt{5}$ *if and only if*

$$(l_1,l_2,l_3)\in\mathcal{S}\backslash(\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3\cup\mathcal{T}_4).$$

□

Lemma 3.3.([8]) *Let G be a graph that is not complete. For any vertex $v \in V(G)$, the set $\mathcal{G}_v = \{K|K$ is a complete subgraph of G containing the vertex $v)$. Then*

$$h(G,x) = x \sum_{K\in \mathcal{G}_v} h(G-V(K),x),$$

where $G - V(K)$ is the subgraph obtained from G by deleting the vertices in $V(K)$ and together with their incident edges.

Corollary 3.4. *The adjoint polynomial of $T(l_1, l_2, \ldots, l_k)$ is given by*

$$x\prod_{i=1}^{k} h(P_{l_i},x)\left(1+\sum_{i=1}^{k}\frac{h(P_{l_i-1},x)}{h(P_{l_i},x)}\right).$$

Proof. The proof follows directly from Lemma 3.5. □

Theorem 3.2. *For $k \geq 3$ and $l \geq 1$, the star-tree $T(k\cdot l)$ is adjointly integral if and only if $l \in \{1, 2\}$.*

Proof. By Corollary 3.4 and calculation, we arrive at $h(T(k\cdot 1),x) = x^k(x+k)$ and $h(T(k\cdot 2),x) = x^{k+1}(x+1)^{k-1}(x+k+1)$, which imply the sufficiency of the theorem holds.

For the proof of necessity, the following cases are taken into account.

Case 1. $l = 3$.

From Corollary 3.4, we arrive at $h(T(k\cdot 3),x) = x^{2k}(x+2)^{k-1}(x^2+(k+2)x+k)$. It is obvious that $\dfrac{-(k+1)\pm\sqrt{k^2+4}}{2}$ are two adjoint roots of $T(k\cdot 3)$. Suppose that $\sqrt{k^2+4} = a$, where a is a positive integer. Then $a^2-k^2 = (a+k)(a-k) = 4$. Recall that k is a positive integer and $k \geq 3$, then $a+k > 0$ and $a-k > 0$, which leads to $a > k \geq 3$. Hence, $a+k \geq 7$ that results in $0 < a-k < 1$ which is impossible. Thus, $T(k\cdot 3)$ is not adjointly integral.

Case 2. $l = 5$.

In view of Corollary 3.4, we arrive at $h(T(k\cdot 5),x) = x^3h^{k-1}(P_5)g_1(x)$, where $g_1(x) = x^3+(k+4)x^2+(3k+3)x+k$. By a transformation $x \mapsto y - \frac{k-4}{3}$, we obtain, from $g_1(x)$, that $g_2(y) = y^3+by+c$, where $b = 3k+3-\frac{(k+4)^4}{3}$ and $c = \frac{2(k+4)^2}{27} - (k+1)(k+4)+k$. According to the formula of solving the roots of cubic equations (see [2] pp: 4), we get that the roots of $g_2(y)$ are $\alpha+\beta$ and $-(\frac{\alpha+\beta}{2}) \pm i\sqrt{3}(\frac{\alpha-\beta}{2})$ which indicate that the roots of $g_1(x)$ are $\alpha+\beta-\frac{(k+4)^4}{3}$ and $-(\frac{\alpha+\beta}{2}) \pm i\sqrt{3}(\frac{\alpha-\beta}{2}) - \frac{(k+4)^4}{3}$, where $\alpha = \sqrt[3]{\frac{-c}{2}+\sqrt{\frac{c^2}{2}+\frac{b^3}{27}}}$ and $\beta = \sqrt[3]{\frac{-c}{2}-\sqrt{\frac{c^2}{2}+\frac{b^3}{27}}}$. If $T(k\cdot 5)$ is adjointly integral, then $\alpha - \beta = 0$ which shows by Mathematica that $k \in \{-7.74345 \pm 2.24615i, -1.36296 \pm 1.12805i, -0.574083, 21.7869\}$ which contradicts k being an integer. So $T(k\cdot 5)$ is not adjointly integral.

Case 3. $l = 4$ or $l \geq 6$.

From Corollary 3.4, we know the adjoint roots of $T(k \cdot l)$ contains the adjoint roots of P_l. By Corollary 3.1, P_l is not adjointly integral, so neither is $T(k \cdot l)$. □

Lemma 3.4.([11]) *Let T be a tree and the adjoint roots of T and $T - v$ be, respectively, $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_n$ and $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{n-1}$, where $v \in V(T)$ and $|V(T)| = n$.*

(1) *If $n = 2k$, then*

$$\eta_1 \leq \theta_1 \leq \eta_2 \leq \theta_2 \leq \cdots \leq \eta_{k-1} \leq \theta_{k-1} \leq \eta_k < 0.$$

(2) *If $n = 2k+1$, then*

$$\eta_1 \leq \theta_1 \leq \eta_2 \leq \theta_2 \leq \cdots \leq \theta_{k-1} \leq \eta_k \leq \theta_k < \eta_k = 0.$$

