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Half-cycles and chaplets

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Abstract

For n odd, a **half-cycle** for $\mathbb{Z}_n$ is a cycle $[a_1, a_2, \ldots, a_m]$ of distinct elements from $\mathbb{Z}_n$ such that (a) $m = (n-1)/2$, (b) the elements a_i $(i = 1, 2, \ldots, m)$ are all distinct, and (c) the differences $a_{i+1} - a_i$ $(i = 1, 2, \ldots, m$, with $a_{m+1} = a_1)$ are all distinct and no two of them are the negatives of one another, modulo n. Similarly, a **chaplet** for $\mathbb{Z}_n$ is now newly defined to be a cycle $[a_1, a_2, \ldots, a_m]$ of distinct *units* from $\mathbb{Z}_n$ such that (a) $m = \phi(n)/2$ where Euler's totient function $\phi(n)$ gives the number of units in $\mathbb{Z}_n$, and conditions (b) and (c) are satisfied as before. Thus the relationship between half-cycles and chaplets is analogous to that between **full cycles** and the **daisy chains** defined in a previous paper. Methods of construction are given for both half-cycles and chaplets, with emphasis on methodology that is fruitful in the range $5 < n < 300$. Some of the methods are adaptations of constructions for daisy chains. Most results concern *robust* chaplets, where either (i) the set of elements a_i is identical to the set of differences $a_{i+1} - a_i$ or (ii) the two sets are the negatives of one another. Examples are provided liberally, to help with the understanding of a novel subject.

1 Preliminaries

The leader of the highly salaried orchestra placed his violin caressingly against his chin, lowered his eyelids, and floated into a sea of melody.

'Hark!', said most of the diners, 'he is playing The Chaplet*' ... the familiar strains were greeted with the rapture due to a revelation.* [19]

With the rapture due to (merited by) a revelation, this paper introduces the combinatorial concept of a *chaplet* for the units of $\mathbb{Z}_n$ where n is odd. We defer the formal definition to §2, so that we can first outline the context in which the topic arises.

If n is an odd prime, a chaplet for the units of $\mathbb{Z}_n$ is a *half-cycle* (Azaïs [4]) of the elements of $\mathbb{Z}_n \setminus \{0\}$; thus, in the terminology of, for example, Buratti and Del Fra [6], it is the generator of a cyclic k-cycle system of K_{2k+1} where $k = (n-1)/2$. However, for odd values of n that are not prime, the chaplet, which contains half of the units of $\mathbb{Z}_n$ instead of half of the elements of $\mathbb{Z}_n \setminus \{0\}$, seems to be a new concept. (The non-mathematical meanings of 'chaplet' include 'garland or wreath for the head', 'circlet', 'string of beads'. The word survives in the antiques business.) The relationship between chaplets and half-cycles of $\mathbb{Z}_n$ is the same as that between daisy chains [15] and the *total cycles* of Azaïs [4].

Although the concept of chaplets, like that of daisy chains, arose in the construction of power sequence terraces [1, 2, 3], we now introduce it independently of its provenance, with a brief illustration of the links between chaplets, terraces and graph decompositions in the final section of this paper. Our emphasis is on constructing chaplets for $\mathbb{Z}_n$ where $5 < n < 300$.

When we have elements $a_1, a_2, \ldots, a_s$ of $\mathbb{Z}_n$ that are arranged, in that order, around a circle to form a cycle, we follow the convention of using square brackets to write the cycle as $[a_1, a_2, \ldots, a_s]$. Then, as necessary, we interpret a_{s+1} to be a_1, and a_{s+2} to be a_2, and so on. A *translate* of the cycle is obtained by adding an element c to each member of the cycle, modulo n. For convenience, we write a displayed cycle in linear form, without square brackets and commas:

$$\hookrightarrow \; a_1 \;\; a_2 \;\; a_3 \;\; \ldots \;\; a_s \; \hookleftarrow \quad (\mathrm{mod}\ n)\ .$$

Here, as in [15], the symbols $\hookrightarrow$ and $\hookleftarrow$ are reminders that the two ends of the linear form are joined. We always regard the entry after the symbol $\hookrightarrow$ as being a_1. Some construction methods are such that the constructed cycles fall naturally into segments, all of the same length; we separate the segments by *fences* | , *e.g.*

$$\hookrightarrow \; a_1 \;\; a_2 \;\; a_3 \mid a_4 \;\; a_5 \;\; a_6 \mid \cdots \mid a_{3\ell-2} \;\; a_{3\ell-1} \;\; a_{3\ell} \mid \; \hookleftarrow \quad (\mathrm{mod}\ n)\ .$$

Any positive integer n has a prime-power decomposition $n = p^i q^j r^k \cdots (i, j, k \geq 1)$ where p, q, r, ... are finitely many distinct primes. In standard number-theoretic terminology, the *units* of the corresponding group $\mathbb{Z}_n$ are those elements of $\mathbb{Z}_n \setminus \{0\}$ that are coprime with n (*e.g.* [11, p. 84]). The number of units in $\mathbb{Z}_n$ is given by Euler's totient function

$$\phi(n) = (p-1)p^{i-1}(q-1)q^{j-1}(r-1)r^{k-1}\cdots$$

(*e.g.* [11, p. 87]). We write $\mathbb{U}_n$ for the set of units of $\mathbb{Z}_n$. Thus $|\,\mathbb{U}_n\,| = \phi(n)$. If $z \in \mathbb{Z}_n$ or $\mathbb{U}_n$, we write $\mathrm{ord}_n(z)$ for the order of z, modulo n, in $\mathbb{Z}_n$ or $\mathbb{U}_n$ respectively.

For values of n that are odd prime powers, primitive roots of n can be used in constructing chaplets (see Theorem 3.2 below). However, other odd integers greater

than 1 do not have primitive roots. For these other values of n, the "next best thing" to a primitive root is a *primitive λ-root* of n, which is a unit of $\mathbb{Z}_n$ that is of maximum order [8, 9], that order being given by Carmichael's λ-function. As the literature of primitive λ-roots is sparse, notes on them have been placed on the Web [7]. As in those notes, we write $\lambda(n)$ for the order of a primitive λ-root of n, although some authors use $e(n)$ instead, and we write $\xi(n) = \phi(n)/\lambda(n)$. For any composite odd n, the value of $\xi(n)$ is even [7, §6]. A primitive λ-root x of n is *inward* if $x-1$ is a unit of $\mathbb{Z}_n$, and is *outward* otherwise. A primitive λ-root x of n is *negating* if $-1 \in \langle x \rangle$, and is *non-negating* otherwise. A primitive λ-root of n is *strong* [7, §11] if it is both inward and non-negating. More generally, we now describe any unit y from $\mathbb{Z}_n$ to be *negating* if $-1 \in \langle y \rangle$ and *non-negating* otherwise, to be *inward* if $y-1$ is a unit of $\mathbb{Z}_n$ and *outward* otherwise, and to be *strong* if it is both inward and non-negating.

2 Definitions of types of cycle

In [4], Azaïs defined a *total cycle* for $\mathbb{Z}_n$ to be what is now more commonly called a directed R-terrace for $\mathbb{Z}_n$ [13, p. 252], namely an arrangement $[a_1, a_2, \ldots, a_{n-1}]$ of the elements of $\mathbb{Z}_n \setminus \{0\}$ such that the set of differences $a_{i+1} - a_i$ $(i = 1, 2, \ldots, n-1)$ is itself the set of all elements of $\mathbb{Z}_n \setminus \{0\}$. In [15], we likewise defined a *daisy chain* for the **units** of $\mathbb{Z}_n$, where n is odd, to be a cycle $[a_1, a_2, \ldots, a_{\phi(n)}]$ of units such that the set of differences $a_{i+1} - a_i$ $(i = 1, 2, \ldots, \phi(n))$ is itself the set of units. Analogous to total cycles and daisy chains, we now define half-cycles and chaplets as types of half-length cycle, and we define subclasses of them that are of special interest.

