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A note on distance domination numbers of graphs*

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Abstract

Let k be a positive integer and $G = (V, E)$ a connected graph of order n. A set $D \subseteq V$ is called a distance k-dominating set of G if each $x \in V(G) - D$ is within distance k from some vertex of D. The k-domination number of G, denoted by $\gamma_k(G)$, is the minimum cardinality over all distance k-dominating sets. Determining $\gamma_k(G)$ has a significant impact on an efficient design of routing protocols in networks and, moreover, computing $\gamma_k(G)$ is NP-hard. This paper establishes some upper bounds for $\gamma_k(G)$ and improves some known results.

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [12]. Let k be a positive integer; for every vertex $x \in V(G)$, the k-neighborhood $N_k(x)$ is defined by $N_k(x) = \{y \in V(G) : d_G(x, y) \leq k\}$.

A subset $D \subseteq V(G)$ is called a distance k-dominating set of G if every vertex in $V(G) - D$ is within distance k from some vertex of D. The minimum cardinality among all distance k-dominating sets of G is called the distance k-domination number

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of G and is denoted by $\gamma_k(G)$. For the special case of $k = 1$, $\gamma_1(G)$ is the classic domination number of G. For any graph G, to determine $\gamma_k(G)$ is an NP-hard problem even if G is bipartite [6]. The concept of k-dominating set has been studied extensively by several authors to consider the distance parameters in many situations and structures which give rise to graphs; see, for example, [6, 5, 3, 7, 9].

A general lower bound [2] for $\gamma_k(G)$ is $\gamma_k(G) \geq \left\lceil \frac{n}{\Delta_k+1} \right\rceil$, where Δ_k is the maximum k-degree. Meir and Moon [8] established an upper bound for $\gamma_k(G)$, that is, $\gamma_k(G) \leq \left\lfloor \frac{n}{k+1} \right\rfloor$ for any connected graph of order n with $n \geq k+1$. Sridharan, Subramanian and Elias [10] considered the special case of $k = 2$, independently, and also established $\gamma_2(G) \leq \lfloor \frac{n}{3} \rfloor$ for $n \geq 3$. In this paper, we will prove some new upper bounds for $\gamma_k(G)$ using the minimum degree δ and the maximum degree Δ, which improves some known results.

2 Main Results

For an event A and for a random variable Z of an arbitrary probability space, $P[A]$ and $E[Z]$ denote the probability of A and the expectation of Z, respectively. It is clear that $\gamma_k(G) = 1$ for any integer k not less than $d(G)$, the diameter of G. Thus, let $1 \leq k < d(G)$ in this paper below.

Let $Q^n = \{(p_1, p_2, \ldots, p_n) \mid 0 < p_i < 1, i = 1, 2, \ldots, n\}$ and $V = V(G) = \{1, 2, \ldots, n\}$. Let us pick, randomly and independently, each vertex i of V with probability p_i. Let $X \subset V(G)$ be the set of vertices picked. Let $Y \subset V(G-X)$ such that $|N_k(Y) \cap X| = 0$. In order to get the upper bound of $\gamma_k(G)$, we can assume that $Y \neq \emptyset$. Let $|X| = \alpha$ and $|Y| = \beta$. By simple discussions, we immediately get the following theorem.

Theorem 2.1 *Let G be a connected graph with vertex set $V = \{1, 2, \ldots, n\}$. Then*

$$\gamma_k(G) \leq \min_{(p_1,p_2,\ldots,p_n)\in Q^n} f_G(p_1, p_2, \ldots, p_n), \tag{1}$$

where $f_G(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} \left(p_i + (1-p_i) \prod_{j\in N_k(i)} (1-p_j) \right)$.

Proof Since α can be written as a sum of n indicator random variables χ_i, where $\chi_i = 1$ if $i \in X$ and $\chi_i = 0$ otherwise, it follows that the expectation of α satisfies $E[\alpha] = \sum_{i=1}^{n} \chi_i P(i \in X) = \sum_{i=1}^{n} p_i$. Likewise, we have $E[\beta] = \sum_{i=1}^{n} \chi_i P(i \in Y) = \sum_{i=1}^{n} (1-p_i) \prod_{j\in N_k(i)} (1-p_j)$.