Lemma 3.5.([13]) *Let H be a proper subgraph of a connected graph G, then*

$$\beta(G) < \beta(H).$$

Theorem 3.3. (1) *If $1 \leq l_i \leq 2$ $(1 \leq i \leq k)$ and $(l_1, l_2, \ldots, l_k) \notin \{(k \cdot 1), (k \cdot 2)\}$, then $T(l_1, l_2, \ldots, l_k)$ is not adjointly integral.*

(2) *If $l_1 \geq 12$, then $T(l_1, l_2, \ldots, l_k)$ is not adjointly integral.*

Proof. (1) Under the conditions of the theorem, we have $T(k \cdot 1) \subset T(l_1, l_2, \ldots, l_k) \subset T(k \cdot 2)$. By $\beta(T(k \cdot 1)) = -k$ and $\beta(T(k \cdot 2)) = -k-1$, we obtain from Lemma 3.5 that $-k-1 < \beta(T(l_1, l_2, \ldots, l_k)) < -k$ which indicates that $T(l_1, l_2, \ldots, l_k)$ is not adjointly integral.

(2) By u we denote the vertex of degree of k in $T(l_1, l_2, \ldots, l_k)$. From (1) of Corollary 2.1, the path P_{l_1} has the adjoint roots $x_1 = -2 - 2\cos\frac{2\pi}{l_1+1}$ and $x_2 = -2 - 2\cos\frac{4\pi}{l_1+1}$. Since $l_1 \geq 12$, then $0 < \frac{2\pi}{l_1+1} < \frac{4\pi}{l_1+1} < \frac{\pi}{3}$, which leads to $-4 < x_1 < x_2 < -3$. From Lemma 3.4 and $T(l_1, l_2, \ldots, l_k) - u = P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_k}$, we have that $T(l_1, l_2, \ldots, l_k)$ has at least one adjoint root that lies in the interval $[x_1, x_2]$, which implies that this adjoint root is not an integer. And so the result holds. □

On the basis of the above theorems, it seems to be difficult to give an necessary and sufficient condition for the star-tree to be adjointly integral. However, we have the following conjecture:

Conjecture. *For $k \geq 3$, the star-tree $T(l_1, l_2, \ldots, l_k)$ is adjointly integral if and only if $(l_1, l_2, \ldots, l_k) \in \{(k \cdot 1), (k \cdot 2)\}$.* □

From Corollary 3.2, we see that if a tree is integral then it is also adjointly integral. However, the converse is not true. For example, the path P_3 is adjointly integral but is not integral. Thus it is interesting to study the relationships between integral graphs and adjointly integral graphs. We end this paper by posing the following problem:

Problem. *Which graphs are adjointly integral?* □

Acknowledgement

The authors would like to thank the referee for giving a lot of comments on improving this paper.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland, Amsterdam, 1976.

[2] P. Borwein and T. Erdelyi, *Polynomial and Polynomial Inequalities*, Springer-Verlag, New York, Berlin, Heidelberg, 1993.

[3] F.M. Dong, K.L. Teo, C.H.C. Little and M.D. Hendy, Zeros of adjoint polynomials of paths and cycles, *Australas. J. Combin.* **25** (2002), 167–174.

[4] K.M. Koh and K.L. Teo, The search for chromatically unique graphs. *Graphs Combin.* **6** (1990), 259–285.

[5] K.M. Koh and K.L. Teo, The search for chromatically unique graphs-II, *Discrete Math.* **172** (1997), 59–78.

[6] R.R. Korfhag, σ-polynomials and graph coloring, *J. Combin. Theory Ser. B* **24** (1978), 137–153.

[7] M. Lepović and I. Gutman, Some spectral properties of starlike trees, *Bull. T.CXXII de l'Académie Serbe des Sciences et des Arts-2001 Classe des Sciences mathématiques et naturelles Sciences mathématiques* **26** (2001), 99–105.

[8] R.Y. Liu , Several results on adjoint polynomials of graphs. *J. Qinghai Normal Univ.* **10** (1992), 1–6 (in Chinese).

[9] R.Y. Liu, On the irreducible graph, *J.Qinghai Normal Univ. (Natur. Sci.)* **4** (1993), 29–33 (in Chinese).

[10] R.Y. Liu, Adjoint polynomials and chromatically unique graphs. *Discrete Math.* **172** (1997), 85–92.

[11] J.F. Wang, R.Y. Liu, C.F. Ye and Q.X. Huang, A complete solution to the chromatic equivalence class of graph $\overline{B_{n-7,1,3}}$, *Discrete Math.* **308** (2008), 3607–3623.

[12] H.X. Zhao and R.Y. Liu, A necessary and sufficient condition for irreducible paths, *J. Northeast Normal Univ. (Natur. Sci.)* **33** (2001) No. 2, 18–21 (in Chinese).

[13] H.X. Zhao, X.L. Li, S.G. Zhang and R.Y. Liu. On the minimum real roots of the σ-polynomial and chromatic uniqueness of graphs, *Discrete Math.* **281** (2004), 277–294.

(Received 3 Nov 2007; revised 4 Jan 2008)