For n odd, Azaïs [4] defined a cycle $[a_1, a_2, \ldots, a_{(n-1)/2}]$ of elements of $\mathbb{Z}_n$ to be a *half-cycle* if

(i) the elements a_i $(i = 1, 2, \ldots, (n-1)/2))$ are all distinct, and

(ii) the differences $a_2 - a_1$, $a_3 - a_2$, $\ldots$, $a_{(n+1)/2} - a_{(n-1)/2}$ are all distinct and no two of them are the negatives of one another, modulo n.

We now define a half-cycle to be *strong* if none of the values a_i equals 0 (mod n) and no two of the values a_i are the negatives of one another, modulo n. We further define a strong half-cycle to be a *robust half-cycle* if the set of values a_i is identical to the set of values $a_{i+1} - a_i$ or to the set of values $a_i - a_{i+1}$ $(i = 1, 2, \ldots, (n-1)/2))$. We say that a robust half-cycle is a *champion half-cycle* if, when i and c are values such that $a_{i+c} \equiv \pm(a_{i+1} - a_i)$, then $a_{i+c+j} \equiv \pm(a_{i+1+j} - a_{i+j})$, modulo n, for all j in $\{0, 1, \ldots, (n-3)/2\}$.

As the values $a_{i+1} - a_i$ $(i = 1, 2, \ldots, (n-1)/2))$ sum to zero, modulo n, a necessary condition for a strong half-cycle to be robust is that its elements $(i = 1, 2, \ldots, (n-1)/2))$ sum to zero, modulo n.

Half-cycles generate balanced circuit designs [18], these being neighbour designs that have been variously named (see [14]).

Turning now to cycles of elements of $\mathbb{U}_n$ instead of elements of $\mathbb{Z}_n$ or $\mathbb{Z}_n \setminus \{0\}$, we define a cycle $[a_1, a_2, \ldots, a_{\phi(n)/2}]$ of elements of $\mathbb{U}_n$ to be a *chaplet* for $\mathbb{Z}_n$ if

(i) the elements a_i $(i = 1, 2, \ldots, \phi(n)/2))$ are all distinct, and

(ii) the differences $a_2 - a_1$, $a_3 - a_2$, ... , $a_{(\phi(n)+2)/2} - a_{\phi(n)/2}$ are all distinct and no two of them are the negatives of one another, modulo n.

We further define *strong chaplet*, *robust chaplet* and *champion chaplet* exactly analogously to strong, robust and champion half-cycles, respectively. Thus, if n is prime, a chaplet for $\mathbb{Z}_n$ is a half-cycle for $\mathbb{Z}_n$ that does not contain 0, and the concepts of strong, robust and champion chaplets are identical to the respective concepts for half-cycles.

The following are half-cycles and chaplets for $\mathbb{Z}_{13}$:

(a) $\hookrightarrow 1 \;\; 2 \;\; 7 \;\; 9 \;\; 5 \;\; 8 \hookleftarrow$

(b) $\hookrightarrow 1 \;\; 2 \;\; 8 \;\; 10 \;\; 6 \;\; 9 \hookleftarrow$

(c) $\hookrightarrow 1 \;\; 8 \;\; 9 \;\; 7 \;\; 3 \;\; 11 \hookleftarrow$

Of these, (a) is not strong, nor is any of its translates that are half-cycles and chaplets, whereas (b) is strong but not robust, and (c), which is adapted from a cycle in [12], is robust but not champion. The cycle

$$\hookrightarrow \;\; 1 \;\; 9 \;\; 13 \;\; 10 \;\; 11 \;\; 5 \;\; 15 \;\; 3 \;\; \hookleftarrow,$$

given in 1967 by Rees [17, p. 790], is a strong (but not robust) half-cycle and chaplet for $\mathbb{Z}_{17}$; it is an arrangement of the elements $3^0, 3^1, \ldots, 3^7 \pmod{17}$. The following is a chaplet for $\mathbb{Z}_{21}$, but no translate of it is a strong chaplet:

$$\hookrightarrow \;\; 1 \;\; 17 \;\; 4 \;\; 8 \;\; 19 \;\; 20 \;\; \hookleftarrow \;\; .$$

Chaplets do not exist for $\mathbb{Z}_3$ and $\mathbb{Z}_5$. Also, for $n = 3^r$ $(r > 1)$, the 3^{r-1} elements in a chaplet for $\mathbb{Z}_n$ would have to be alternately congruent to 1 and 2 (mod 3), which is impossible as 3^{r-1} is odd, so chaplets do not exist where n is any power of 3. However, no such argument excludes powers of 5. Indeed, the following is a strong chaplet for $\mathbb{Z}_{25}$:

$$\hookrightarrow \;\; 1 \;\; 2 \;\; 8 \;\; 21 \;\; 18 \;\; 9 \;\; 13 \;\; 6 \;\; 14 \;\; 3 \;\; \hookleftarrow \;\; .$$

The Azaïs constructions [4] for half-cycles for $\mathbb{Z}_n$ (where n is odd, $n > 7$) differ slightly from one another according as $n \equiv 1$, 3, 5 or 7 (mod 8), and do not produce strong half-cycles. So instead we now give Buratti and Del Fra's simple construction [6, p. 116] that works for any odd value n. Strong half-cycles obtained via this construction consist of the integers $1, 2, \ldots, (n-1)/2$, taken in order, save that $1 + 2\lceil (n-9)/8 \rceil$ of them are negated, modulo n.

Theorem 2.1 [6] *For $i = 1, 2, \ldots, (n-1)/2$ where n is odd, $n > 5$, write*

$$a_i = \begin{cases} i(-1)^{i+1} & \text{if } i < (n-1)/4 \\ i(-1)^i & \text{if } i \geq (n-1)/4 \end{cases}$$

Then the cycle $[a_1, a_2, \ldots, a_{(n-1)/2}]$ is a strong half-cycle for $\mathbb{Z}_n$.

Example 2.1: Theorem 2.1 yields the following strong half-cycle for $\mathbb{Z}_{19}$:

$$\hookrightarrow \ 1 \ \ 17 \ \ 3 \ \ 15 \ \ 14 \ \ 6 \ \ 12 \ \ 8 \ \ 10 \ \ \hookleftarrow \ .$$

Note 2.1: For $n \equiv 7 \pmod 8$, the elements $a_1, a_2, \ldots, a_{(n-1)/2}$ in Theorem 2.1 sum to 0 (mod n), but the strong half-cycle that they produce is not robust for $n > 7$.

3 n prime or a prime power

We now turn to some theorems that are similar to theorems given previously [15] for daisy chains. Our first three theorems are for $n = p^k$ where p is a prime satisfying $p \equiv 3 \pmod 4$ and $p > 3$.