Note that $X \cup Y$ is a k-dominating set of G. Thus,

$$\gamma_k(G) \leq |X \cup Y| \leq E[|X \cup Y|] = E[\alpha] + E[\beta] \leq \min_{(p_1,p_2,\ldots,p_n)\in Q^n} f_G(p_1, p_2, \ldots, p_n),$$

where $f_G(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} \left(p_i + (1-p_i) \prod_{j\in N_k(i)} (1-p_j) \right)$. ∎

Lemma 2.2 *$|N_k(i)| \geq m(\delta+1)+1-t$ for any $i \in V$, where $m = \left\lceil \frac{k}{3} \right\rceil$ and $t = 3\left\lceil \frac{k}{3} \right\rceil - k$.*

Proof Let $k = 3m-t$, where $m \geq 1$, $0 \leq t \leq 2$. By the choice of Y, we have $Y \neq \emptyset$ and $d_G(X,Y) \geq k+1$. Now let $a \in X$, $b \in Y$ be two vertices whose distance in G is as small as possible, that is, $d_G(a,b) = d_G(X,Y)$. Let P be any shortest path from a to b and let v be the second to last vertex on P. Then $v \notin Y$. If $d_G(a,b) \geq k+2$, then v has no k-neighbors in X. By definition of Y, we get $v \in Y$, a contradiction. Thus, $d_G(X,Y) = k+1$.

Now let $X_\ell(i) = \{u \in V(G) : \ d_G(u,i) = \ell\}$.

If $i \in X \cup Y$, then since $d_G(X,Y) = k+1$ and G is connected, we have $X_\ell(i) \neq \emptyset$ for $\ell = 1, \ldots, k$. Clearly, $|X_1(i)| \geq \delta$. For $2 \leq \ell \leq k-2$, we have that $|X_\ell(i)| + |X_{\ell+1}(i)| + |X_{\ell+2}(i)| \geq \delta+1$. In fact, for any $u \in X_{\ell+1}(i)$, $N_1(u) \subseteq X_\ell(i) \cup X_{\ell+1}(i) \cup X_{\ell+2}(i)$, thus, $|X_\ell(i)| + |X_{\ell+1}(i)| - 1 + |X_{\ell+2}(i)| \geq \delta$. So, we have

$$
\begin{aligned}
|N_k(i)| &= |X_1(i)| + |X_2(i)| + \cdots + |X_k(i)| \\
&\geq \delta + \left\lfloor \frac{k-1}{3} \right\rfloor (\delta+1) + \left(k-1-3\left\lfloor \frac{k-1}{3} \right\rfloor\right) \\
&= \delta + (m-1)(\delta+1) + (2-t) \\
&= m(\delta+1)+1-t.
\end{aligned}
$$

Let $i \in V(G)-(X\cup Y)$. If $d_G(i,Y) \geq k$ or $d_G(i,X) \geq k$, using the same discussion as above, we get $|N_k(i)| \geq m(\delta+1)+1-t$. Now suppose that $d_G(i,Y) < k$ and $d_G(i,X) < k$. Since $d_G(X,Y) = k+1$, there must exist a shortest path between a vertex $a \in X$ and a vertex $b \in Y$ through i such that $d_G(a,b) \geq k+1$, $d_G(i,b) < k$ and $d_G(a,i) < k$. We only consider the worst case $d_G(a,b) = k+1$, and let P_{ab} denote the shortest path from a to b passing through i.

Let v_1 and v_2 be two neighbors of i on P_{ab} from b to i and from a to i, respectively. Let $d_G(b,v_1) = \ell_1$, $d_G(a,v_2) = \ell_2$. Thus, $\ell_1 + \ell_2 = k-1$. By symmetry, we consider the following six cases.

If $\ell_1 \equiv 1 \pmod 3$, $\ell_2 \equiv 1 \pmod 3$, then $k \equiv 0 \pmod 3$, that is, $k = 3m$, $t = 0$.

$$
\begin{aligned}
|N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1)+2 \\
&= \delta + \frac{\ell_1+\ell_2-2}{3}(\delta+1)+2 \\
&= \delta + \frac{k-3}{3}(\delta+1)+2 \\
&= \delta + (m-1)(\delta+1)+2 \\
&= m(\delta+1)+1-t.
\end{aligned}
$$

If $\ell_1 \equiv 1 \pmod 3$, $\ell_2 \equiv 2 \pmod 3$, then $k \equiv 1 \pmod 3$, that is, $k = 3m-2$, $t = 2$. Since $\ell_2 \equiv 2 \pmod 3$ and $\ell_2 < k$, we have $N_1(a) \subseteq N_k(i)$, and so

$$|X_{\ell_2-1}(v_2)| + |X_{\ell_2}(v_2)| + |X_{\ell_2+1}(v_2)| \geq \delta+1.$$

Thus

$$\begin{aligned} |N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 1 + (\delta+1) \\ &= \delta + \frac{\ell_1 - 1 + \ell_2 - 2}{3}(\delta+1) + \delta + 2 \\ &= \delta + \frac{k-4}{3}(\delta+1) + \delta + 2 \\ &= m(\delta+1) \\ &> m(\delta+1) + 1 - t. \end{aligned}$$

If $\ell_1 \equiv 2 \pmod 3$, $\ell_2 \equiv 2 \pmod 3$, then $k \equiv 2 \pmod 3$, that is, $k = 3m-1$, $t = 1$. By the discussion as above, we also get $|X_{\ell_1-1}(v_1)| + |X_{\ell_1}(v_1)| + |X_{\ell_1+1}(v_1)| \geq \delta + 1$. Thus, we have,

$$\begin{aligned} |N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 2(\delta+1) \\ &= \delta + \frac{\ell_1 - 2 + \ell_2 - 2}{3}(\delta+1) + 2\delta + 2 \\ &= \delta + \frac{k-5}{3}(\delta+1) + 2\delta + 2 \\ &= m(\delta+1) + \delta \\ &> m(\delta+1) + 1 - t. \end{aligned}$$

If $\ell_1 \equiv \ell_2 \equiv 0 \pmod 3$, then $k \equiv 1 \pmod 3$, that is, $k = 3m-2$, $t = 2$.

$$\begin{aligned} |N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) \\ &= \delta + \frac{\ell_1 + \ell_2}{3}(\delta+1) \\ &= \delta + (m-1)(\delta+1) \\ &= m(\delta+1) - 1 \\ &= m(\delta+1) + 1 - t. \end{aligned}$$

If $\ell_1 \equiv 0 \pmod 3$, $\ell_2 \equiv 1 \pmod 3$, then $k \equiv 2 \pmod 3$, that is, $k = 3m-1$, $t = 1$.

$$\begin{aligned} |N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 1 \\ &= \delta + \frac{\ell_1 + \ell_2 - 1}{3}(\delta+1) + 1 \\ &= \delta + (m - 1 + \frac{1}{3})(\delta+1) + 1 \\ &= m(\delta+1) + \frac{1}{3}(\delta+1) \\ &> m(\delta+1) + 1 - t. \end{aligned}$$

If $\ell_1 \equiv 0 \pmod 3$, $\ell_2 \equiv 2 \pmod 3$, then $k \equiv 0 \pmod 3$, that is, $k = 3m$, $t = 0$.

$$\begin{aligned}|N_k(i)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + (\delta+1) \\ &= \delta + \frac{\ell_1+\ell_2-2}{3}(\delta+1) + (\delta+1) \\ &= \delta + (m-1)(\delta+1) + (\delta+1) \\ &= m(\delta+1) + \delta \\ &\geq m(\delta+1) + 1 - t.\end{aligned}$$

∎

Theorem 2.3 *Let G be a connected graph and k a positive integer. Then $\gamma_k(G) \leq n\frac{1+\ln(m(\delta+1)+2-t)}{m(\delta+1)+2-t}$, where $m = \lceil \frac{k}{3} \rceil$ and $t = 3\lceil \frac{k}{3} \rceil - k$.*

Proof By Theorem 2.1, we have $\gamma_k(G) \leq \min_{0<p<1} f_G(p, p, \ldots, p)$, where $f_G(p) = \left(np + \sum_{i=1}^{n}(1-p)^{|N_k(i)|+1}\right)$. By Lemma 2.2, $f_G(p) \leq np + ne^{-p\left(m(\delta+1)+2-t\right)}$ which gets its minimum at $p = \frac{\ln(m(\delta+1)+2-t)}{m(\delta+1)+2-t}$. ∎

Corollary 2.4 (Alon and J.H. Spencer [1]) *For any connected graph G, $\gamma_1(G) \leq n\frac{1+\ln(\delta+1)}{\delta+1}$.*

Since determining $\gamma_k(G)$ is an NP-hard problem even if G is bipartite, we consider G to be a bipartite graph with bipartition V_1 and V_2, $|V_j| = n_j$ and $\delta_j = \min\{\deg(i) \mid i \in V_j\}$. Let us pick the vertices in V_j with probability $0 < p_j < 1$ for $j = 1, 2$.