Theorem 3.1 *Let n be an odd prime power of the form $n = p^k$ $(k \geq 1)$ where the prime p satisfies $p \equiv 3 \pmod 4$ and $p > 3$. Then $\phi(n) = p^{k-1}(p-1)$, and the cycle $[a_1, a_2, \ldots, a_{\phi(n)/2}]$ is a champion chaplet for $\mathbb{Z}_n$ if $a_i = x^{i-1}$ $(i = 1, 2, \ldots, \phi(n)/2)$ where x is a strong unit of order $\phi(n)/2$ in $\mathbb{Z}_n$.*

Proof: Obvious. (For n prime, see [17, pp. 784–785].) □

Example 3.1(a): With $n = 19$, we can take $x = 5$ in Theorem 3.1 to obtain the following champion chaplet for $\mathbb{Z}_{19}$:

$$\hookrightarrow \ 1 \ \ 5 \ \ 6 \ \ 11 \ \ 17 \ \ 9 \ \ 7 \ \ 16 \ \ 4 \ \ \hookleftarrow \ .$$

Example 3.1(b): With $n = 49$, we can take $x = 46$ in Theorem 3.1 to obtain the following champion chaplet for $\mathbb{Z}_{49}$:

$$\hookrightarrow \ 1 \ \ 46 \ \ 9 \ \ 22 \ \ 32 \ \ 2 \ \ 43 \ \ 18 \ \ 44 \ \ 15 \ \ 4 \ \ 37 \ \ 36 \ \ 39 \ \ 30 \ \ 8 \ \ 25 \ \ 23 \ \ 29 \ \ 11 \ \ 16 \ \ \hookleftarrow \ .$$

Theorem 3.2 *Let n be an odd prime power of the form $n = p^k$ $(k \geq 1)$ where the prime p satisfies $p \equiv 3 \pmod 4$ and $p > 3$. Suppose that $\phi(n) = p^{k-1}(p-1) = 2\pi\omega$ where π and ω are coprime $(2 < \pi$ and $2 < \omega)$. Suppose further that y and v are units of $\mathbb{Z}_n$ such that $\mathrm{ord}_n(y) = \pi\omega$ and $\mathrm{ord}_n(v) = \omega$. Then*

$$\begin{array}{l} \hookrightarrow \ v^0y^0 \ \ v^0y^1 \ \ \ldots \ \ v^0y^{\pi-1} \ \mid \ v^1y^0 \ \ v^1y^1 \ \ \ldots \ \ v^1y^{\pi-1} \ \mid \ \cdots \ \mid \\ \qquad\qquad v^{\omega-1}y^0 \ \ v^{\omega-1}y^1 \ \ \ldots \ \ v^{\omega-1}y^{\pi-1} \ \mid \ \hookleftarrow \quad (\mathrm{mod}\ n) \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$ if
(a) *$y - 1 \in y\langle v \rangle$ and $v - y^{\omega-1} \in \langle v \rangle$, or*
(b) *$y - 1 \in -y\langle v \rangle$ and $v - y^{\omega-1} \in -\langle v \rangle$.*

Proof: (a) As $y - 1 \in y\langle v \rangle$, the successive differences $v^0y^i(y-1)$ from the first segment $(i = 0, 1, \ldots, \pi - 2)$ are equal to the successive entries (excluding the first entry) in one of the subsequent segments. As $v - y^{\omega-1} \in \langle v \rangle$, the difference $v - y^{\omega-1}$

at the first fence is the also the first entry in one of the segments. The result follows. Case (b) is proved similarly. □

Example 3.2(a): For $n = 43$ with $(\pi, \omega) = (3, 7)$ we can take $y = 15$ and $v = 11$ in Theorem 3.2 to obtain the following robust half-cycle and chaplet for $\mathbb{Z}_{43}$:

$$\hookrightarrow \ 1 \ \ 15 \ \ 10 \mid 11 \ \ 36 \ \ 24 \mid 35 \ \ 9 \ \ 6 \mid 41 \ \ 13 \ \ 23 \mid 21 \ \ 14 \ \ 38 \mid 16 \ \ 25 \ \ 31 \mid 4 \ \ 17 \ \ 40 \mid \ \hookleftarrow \ .$$

Here $v \neq y^3$, but we could of course take $v = y^3$ to produce the robust half-cycle obtainable from Theorem 3.1.

Example 3.2(b): For $n = 49$ with $(\pi, \omega) = (3, 7)$ we can take $y = 46$ in Theorem 3.2, along with any v satisfying $v \equiv 1 \pmod 7$ and $v > 1$, to obtain a robust chaplet for $\mathbb{Z}_{49}$. Taking $v = y^3 = 22$ gives the chaplet from Example 3.1(b). We can also take v to be any other value ($v > 1$) satisfying $v \equiv 1 \pmod 7$. Taking $v = 43$ gives

$$\hookrightarrow \ 1 \ \ 46 \ \ 9 \mid 43 \ \ 18 \ \ 44 \mid 36 \ \ 39 \ \ 30 \mid 29 \ \ 11 \ \ 16 \mid 22 \ \ 32 \ \ 2 \mid 15 \ \ 4 \ \ 37 \mid 8 \ \ 25 \ \ 23 \mid \ \hookleftarrow \ .$$

Theorem 3.3 *Let n be an odd prime power of the form $n = p^k$ ($k \geq 1$) where the prime p satisfies $p \equiv 3 \pmod 4$ and $p > 3$. Suppose that $\phi(n) = p^{k-1}(p-1) = 2\pi\omega$ where π and ω are coprime ($2 < \pi$ and $2 < \omega$) and ω is odd. Suppose further that $\mathbb{Z}_n$ contains non-negating units v and $y = (v+1)^{-1}$ such that $\mathrm{ord}_n(v) = \omega$ and $\mathrm{ord}_n(y) = \pi$. Then*

$$\begin{array}{llllllllllll} \hookrightarrow & v^0y^0 & v^0y^1 & \ldots & v^0y^{\pi-1} & \mid & v^1y^0 & v^1y^1 & \ldots & v^1y^{\pi-1} & \mid & \cdots \ \mid \\ & & & & v^{\omega-1}y^0 & v^{\omega-1}y^1 & \ldots & v^{\omega-1}y^{\pi-1} & \mid & \hookleftarrow & & \pmod n \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Now write $x = v + 1 = y^{-1}$ and $z = -v^{-1} = -(x-1)^{-1}$, so that $\mathrm{ord}_n(x) = \pi$ and $\mathrm{ord}_n(z) = 2\omega$ [not ω]. Then

$$\begin{array}{llllllllllll} \hookrightarrow & x^0z^0 & x^0z^1 & \ldots & x^0z^{\omega-1} & \mid & x^1z^0 & x^1z^1 & \ldots & x^1z^{\omega-1} & \mid & \cdots \ \mid \\ & & & & x^{\pi-1}z^0 & x^{\pi-1}z^1 & \ldots & x^{\pi-1}z^{\omega-1} & \mid & \hookleftarrow & & \pmod n \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As $y = (v+1)^{-1}$, we have $y - 1 = -vy$, whence the successive differences $v^0y^i(y-1)$ from the first segment of the first cycle ($i = 0, 1, \ldots, \pi - 2$) are equal to the values $-vy^{i+1}$, which are the negatives of all the second segment entries save the first. The difference at the first fence is $v - y^{-1} = -1$, which is the negative of the first entry in the first segment.

The differences from the first segment of the second cycle are $z^i(z-1)$ ($i = 0, 1, \ldots, \omega - 2$), whereas the elements in the second segment, save the first, are likewise $z^i \cdot xz$. These two sets of values are identical, as the congruence $vz \equiv -1 \pmod n$ implies $xz \equiv z - 1 \pmod n$. The difference at the first fence is $x - z^{\omega-1} \equiv x + z^{-1} \equiv x - v \equiv 1 \pmod n$, which is the first entry in the first segment. □

Example 3.3(a): For $n = 31$ with $(\pi, \omega) = (3, 5)$ we can take $v = 4$ along with $y = 5^{-1} = 25$ in Theorem 3.3 to obtain the following robust chaplet for $\mathbb{Z}_{31}$:

$$\hookrightarrow \ 1 \ \ 25 \ \ 5 \mid 4 \ \ 7 \ \ 20 \mid 16 \ \ 28 \ \ 18 \mid 2 \ \ 19 \ \ 10 \mid 8 \ \ 14 \ \ 9 \mid \ \hookleftarrow \ .$$

Then, taking $x = 5$ and $z = 23$, we obtain the further robust chaplet

$$\hookrightarrow \ 1 \ \ 23 \ \ 2 \ \ 15 \ \ 4 \mid 5 \ \ 22 \ \ 10 \ \ 13 \ \ 20 \mid 25 \ \ 17 \ \ 19 \ \ 3 \ \ 7 \mid \ \hookleftarrow \ .$$