Lemma 2.5 *If G is a bipartite graph with bipartition V_1 and V_2, k is a positive integer, and $V - N_k(i) \neq \emptyset$ for any $i \in V$, then $|N_k(i) \cap V_1| \geq (\mathscr{M} - 1)(\delta_2 + 1)$, $|N_k(i) \cap V_2| \geq \mathscr{M}(\delta_1 + 1) - 1$ for any $i \in V_1$; and $|N_k(i) \cap V_1| \geq \mathscr{M}(\delta_2 + 1) - 1$, $|N_k(i) \cap V_2| \geq (\mathscr{M} - 1)(\delta_1 + 1)$ for any $i \in V_2$, where $\mathscr{M} = \lceil \frac{k}{6} \rceil$.*

Proof Let $k = 6\mathscr{M} - \mathscr{T}$, where $\mathscr{M} \geq 1$, $0 \leq \mathscr{T} \leq 5$. Without loss of generality, we only prove this statement for $i \in V_1$. Let $X_\ell(i) = \{u \in V(G) : d_G(u,i) = \ell\}$. Since $V - N_k(i) \neq \emptyset$, we have $X_\ell(i) \neq \emptyset$ for $\ell = 1, \ldots, k$. Clearly, $X_1(i) \geq \delta_1$. For $2 \leq \ell \leq k-2$ and any vertex $u \in X_{\ell+1}(i)$, we have $N_1(u) \subseteq X_\ell(i) \cup X_{\ell+2}(i)$, and so $|X_\ell(i)| + |X_{\ell+2}(i)| \geq \delta_j$ as $u \in V_j$ and $j = 1, 2$.

So we have

$$|N_k(i) \cap V_2| = \sum_{\mathrm{odd}\,\ell} |X_\ell(i)| \geq \delta_1 + \left\lfloor \frac{k-1}{6} \right\rfloor (\delta_1+1) = \delta_1 + (\mathscr{M}-1)(\delta_1+1) = \mathscr{M}(\delta_1+1) - 1$$

and

$$|N_k(i) \cap V_1| = \sum_{\mathrm{even}\,\ell} |X_\ell(i)| \geq \left\lfloor \frac{k-1}{6} \right\rfloor (\delta_2+1) = (\mathscr{M}-1)(\delta_2+1).$$

∎

Definition 2.6 *A connected bipartite graph G is said to be perfect if $\delta_1\delta_2 > 1$, $n_1\big[\mathscr{M}(\delta_1+1)-1\big] > n_2\big[(\mathscr{M}-1)(\delta_2+1)+1\big]$ and $n_2\big[\mathscr{M}(\delta_2+1)-1\big] > n_1\big[(\mathscr{M}-1)(\delta_1+1)+1\big]$, where $\mathscr{M}=\lceil\frac{k}{6}\rceil$.*

Theorem 2.7 *Let G be a perfect bipartite graph with*

$$0 < p_1 = \frac{\big[(\mathscr{M}-1)(\delta_1+1)+1\big]\ln u - \big[\mathscr{M}(\delta_1+1)-1\big]\ln v}{(2\mathscr{M}-1)(\delta_1\delta_2-1)} < 1,$$

and

$$0 < p_2 = \frac{\big[(\mathscr{M}-1)(\delta_2+1)+1\big]\ln v - \big[\mathscr{M}(\delta_2+1)-1\big]\ln u}{(2\mathscr{M}-1)(\delta_1\delta_2-1)} < 1,$$

where

$$u = \frac{n_2\big[\mathscr{M}(\delta_2+1)-1\big] - n_1\big[(\mathscr{M}-1)(\delta_1+1)+1\big]}{n_1(2\mathscr{M}-1)(\delta_1\delta_2-1)},$$

and

$$v = \frac{n_1\big[\mathscr{M}(\delta_1+1)-1\big] - n_2\big[(\mathscr{M}-1)(\delta_2+1)+1\big]}{n_2(2\mathscr{M}-1)(\delta_1\delta_2-1)};$$

then

$$\gamma_k(G) \le h(p_1,p_2) \le \min_{0<p<1} h(p,p) \le \frac{n\big(1+\ln(2\mathscr{M}-1)(\delta+1)\big)}{(2\mathscr{M}-1)(\delta+1)}.$$

where $\mathscr{M}=\lceil\frac{k}{6}\rceil$ and

$$\begin{aligned} h(p_1,p_2) &= \sum_{i=1}^{2} n_ip_i + n_1e^{-p_1\big[(\mathscr{M}-1)(\delta_2+1)+1\big]-p_2\big[\mathscr{M}(\delta_1+1)-1\big]} \\ &\quad + n_2e^{-p_1\big[(\mathscr{M}(\delta_2+1)-1\big]-p_2\big[(\mathscr{M}-1)(\delta_1+1)+1\big]}. \end{aligned}$$

Proof By Theorem 2.1 and Lemma 2.5, it follows that

$$\begin{aligned} \gamma_k(G) &\le \min_{(p_1,p_2)\in(0,1)^2}\Bigg(\sum_{i=1}^{2} n_ip_i + \sum_{i\in V_1}(1-p_1)^{|N_k(i)\cap V_1|+1}(1-p_2)^{|N_k(i)\cap V_2|} \\ &\qquad + \sum_{i\in V_2}(1-p_2)^{|N_k(i)\cap V_2|+1}(1-p_1)^{|N_k(i)\cap V_1|}\Bigg) \\ &\le \min_{(p_1,p_2)\in(0,1)^2}\Bigg(\sum_{i=1}^{2} n_ip_i + n_1(1-p_1)^{(\mathscr{M}-1)(\delta_2+1)+1}(1-p_2)^{\mathscr{M}(\delta_1+1)-1} \\ &\qquad + n_2(1-p_1)^{\mathscr{M}(\delta_2+1)-1}(1-p_2)^{(\mathscr{M}-1)(\delta_1+1)+1}\Bigg) \\ &\le \min_{(p_1,p_2)\in(0,1)^2} h(p_1,p_2) \end{aligned}$$

where,

$$\begin{aligned} h(p_1,p_2) &= \sum_{i=1}^{2} n_ip_i + n_1e^{-p_1\big[(\mathscr{M}-1)(\delta_2+1)+1\big]-p_2\big[\mathscr{M}(\delta_1+1)-1\big]} \\ &\quad + n_2e^{-p_1\big[(\mathscr{M}(\delta_2+1)-1\big]-p_2\big[(\mathscr{M}-1)(\delta_1+1)+1\big]}. \end{aligned}$$

Let

$$u = e^{-p_1\left[(\mathscr{M}-1)(\delta_2+1)+1\right]-p_2\left[\mathscr{M}(\delta_1+1)-1\right]} \text{ and } v = e^{-p_1\left[(\mathscr{M}(\delta_2+1)-1\right]-p_2\left[(\mathscr{M}-1)(\delta_1+1)+1\right]},$$

and consider

$$\begin{cases} \dfrac{\partial h(p_1,p_2)}{\partial p_1} = 0, \\ \dfrac{\partial h(p_1,p_2)}{\partial p_2} = 0. \end{cases}$$

Thus we have

$$\begin{cases} n_1 u\left[(\mathscr{M}-1)(\delta_2+1)+1\right] + n_2 v\left[\mathscr{M}(\delta_2+1)-1\right] = n_1, \\ n_1 u\left[\mathscr{M}(\delta_1+1)-1\right] + n_2 v\left[(\mathscr{M}-1)(\delta_1+1)+1\right] = n_2. \end{cases}$$

Since G is a perfect bipartite graph, we get the solutions of the above two equations

$$\begin{cases} u = \frac{n_2[\mathscr{M}(\delta_2+1)-1]-n_1[(\mathscr{M}-1)(\delta_1+1)+1]}{n_1(2\mathscr{M}-1)(\delta_1\delta_2-1)}, \\ v = \frac{n_1[\mathscr{M}(\delta_1+1)-1]-n_2[(\mathscr{M}-1)(\delta_2+1)+1]}{n_2(2\mathscr{M}-1)(\delta_1\delta_2-1)}. \end{cases}$$