Example 3.3(b): For $n = 49$ with $(\pi, \omega) = (3, 7)$ we can take $(v, y) = (29, 18)$ and $(x, z) = (30, 27)$ to obtain the following robust chaplets for $\mathbb{Z}_{49}$:

$$\hookrightarrow \ 1 \ \ 18 \ \ 30 \mid 29 \ \ 32 \ \ 37 \mid 8 \ \ 46 \ \ 44 \mid 36 \ \ 11 \ \ 2 \mid 15 \ \ 25 \ \ 9 \mid 43 \ \ 39 \ \ 16 \mid 22 \ \ 4 \ \ 23 \mid \ \hookleftarrow$$

and

$$\hookrightarrow \ 1 \ \ 27 \ \ 43 \ \ 34 \ \ 36 \ \ 41 \ \ 29 \ \mid \ 30 \ \ 26 \ \ 16 \ \ 40 \ \ 2 \ \ 5 \ \ 37 \ \mid \ 18 \ \ 45 \ \ 39 \ \ 24 \ \ 11 \ \ 3 \ \ 32 \ \mid \ \hookleftarrow \ .$$

Note 3.3(a): For prime powers n in the range $5 < n < 300$, Theorem 3.3 provides robust chaplets as follows, where the values of x and z are given in order to permit comparison with results from Theorem 3.4 below:

n	31	43	49	79	131			211		239
π	3	3	3	3	5		13	3	5	17
ω	5	7	7	13	13		5	35	21	7
v	4	35	29	22	52	60	61	13	54	100
y	25	6	18	55	89	58	112	196	188	71
x	5	36	30	23	53	61	62	14	55	101
z	23	27	27	61	68	24	73	146	168	141

Note 3.3(b): For n-values as in Theorem 3.3, the relationship $y = (v + 1)^{-1}$ is far from being the only one that can be used, along with $\mathrm{ord}_n(y) = \pi$ and $\mathrm{ord}_n(v) = \omega$, to produce a robust chaplet of the first type given in Theorem 3.3. (We now allow the set of differences to be identical either to the set of entries in the chaplet or to the negative of the set of entries.) For example, for $n = 103$, the relationship $y = (v + 1)^{-1}$ cannot be satisfied, but we can use instead $(v, y) = (61, 56)$ with $(\pi, \omega) = (3, 17)$ to obtain the following robust chaplet for $\mathbb{Z}_{103}$:

$$\hookrightarrow \ 1 \ \ 56 \ \ 46 \mid 61 \ \ 17 \ \ 25 \mid \cdots \mid 72 \ \ 15 \ \ 16 \mid \cdots \mid 93 \ \ 58 \ \ 55 \mid \cdots \mid 76 \ \ 33 \ \ 97 \mid \ \hookleftarrow \ .$$

The first 3 differences here are 55, 93 and 15, and by examining the positions of these elements in the segments of the cycle, we see at once that the chaplet is robust. For $n = p^k$ with $p \equiv 3 \pmod 4$, with the requirement $y = (v + 1)^{-1}$ abandoned, robust chaplets for $\mathbb{Z}_n$ for some pairs (n, π) absent from the table in Note 3.3(a) can be obtained using the sets of parameters in the following table, where the value marked ‡ may be replaced by any of the values 111^i $(i = 2, 3, \ldots, 10)$, all of these being congruent to 1, modulo 11:

n	103	121	127	139	151	191	199	223	239	271	283
π	3	5	7	3	3	5	11	3	7	5	3
ω	17	11	9	23	25	19	9	37	17	27	47
v	61	111‡	52	36	9	6	175	16	211	32	161
y	56	81	2	42	32	109	63	39	10	187	44

We now give a corollary which, although of only slight use in itself, is a prototype for the very fruitful corollary to Theorem 4.2 below.

Corollary to Theorem 3.3 *Let n, p, k, π, ω, v and y be as in* Theorem 3.3. *Suppose that $\mathbb{Z}_n$ contains a unit u such that $u \in \langle v \rangle$ and $u + 1 \equiv v^{-\tau}y^{-1}u \pmod{n}$ where $1 < \tau < \omega$ and the values τ and ω are coprime. Write $t = v^\tau$. Then*

$$\hookrightarrow\ t^0y^0\ \ t^0y^1\ \ \ldots\ t^0y^{\pi-1} \mid t^1y^0\ \ t^1y^1\ \ \ldots\ t^1y^{\pi-1} \mid \\ \cdots \mid t^{\omega-1}y^0\ \ t^{\omega-1}y^1\ \ \ldots\ t^{\omega-1}y^{\pi-1} \mid \hookleftarrow \pmod{n}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: The segments of the chaplet in the corollary are the same as those in the first chaplet of Theorem 3.3, but they are now in a different order. The difference at the first fence is now

$$\begin{aligned} t - y^{\pi-1} = v^\tau - y^{-1} &= v^\tau(1 - v^{-\tau}y^{-1}) \\ &\equiv -u^{-1}v^\tau \pmod{n}\,, \end{aligned}$$

which is the same as the difference at one of the fences of the first chaplet in Theorem 3.3. □

Example 3.3(c): For the set of parameters given for $n = 79$ in Note 3.3(a), we can take $u = 8 = v^{11}$, with $u + 1 = 9 = y^{-1}v^4 = v^{-7}y^{-1}u$, whence we take $\tau = 7$ and thus $t = v^7 = 38$. This gives the following robust chaplet for $\mathbb{Z}_{79}$:

$$\begin{array}{l} \hookrightarrow\ 1\ \ 55\ \ 23 \mid 38\ \ 36\ \ 5 \mid 22\ \ 25\ \ 32 \mid 46\ \ 2\ \ 31 \mid \\ \quad 10\ \ 76\ \ 72 \mid 64\ \ 44\ \ 50 \mid 62\ \ 13\ \ 4 \mid 65\ \ 20\ \ 73 \mid 21\ \ 49\ \ 9 \mid \\ \qquad 8\ \ 45\ \ 26 \mid 67\ \ 51\ \ 40 \mid 18\ \ 42\ \ 19 \mid 52\ \ 16\ \ 11 \mid \hookleftarrow\ . \end{array}$$

Example 3.3(d): In the range $5 < n < 300$, the parameter sets corresponding to chaplets obtainable from the corollary to Theorem 3.3 are as follows:

n			49			79	131	211
π			3			3	5	3
ω			7			13	13	35
v			29			22	60	13
y			18			55	58	196
	⌢	⌢	⌢	⌢	⌢			
u	1	8	15	36	43	8	80	5
τ	4	5	6	2	3	7	3	22
t	15	43	22	8	36	38	112	82

We now move on to a theorem for primes n satisfying $n \equiv 1 \pmod 4$. This theorem embodies the part of Theorem 3.3 that still applies in the changed circumstances.

Theorem 3.4 *Let n be a prime with $n \equiv 1 \pmod 4$. Suppose that $n - 1 = 2\pi\omega$ where π is odd, ω is even, and π and ω are coprime. Suppose further that $\mathbb{Z}_n$ contains elements x and $z = -(x-1)^{-1}$ with $\mathrm{ord}_n(x) = \pi$ and $\mathrm{ord}_n(z) = 2\omega$, as in* Theorem 3.3. *Then*

$$\begin{array}{lccccccccccccc} \hookrightarrow & x^0z^0 & x^0z^1 & \dots & x^0z^{\omega-1} & | & x^1z^0 & x^1z^1 & \dots & x^1z^{\omega-1} & | & \cdots & | \\ & & & & x^{\pi-1}z^0 & x^{\pi-1}z^1 & \dots & x^{\pi-1}z^{\omega-1} & | & \hookleftarrow & & \pmod n \end{array}$$

is a robust half-cycle and robust chaplet for $\mathbb{Z}_n$.