Now, we prove the Hessian matrix of $h(p_1,p_2)$ is definite, that is, the principal minor sequences of it are positive. Firstly, $\frac{\partial^2 h}{\partial p_1^2} = n_1\left[(\mathscr{M}-1)(\delta_2+1)+1\right]^2 u + n_2\left[\mathscr{M}(\delta_2+1)-1\right]^2 v > 0$; secondly,

$$\begin{vmatrix} \frac{\partial^2 h}{\partial p_1^2} & \frac{\partial^2 h}{\partial p_1 \partial p_2} \\ \frac{\partial^2 h}{\partial p_2 \partial p_1} & \frac{\partial^2 h}{\partial p_2^2} \end{vmatrix} = n_1 n_2 uv\left[(2\mathscr{M}-1)(1-\delta_1\delta_2)\right]^2 > 0$$

Thus, we get

$$\gamma_k(G) \leq h(p_1,p_2),$$

where

$$p_1 = \frac{\left[(\mathscr{M}-1)(\delta_1+1)+1\right]\ln u - \left[\mathscr{M}(\delta_1+1)-1\right]\ln v}{(2\mathscr{M}-1)(\delta_1\delta_2-1)}$$

and

$$p_2 = \frac{\left[(\mathscr{M}-1)(\delta_2+1)+1\right]\ln v - \left[\mathscr{M}(\delta_2+1)-1\right]\ln u}{(2\mathscr{M}-1)(\delta_1\delta_2-1)}.$$

Take $\delta = \min\{\delta_1,\delta_2\}$ and $p_1 = p_2 = p$; then $h(p,p)$ gets its minimum at $p = \dfrac{\ln(2\mathscr{M}-1)(\delta+1)}{(2\mathscr{M}-1)(\delta+1)}$. ∎

Remark 2.8 *We now construct a graph to show that the bound $h(p_1,p_2)$ in Theorem 2.7 can be arbitrarily smaller than the bound given in Theorem 2.3.*

Take k copies of K_{a,a^2}, denoted by ${}^jK_{a,a^2} = V_{1,j} \cup V_{2,j}$, where $1 \leq j \leq k$, and $V_{1,j}$ is the independent set with a vertices, $V_{2,j}$ is the independent set with a^2 vertices. For $1 \leq j \leq k-1$, we join one vertex in $V_{1,j}$ to one vertex in $V_{2,j+1}$.

Thus, $\delta_1 = a^2$, $\delta_2 = a$, $n_1 = ka$ and $n_2 = ka^2$. For a sufficiently large positive integer a, we can verify that the graph constructed as above is perfect, and the bound in Theorem 2.3 is $\frac{2a}{\lceil \frac{k}{3} \rceil}\left(1 + \ln \lceil \frac{k}{3} \rceil (a+1)\right)$, but the bound $h(p_1, p_2) < \frac{5k}{2\lceil \frac{k}{6} \rceil - 1} \ln(2\lceil \frac{k}{6} \rceil - 1)(a+1) + \frac{2k}{2\lceil \frac{k}{6} \rceil - 1}$.

Lastly, we give an upper bound for $\gamma_k(G)$ with the maximum degree Δ for a connected graph G. For n, Δ and k satisfying $\Delta - k + 1 > 0$ and $n < (k+1)(\Delta - k + 1)$, the proof of the upper bound for $\gamma_k(G)$ given in Theorem 2.10 is better than the bounds in [8, 11].

Lemma 2.9 (Meir and Moon [8]) *For any positive integer k, G is a graph with order n and each component contains at least $k+1$ vertices, $\gamma_k(G) \leq \lfloor \frac{n}{k+1} \rfloor$.*

Theorem 2.10 *Let G be a connected graph of order n. Then $\gamma_k(G) \leq \left\lfloor \frac{n - \Delta + k - 1}{k} \right\rfloor$ for any positive integer k.*

Proof If $\Delta = n - 1$, then $\gamma_k(G) = 1$. Thus, the statement is true. Suppose $\Delta < n - 1$ below.

Let u be a vertex satisfying $\deg(u) = \Delta$ and $X_\ell(u) = \{i \in V(G) : d_G(u, i) = \ell\}$. If $X_k(u) = \emptyset$, then $\{u\}$ is a k-dominating set of G, that is $\gamma_k(G) = 1$, and the statement follows. Thus, assume $X_k(u) \neq \emptyset$ below. Let $\mathscr{N}(u) = \{u\} \cup X_1(u) \cup \cdots \cup X_k(u)$, and $\overline{V} = V(G) \setminus \mathscr{N}(u)$. If $\overline{V} = \emptyset$, then $\gamma_k(G) = 1$ and the statement holds. Thus $\overline{V} \neq \emptyset$ below.

Let H be the set of vertices in all connected components with at least $k+1$ vertices in $G[\overline{V}]$ and let B be a γ_k-set in $G[H]$. Then, by Lemma 2.9, we have that $|B| \leq \left\lfloor \frac{|H|}{k+1} \right\rfloor$. If $H = \overline{V}$, then $\{u\} \cup B$ is a k-dominating set in G and

$$\begin{aligned} \gamma_k(G) &\leq 1 + |B| \leq 1 + \frac{|H|}{k+1} \leq 1 + \frac{n - \Delta - k}{k+1} \\ &= \frac{n - \Delta + 1}{k+1} < \frac{n - \Delta + k - 1}{k}. \end{aligned}$$

Now suppose $H \subset \overline{V}$. Let I be the set of orders of connected components in $G[\overline{V} \setminus H]$. For each $i \in I$, let H_i be the set of vertices of all connected components of order i in $G[\overline{V} \setminus H]$. Then $G[\overline{V} \setminus H] = \bigcup_{i \in I} G[H_i]$. Since G is connected, $X_k(u) \cap X_1(H_i) \neq \emptyset$ for each $i \in I$. Choose $M_k^i(H_i) \subseteq X_k(u) \cap X_1(H_i)$ to be the set with minimum cardinality such that each vertex of $M_k^i(H_i)$ is adjacent to at least one vertex of H_i. For each j with $i \leq j \leq k-1$, let $M_j^i(H_i) \subseteq X_j(u) \cap X_1(M_{j+1}^i(H_i))$ be the set with minimum cardinality such that each vertex of $M_j^i(H_i)$ is adjacent to at least one vertex of $M_{j+1}^i(H_i)$. Then $|M_i^i(H_i)| \leq \cdots \leq |M_{k-1}^i(H_i)| \leq |M_k^i(H_i)| \leq |H_i|$.

Suppose $H_1 = \emptyset$ or all the vertices in H_1 can be connected to some vertices in $M_j^i(H_i)$ by some shortest paths for $2 \leq i, j \leq k$. Let $\mathscr{P}$ denote the vertices on these shortest paths and

$$M = \bigcup_{i \in I,\ 2 \leq i \leq j \leq k} M_j^i(H_i) \cup H_i \cup H_1 \cup \mathscr{P}.$$

Then each component of $G[M]$ has at least $k+1$ vertices, and by Lemma 2.9, we have that $\gamma_k(G[M]) \leq \left\lfloor \frac{|M|}{k+1} \right\rfloor$. Let D' be a γ_k-set of $G[M]$. Since $X_1(u) \cap M = \emptyset$, we have $|H| + |M| \leq n - |\{u\} \cup X_1(u)| = n - 1 - \Delta$. Since $D = \{u\} \cup B \cup D'$ is a k-dominating set in G, we have

$$\begin{aligned} \gamma_k(G) &\leq |D| = 1 + |B| + |D'| \leq 1 + \frac{|H|}{k+1} + \frac{|M|}{k+1} \\ &\leq 1 + \frac{n-\Delta-1}{k+1} = \frac{n-\Delta+k}{k+1} < \frac{n-\Delta+k-1}{k}. \end{aligned}$$

Otherwise, if there exist vertices in H_1, denoted by H_1', that cannot be connected to any vertex in $M_j^i(H_i)$ by some paths for $2 \leq i,\, j \leq k$, then we have $M_j^1(H_1') \cap M_j^i(H_i) = \emptyset$ for any $2 \leq i,\, j \leq k$ and $M_1^1(H_1') \neq \emptyset$. Thus, $D = \{u\} \cup B \cup D' \cup M_1^1(H_1')$ is a k-dominating set of G. By the definition of H_1', and since $|M_1^1(H_1')| \leq \cdots \leq |M_{k-1}^1(H_1')| \leq |M_k^1(H_1')| \leq |H_1'|$, we have

$$\begin{aligned} \gamma_k(G) &= |D| \leq 1 + |B| + |D'| + |M_1^1(H_1')| \\ &\leq 1 + \frac{|H|}{k+1} + \frac{|M|}{k+1} + \frac{|M_2^1(H_1')| + \cdots + |M_k^1(H_1')| + |H_1'|}{k} \\ &\leq 1 + \frac{|H|+|M|}{k+1} + \frac{n - |H| - |M| - \Delta - 1}{k} \\ &\leq \frac{n-\Delta+k-1}{k}. \end{aligned}$$

The theorem follows. ∎

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