Proof: As for the second part of the proof of Theorem 3.3. □

Example 3.4: For $n = 89$, taking $(x, z) = (78, 52)$ in Theorem 3.4 yields the following chaplet for $\mathbb{Z}_{89}$:

$$\begin{array}{l} \hookrightarrow\ 1\ \ 52\ \ 34\ \ 77\ |\ 78\ \ 51\ \ 71\ \ 43\ |\ 32\ \ 62\ \ 20\ \ 61\ | \\ \quad 4\ \ 30\ \ 47\ \ 41\ |\ 45\ \ 26\ \ 17\ \ 83\ |\ 39\ \ 70\ \ 80\ \ 66\ |\ 16\ \ 31\ \ 10\ \ 75\ | \\ \quad\ 2\ \ 15\ \ 68\ \ 65\ |\ 67\ \ 13\ \ 53\ \ 86\ |\ 64\ \ 35\ \ 40\ \ 33\ |\ 8\ \ 60\ \ 5\ \ 82\ |\ \hookleftarrow\ . \end{array}$$

Note 3.4: in the range $5 < n < 300$, specimen sets of values satisfying the conditions of Theorem 3.4 are as follows:

n	13	37	53	61	89	97	101	113	157	173	229	241	277
π	3	9	13	15	11	3	25	7	13	43	3	3	3
ω	2	2	2	2	4	16	2	8	6	2	38	40	46
x	9	7	24	12	78	35	92	49	108	81	94	225	160
z	8	6	23	11	52	77	91	40	22	80	32	156	54

The presence of $n = 13$ and 97 in this table, in conjunction with the absence of $n = 193$, shows that the theorem does not cover all primes n satisfying $n = 3 \cdot 2^i + 1$ and $n > 7$.

As Theorems 3.3 and 3.4 give restricted coverage of primes n with, respectively, $n \equiv 3$ and $n \equiv 1 \pmod 4$, we now proceed to a theorem which, although not a generalisation of the two previous theorems, has conditions that are less exacting.

Theorem 3.5 *Let n be a prime with $n > 11$. Suppose that $n - 1 = 2\pi\omega$ where π is odd, and π and ω are coprime. Let x be an element of $\mathbb{Z}_n$ with $\mathrm{ord}_n(x) = \pi$, and let z be a primitive root of n. Then*

$$\begin{array}{lccccccccccccc} \hookrightarrow & x^0z^0 & x^0z^1 & \dots & x^0z^{\omega-1} & | & x^1z^0 & x^1z^1 & \dots & x^1z^{\omega-1} & | & \cdots & | \\ & & & & x^{\pi-1}z^0 & x^{\pi-1}z^1 & \dots & x^{\pi-1}z^{\omega-1} & | & \hookleftarrow & & \pmod n \end{array}$$

is a robust half-cycle and robust chaplet for $\mathbb{Z}_n$ *if*
(a) $z-1 \in z\langle x\rangle$ *and* $x - z^{\omega-1} \in \langle x\rangle$, *or*
(b) $z-1 \in -z\langle x\rangle$ *and* $x - z^{\omega-1} \in -\langle x\rangle$.

Proof: Immediate. □

Example 3.5: For $n = 41$, taking $(x, z) = (18, 30)$ in Theorem 3.5 satisfies the conditions (a) and yields the following robust chaplet for $\mathbb{Z}_{41}$:

$$\hookrightarrow \ 1 \ \ 30 \ \ 39 \ \ 22 \mid 18 \ \ 7 \ \ 5 \ \ 27 \mid 37 \ \ 3 \ \ 8 \ \ 35 \mid 10 \ \ 13 \ \ 21 \ \ 15 \mid 16 \ \ 29 \ \ 9 \ \ 24 \mid \ \hookleftarrow \ .$$

Like many type (a) chaplets obtainable from Theorem 3.5, this one satisfies $x+z \equiv xz \pmod{n}$, so its second element equals the first difference in the second segment.

Note 3.5(a): In general, for a fixed admissible pair (n, π), Theorem 3.5 may yield solutions of just one type, or both, or neither, and may yield two or more solutions of the same type. Thus, for $n = 41$ with $\pi = 5$, solutions of type (a) are obtainable from $(x, z) = (18, 30)$ and $(16, 12)$, but no solution of type (b) exists. Even for fixed n, π and x, or fixed n, π and z, there may be more than one solution of the same type. For example, for $(n, \pi) = (89, 11)$, there are type (a) solutions with $(x, z) = (4, 24)$, $(4, 66)$, $(4, 38)$ and $(64, 38)$. For the range $5 < n < 300$, specimen sets of values satisfying the conditions of Theorem 3.5 are given in Table 1.

Note 3.5(b): If, in Theorem 3.5, we change the condition $\mathrm{ord}_n(z) = 2\pi\omega$ to $\mathrm{ord}_n(z) = 2\omega$, then further robust half-cycles are obtainable, particularly type (a) half-cycles for parameter sets (n, π, ω, x, z) that satisfy the conditions of Theorems 3.3 and 3.4. This enables us to provide a solution for $n = 97$ (see Note 3.4), which is an n-value absent from Table 1. Other possibilities arise if k is a proper factor of π and we change the same condition to $\mathrm{ord}_n(z) = 2k\omega$. The details of all this seem to be of insufficient interest to be given in full, so we now merely give the parameters of a few half-cycles now obtainable (with asterisks * again indicating solutions with $x + z \equiv xz \pmod{n}$):

n	37		61			73	101		109			
π	9		15			9	25		27			
ω	2		2			4	2		2			
$\mathrm{ord}_n(z)$	12		12		20	24	20		12		36	
x	33	7	25	16	25	12	55	78	56	25	89	89	5
z	23	14	29	32	24	8	30	62	39	41	68	55	92
type	(a)*	(b)	(a)*	(b)	(a)	(b)	(b)	(a)	(b)	(a)	(b)	(a)	(b)

Finally in this section, we come to theorems specifically for $n = p^2$ and $n = p^3$ where p is an odd prime. The theorems could readily be generalised to $n = p^r$ $(r \geq 2)$, but this would not be beneficial for the range $5 < n < 300$.

TABLE 1
Specimen sets of values for robust chaplets from Theorem 3.5 (n prime)
Type (a). An asterisk * indicates a solution with $x + z \equiv xz \pmod{n}$

n	13*	29	31*	37*	41*	43	53*	61*	67*	71	71*	73*	79*	89	101*	103*
π	3	7	5	9	5	7	13	15	11	5	7	9	13	11	25	17
ω	2	2	3	2	4	3	2	2	3	7	5	4	3	4	2	3
x	9	23	4	7	16	41	24	12	14	54	32	16	67	4	92	13
z	6	3	22	32	12	30	31	51	32	53	56	40	7	24	11	44

n	109	113*	127	127	131	137	139	139*	149*	151*	151*	157	157*	173*
π	27	7	7	9	13	17	3	23	37	3	25	13	39	43
ω	2	8	9	7	5	4	23	3	2	25	3	6	2	2
x	25	28	2	37	112	16	96	131	95	118	127	93	9	29
z	18	68	86	92	111	6	61	109	66	112	7	136	60	69

n	181	181*	191	191	197	199	199	211	211	211	211	211	223	229
π	9	45	5	19	49	9	11	5	7	15	21	35	37	19
ω	10	2	19	5	2	11	9	21	15	7	5	3	3	6
x	43	9	109	6	16	162	63	188	144	19	179	25	128	44
z	77	69	187	58	78	164	119	3	175	3	175	167	11	74

n	233*	239*	239*	241	241*	269	271	271	277	277	281*	281*	283*	293
π	29	7	17	5	15	67	5	27	23	69	7	35	47	73
ω	4	17	7	24	8	2	27	5	6	2	20	4	3	2
x	152	44	75	98	94	61	10	5	16	55	165	238	250	55
z	180	190	43	228	185	8	52	236	99	263	13	84	259	19

Type (b).

n	29	37	53	61	61	67	71	73	79	89	101	103	109	127	131	139	149	151
π	7	9	13	5	15	11	7	9	13	11	25	17	27	9	13	23	37	25
ω	2	2	2	6	2	3	5	4	3	4	2	3	2	7	5	3	2	3
x	16	12	28	20	42	59	48	16	8	32	80	30	97	37	112	64	129	20
z	11	19	19	31	55	18	28	31	3	30	35	20	69	116	95	12	75	77

n	157	157	173	181	197	211	211	223	233	239	269	271	277	277	281	283	293
π	13	39	43	45	49	15	35	37	29	17	67	27	23	69	35	47	73
ω	6	2	2	2	2	7	3	3	4	14	2	5	6	2	4	3	2
x	67	81	83	75	70	137	87	16	2	128	14	259	175	230	35	250	69
z	6	24	59	23	31	106	118	90	35	84	28	75	97	110	199	145	8

Theorem 3.6 *Let $n = p^2$ where p is a prime satisfying $p > 5$. Suppose that*

$$\hookrightarrow \; a_1 \quad a_2 \quad \ldots \quad a_{(p-1)/2} \; \hookleftarrow \qquad (\mathrm{mod}\ p)$$

is a strong chaplet for $\mathbb{Z}_p$. Let u and w be units of orders p and $p-1$, respectively, in $\mathbb{Z}_n$. Let $e_1, e_2, \ldots, e_{(p-1)/2}$ be those elements of $\langle w \rangle$ which are congruent to, respectively, $a_1, a_2, \ldots, a_{(p-1)/2}$, modulo p. Then

$$\begin{aligned} \hookrightarrow \; & u^0e_1 \;\; u^0e_2 \;\; \ldots \;\; u^0e_{(p-1)/2} \mid u^1e_1 \;\; u^1e_2 \;\; \ldots \;\; u^1e_{(p-1)/2} \mid \\ & \cdots \mid u^{p-1}e_1 \;\; u^{p-1}e_2 \;\; \ldots \;\; u^{p-1}e_{(p-1)/2} \; \hookleftarrow \qquad (\mathrm{mod}\ n) \end{aligned}$$

is a strong chaplet for $\mathbb{Z}_n$. If the chaplet for $\mathbb{Z}_p$ is robust, then so is the chaplet for $\mathbb{Z}_n$.

Proof: This follows from the fact that all the segments are identical, modulo p, as each of the elements u^i $(i = 0, 1, \ldots, p-1)$ is congruent to 1, modulo p. □

Example 3.6: Taking $(n, p) = (49, 7)$ in Theorem 3.6, we can start from the robust chaplet

$$\hookrightarrow \; 1 \quad 2 \quad 4 \; \hookleftarrow$$

for $\mathbb{Z}_7$. We can take $w = 19$, the successive powers of which are 1, 19, 18, 48, 30, 31, so we need $(e_1, e_2, e_3) = (1, 30, 18)$. We can take u to be any value from $\{8, 15, 22, 29, 36, 43\}$. To illustrate a situation that arises again below (see Ex. 4.3(b)), we choose $u = 36$, so that the first element in each successsive segment of the robust terrace for $\mathbb{Z}_{49}$ is twice the immediately preceding element, modulo n:

$$\hookrightarrow \; 1 \;\; 30 \;\; 18 \mid 36 \;\; 2 \;\; 11 \mid 22 \;\; 23 \;\; 4 \mid 8 \;\; 44 \;\; 46 \mid 43 \;\; 16 \;\; 39 \mid 29 \;\; 37 \;\; 32 \mid 15 \;\; 9 \;\; 25 \mid \; \hookleftarrow \; .$$

Theorem 3.7 *Let $n = p^3$ where p is a prime satisfying $p > 3$. Suppose that*

$$\hookrightarrow \; a_1 \quad a_2 \quad \ldots \quad a_{p(p-1)/2} \; \hookleftarrow \qquad (\mathrm{mod}\ p^2)$$

is a strong chaplet for $\mathbb{Z}_{p^2}$. Let u and w be units of orders p and $p(p-1)$, respectively, in $\mathbb{Z}_n$, such that $u \equiv 1 \pmod{p^2}$. Let $e_1, e_2, \ldots, e_{p(p-1)/2}$ be those elements of $\langle w \rangle$ which are congruent to, respectively, $a_1, a_2, \ldots, a_{p(p-1)/2}$, modulo p. Then

$$\begin{aligned} \hookrightarrow \; & u^0e_1 \;\; u^0e_2 \;\; \ldots \;\; u^0e_{p(p-1)/2} \mid u^1e_1 \;\; u^1e_2 \;\; \ldots \;\; u^1e_{p(p-1)/2} \mid \\ & \cdots \mid u^{p-1}e_1 \;\; u^{p-1}e_2 \;\; \ldots \;\; u^{p-1}e_{p(p-1)/2} \; \hookleftarrow \qquad (\mathrm{mod}\ n) \end{aligned}$$

is a strong chaplet for $\mathbb{Z}_n$. If the chaplet for $\mathbb{Z}_{p^2}$ is robust, then so is the chaplet for $\mathbb{Z}_n$.

Proof: On the same lines as for Theorem 3.6. □

4 $n = p^i q^j$ where $\xi(n) = 2$

Turning now to composite odd values of n, we start with those having $\xi(n) = 2$, *i.e.* those for which a primitive λ-root generates half of the members of $\mathbb{U}_n$. In the range $5 < n < 300$ there are 52 such n-values, of which 41 are of the form $n = pq$ where p and q are distinct odd primes, whereas 9 are of the form $n = p^2q$ and 2 are of the form $n = p^3q$.

Theorem 4.1 *Let $n = p^i q^j$ $(i \geq 1,\ j \geq 1)$ where p and q are distinct odd primes such that $\gcd((p-1)p^{i-1},\ (q-1)q^{j-1}) = 2$. Let x be a strong primitive λ-root of n. Then the cycle $[a_1, a_2, \ldots, a_{\phi(n)/2}]$ is a champion chaplet for $\mathbb{Z}_n$ if $a_k = x^{k-1}$ $(k = 1, 2, \ldots, \phi(n)/2)$.*

Proof: Obvious. The use of a strong primitive λ-root ensures that no two members of the cycle are the negatives of one another (mod n). □

Example 4.1: The value 26 is a strong primitive λ-root of 33, so

$$\hookrightarrow\ 1\ \ 26\ \ 16\ \ 20\ \ 25\ \ 23\ \ 4\ \ 5\ \ 31\ \ 14\ \hookleftarrow$$

is a champion chaplet for $\mathbb{Z}_{33}$.

Note 4.1: All odd composite numbers n in the range $2 < n < 20000$ are known to have strong primitive λ-roots [7].

Our next theorem re-uses the basic technique of Theorem 3.3.

Theorem 4.2 *Let $n = p^i q^j$ $(i \geq 1,\ j \geq 1)$ where p and q are distinct odd primes such that $\gcd((p-1)p^{i-1},\ (q-1)q^{j-1}) = 2$. Suppose that $\mathbb{Z}_n$ contains non-negating units v and $y = (v+1)^{-1}$ such that $\mathrm{ord}_n(v) = \omega$ and $\mathrm{ord}_n(y) = \pi$ $(2 < \pi,\ 2 < \omega)$, where π and ω are coprime, being even and odd respectively, and vy is a strong primitive λ-root of n. Then*

$$\begin{array}{l} \hookrightarrow\ v^0y^0\ \ v^0y^1\ \ \ldots\ \ v^0y^{\pi-1}\ \mid\ v^1y^0\ \ v^1y^1\ \ \ldots\ \ v^1y^{\pi-1}\ \mid\ \cdots\ \mid \\ \qquad\qquad v^{\omega-1}y^0\ \ v^{\omega-1}y^1\ \ \ldots\ \ v^{\omega-1}y^{\pi-1}\ \mid\ \hookleftarrow \quad (\mathrm{mod}\ n) \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$. Now write $x = v + 1 = y^{-1}$ and $z = -v^{-1} = -(v-1)^{-1}$, so that $\mathrm{ord}_n(x) = \pi$ and $\mathrm{ord}_n(z) = 2\omega$. Then

$$\begin{array}{l} \hookrightarrow\ x^0z^0\ \ x^0z^1\ \ \ldots\ \ x^0z^{\omega-1}\ \mid\ x^1z^0\ \ x^1z^1\ \ \ldots\ \ x^1z^{\omega-1}\ \mid\ \cdots\ \mid \\ \qquad\qquad x^{\pi-1}z^0\ \ x^{\pi-1}z^1\ \ \ldots\ \ x^{\pi-1}z^{\omega-1}\ \mid\ \hookleftarrow \quad (\mathrm{mod}\ n) \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As for Theorem 3.3. Here vy is a primitive λ-root of n, with $\mathrm{ord}_n(vy) = \phi(n)/2 = \pi\omega$. □

Example 4.2: For $n = 87 = 3\times 29$, the values $(v, y) = (16, 41)$ satisfy the conditions of Theorem 4.2 with $(\pi, \omega) = (4, 7)$. We thus obtain the following robust chaplets for $\mathbb{Z}_{87}$:

$$\hookrightarrow\ 1\ \ 41\ \ 28\ \ 17\ |\ 16\ \ 47\ \ 13\ \ 11\ |\ 82\ \ 56\ \ 34\ \ 2\ |\ 7\ \ 26\ \ 22\ \ 32\ |$$
$$25\ \ 68\ \ 4\ \ 77\ |\ 52\ \ 44\ \ 64\ \ 14\ |\ 49\ \ 8\ \ 67\ \ 50\ |\ \hookleftarrow\ .$$

$$\hookrightarrow\ 1\ \ 38\ \ 52\ \ 62\ \ 7\ \ 5\ \ 16\ \ |\ \ 17\ \ 37\ \ 14\ \ 10\ \ 32\ \ 85\ \ 11\ \ |$$
$$28\ \ 20\ \ 64\ \ 83\ \ 22\ \ 53\ \ 13\ \ |\ \ 41\ \ 79\ \ 44\ \ 19\ \ 26\ \ 31\ \ 47\ \ |\ \hookleftarrow\ .$$

Corollary to Theorem 4.2 *Let n, p, q, π, ω, v and y be as in* Theorem 4.2. *Suppose that $\mathbb{Z}_n$ contains a unit u such that $u \in \langle v\rangle$ and $u + 1 \equiv v^{-\tau}y^{-1}u \pmod n$ where $1 < \tau < \omega$ and the values τ and ω are coprime. Write $t = v^\tau$. Then*

$$\hookrightarrow\ t^0y^0\ \ t^0y^1\ \ \ldots\ t^0y^{\pi-1}\ |\ t^1y^0\ \ t^1y^1\ \ \ldots\ t^1y^{\pi-1}\ |$$
$$\cdots\ |\ t^{\omega-1}y^0\ \ t^{\omega-1}y^1\ \ \ldots\ t^{\omega-1}y^{\pi-1}\ |\ \hookleftarrow\ \pmod n$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As for the corollary to Theorem 3.3. □

Note 4.2(a): In the range $5 < n < 300$, Theorem 4.2 provides robust chaplets for $\mathbb{Z}_n$ as follows, with daggers † indicating solutions where, using the corollary, the segments of the theorem's first chaplet may be taken in at least one other systematically obtained order; indeed, for every solution with a dagger † we can take $u = 1$ in the corollary, and in many of these instances we have $vy^{-1} = v(v + 1) = 2$:

n	45†	55†	75†	77†	87†	93	95†	123	129	135†	143†	147†	155	175†
π	4	4	4	6	4	6	4	8	6	4	12	6	12	12
ω	3	5	5	5	7	5	9	5	7	9	5	7	5	5
v	16	31	31	64	16	4	36	37	121	106	53	127	66	106
y	8	43	68	32	41	56	18	68	92	53	98	116	118	18
x	17	32	32	65	17	5	37	38	122	107	54	128	67	107
z	14	39	29	6	38	23	29	113	113	14	116	125	54	104

n	183	203†	207†	215	225†	225†	237†	245†	253†	261†	261†	261†	287	295†
π	12	12	22	12	12	20	6	12	22	4	12	28	24	4
ω	5	7	3	7	5	3	13	7	5	21	7	3	5	29
v	142	190	70	121	181	151	22	176	185	16	190	88	78	116
y	32	186	35	178	68	188	134	18	219	215	41	44	109	58
x	143	191	71	122	182	152	23	177	186	17	191	89	79	117
z	125	125	68	199	179	149	140	174	160	212	125	86	195	89

Note 4.2(b): We now have a situation similar to that described in Note 3.3(b). For n-values as in Theorem 4.2, the relationship $y = (v+1)^{-1}$ is not the only one that can be used, in conjunction with $\mathrm{ord}_n(y) = \pi$ and $\mathrm{ord}_n(v) = \omega$, to produce a robust chaplet of the first type given in Theorem 4.2. With this relationship abandoned, robust chaplets for $\mathbb{Z}_n$ for some pairs (n, π) absent from the table in Note 4.2(a) can be obtained by taking $(n, \pi, \omega, v, y) = (159, 4, 13, 121, 83)$ and $(267, 8, 11, 217, 101)$ in the first cycle given in Theorem 4.2.

Theorem 4.3 *Let n be a composite integer satisfying $n = 9p$ where p is prime and $\xi(n) = 2$, so that $p \equiv 5 \pmod 6$. Let y be a strong primitive λ-root of $3p$ such that $y - 1 \in \langle y \rangle$, and write $\omega = \mathrm{ord}_{3p}(y) = p - 1$. Let z be the unit from $\{y,\ y+p,\ y+2p\}$ such that $\mathrm{ord}_n(z) = \omega$. Write x for either $1 + 3p$ or $1 + 6p$. Then*

$$\hookrightarrow \ z^0 \ \ z^1 \ \ \ldots \ \ z^{\omega-1} \mid xz^0 \ \ xz^1 \ \ \ldots \ \ xz^{\omega-1} \mid x^2z^0 \ \ x^2z^1 \ \ \ldots \ \ x^{\omega-1} \mid \ \hookleftarrow \pmod n$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: Similar to the proof of Theorem 3.6. □

Example 4.3(a): Take $p = 11$ in Theorem 4.3, so that $3p = 33$. The strong primitive λ-roots of 33 include 5, and $5 - 1 = 4 \equiv 5^8 \pmod{33}$, so we can take $y = 5$. The order of 5 (mod 33) is 10, which is also the order of $5 + 66 \pmod{99}$, so we take $z = 71$ along with, say, $x = 34$ to obtain the following robust chaplet for $\mathbb{Z}_{99}$:

$$\begin{array}{lcccccccccc} \hookrightarrow & 1 & 71 & 91 & 26 & 64 & 89 & 82 & 80 & 37 & 53 & \mid \\ & 34 & 38 & 25 & 92 & 97 & 56 & 16 & 47 & 70 & 20 & \mid \\ & 67 & 5 & 58 & 69 & 31 & 23 & 49 & 14 & 4 & 86 & \mid \ \hookleftarrow . \end{array}$$

Note 4.3: Theorem 4.3 can readily be extended to provide chaplets for $\mathbb{Z}_n$ where $n = 3^i p$ with $2 \le i$. With y and ω as in the theorem, each chaplet will now have 3^{i-1} segments, each containing ω elements. The first segment will consist of the successive powers of z where z is the sole element from $\mathbb{Z}_n$ with $z \equiv y \pmod{3p}$ and $\mathrm{ord}_n(z) = \omega$. The subsequent segments, in any order, will consist of the first segment multiplied through successively by the successive powers of $1 + 3p$ or, equivalently, by the values $1 + 3kp$ where $k = 1, 2, \ldots, 3^{i-1} - 1$.

Example 4.3(b): Take $p = 5$ and $n = 3^3 p = 135$. Taking $y = 2$, we find that we need $z = 107$, as $\mathrm{ord}_{135}(107) = \mathrm{ord}_{15}(2) = 4$. The robust chaplets obtainable thereby for $\mathbb{Z}_{135}$ include the following, where, as in Ex. 3.6(a), the ordering chosen for the segments is such that the first element of each segment is twice the immediately preceding element, modulo n:

$$\begin{array}{lcccccccccccccc} \hookrightarrow & 1 & 107 & 109 & 53 & \mid & 106 & 2 & 79 & 83 & \mid & 31 & 77 & 4 & 23 & \mid \\ & 46 & 62 & 19 & 8 & \mid & 16 & 92 & 124 & 38 & \mid & 76 & 32 & 49 & 113 & \mid \\ & 91 & 17 & 64 & 98 & \mid & 61 & 47 & 34 & 128 & \mid & 121 & 122 & 94 & 68 & \mid \ \hookleftarrow . \end{array}$$

Further robust chaplets for some values of n with $\xi(n) = 2$ are obtainable from Theorem 8.1 below.

5 Values of n with $\xi(n) = 4$

We come now to composite odd values of n that have $\xi(n) = 4$. In the range $2 < n < 300$ there are 11 of these, all covered by Theorem 5.2 below, and they arise for n-values of the forms p^iq^j and pqr. We start with a very simple construction.

Theorem 5.1 *Let n be a composite integer satisfying $n = p^iq^j$ $(i \geq 1,\ j \geq 1)$ with $\xi(n) = 4$. Suppose that $\mathbb{Z}_n$ contains units x and z such that x is a strong primitive λ-root of n, and $z^2 \equiv -1 \pmod{n}$ where $z = x - 1$. Then*

$$\hookrightarrow\ 1\ \ z \mid x\ \ zx \mid x^2\ \ zx^2 \mid \cdots \mid x^{\lambda(n)}\ \ zx^{\lambda(n)} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: We have $zx = z(z+1) = z^2 + z \equiv z - 1 \pmod{n}$. Thus the first two differences for the first proposed chaplet, namely $z - 1$ and $x - z$, are congruent respectively to zx and $1 \pmod{n}$. The result readily follows. □

Lemma 5.1 *If the conditions of* Theorem 5.1 *are satisfied by $(x, z) = (x_1, z_1)$, then they are also satisfied by $(x, z) = (x_2, z_2)$ where $x_2 \equiv 2 - x_1$ and $z_2 \equiv -z_1 \pmod{n}$.*

Proof: As $x_2 \equiv -z_1x_1 \pmod{n}$ and the order $\lambda(n)$ of x_1 must be a multiple of 4, it follows that x_2 must be a strong primitive λ-root of n. Also $z_2^2 = z_2^{-2} \equiv z_1^2 \equiv -1 \pmod{n}$. □

Example 5.1: For $n = 65$, we can take $x = 19$ in Theorem 5.1 to obtain the following robust chaplet for $\mathbb{Z}_{65}$:

$$\begin{aligned}\hookrightarrow\ 1\ \ 18 \mid 19\ \ 17 \mid 36\ \ 63 \mid 34\ \ 27 \mid 61\ \ 58 \mid 54\ \ 62 \mid \\ 51\ \ 8 \mid 59\ \ 22 \mid 16\ \ 28 \mid 44\ \ 12 \mid 56\ \ 33 \mid 24\ \ 42 \mid \hookleftarrow\ .\end{aligned}$$

Note 5.1: In the range $5 < n < 300$, the conditions of Theorem 5.1 are satisfied by $(n, x) = (65, 19)$, $(85, 39)$, $(145, 13)$, $(185, 69)$, $(221, 22)$, $(265, 84)$. They cannot be satisfied for $n = 205$. For some theory relating to $n = 65$, 145, 185 and 265, see [7, §8.2].

Our next theorem covers not only the value $n = 205$, just mentioned, but also n-values each of which satisfies both $\xi(n) = 4$ and $n = pqr$, where p, q and r are distinct odd primes. These further n-values include 105, 165, 231 and 285.

Theorem 5.2 *Let n be a composite integer for which $\xi(n) = 4$. Suppose that n has a non-negating primitive λ-root x such that $2 \notin \pm\langle x \rangle$ and $x - 2 \in 2\langle x \rangle$. Then*

$$\hookrightarrow\ 1\ \ 2 \mid x\ \ 2x \mid x^2\ \ 2x^2 \mid \cdots \mid x^{\lambda(n)}\ \ 2x^{\lambda(n)} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: Immediate. □

Example 5.2: For $n = 65$, we can take $x = 59$ in Theorem 5.2 to obtain the following robust chaplet for $\mathbb{Z}_{65}$:

$$\hookrightarrow \;\; 1 \;\; 2 \mid 59 \;\; 53 \mid 36 \;\; 7 \mid 44 \;\; 23 \mid 61 \;\; 57 \mid 24 \;\; 48 \mid$$
$$51 \;\; 37 \mid 19 \;\; 38 \mid 16 \;\; 32 \mid 34 \;\; 3 \mid 56 \;\; 47 \mid 54 \;\; 43 \mid \hookleftarrow \; .$$

Note 5.2(a): In the range $5 < n < 300$, the conditions of Theorem 5.2 are satisfied by $(n, x) = (65, 59)$, $(85, 63)$, $(105, 88)$, $(145, 103)$, $(165, 73)$, $(185, 19)$, $(205, 28)$, $(221, 108)$, $(231, 52)$, $(265, 14)$, $(285, 13)$.

Note 5.2(b): Theorem 5.2 provides a crude methodology for obtaining robust chaplets for $\mathbb{Z}_n$ where n is a product of three distinct odd primes such that $\xi(n) = 4$ and therefore $\lambda(n) = \phi(n)/4$. A more subtle method is sometimes available for such n-values, and it enables us — as in Notes 3.3(b) and 4.2(b) — to produce robust chaplets of the form

$$\hookrightarrow \;\; v^0y^0 \;\; v^0y^1 \;\; \ldots \;\; v^0y^{\pi-1} \mid v^1y^0 \;\; v^1y^1 \;\; \ldots \;\; v^1y^{\pi-1} \mid \cdots \mid$$
$$v^{\omega-1}y^0 \;\; v^{\omega-1}y^1 \;\; \ldots \;\; v^{\omega-1}y^{\pi-1} \mid \hookleftarrow \pmod n \; .$$

Now, however, we have $\pi\omega = \phi(n)/2 = 2\lambda(n)$. In the range $5 < n < 300$ we have such solutions for $n = 105$, 165 and 231. If we write $n = pqr$ and $(n, p, q, r) = (105, 3, 5, 7)$, $(165, 3, 5, 11)$ and $(231, 3, 11, 7)$ respectively, our solutions have $(p - 1, q - 1, r - 1) = (2, \omega, \pi)$, so that π and ω are now both even values satisfying $2 < \pi < \lambda(n)$ and $2 < \omega < \lambda(n)$ with $\pi/2$ and $\omega/2$ coprime. Specimen parameter sets are $(n, \pi, \omega, v, y) = (105, 6, 4, 43^i, 74)$, $(165, 10, 4, 133^i, 104)$ and $(231, 6, 10, 211^j, 65)$ respectively, where $i = 1$ or 3, and $j = 1$, 3, 7 or 9. The choice arises because all the segments in any of the chaplets here are identical when reduced modulo r (*cf* Theorem 4.3). With $i = 1$ for $n = 105$, we have the familiar relationship $y \equiv (v + 1)^{-1} \pmod n$. With $i = 1$, the robust chaplet for $\mathbb{Z}_{165}$ is

$$\begin{array}{rrrrrrrrrrl}
\hookrightarrow & 1 & 104 & 91 & 59 & 31 & 89 & 16 & 14 & 136 & 119 & \mid \\
& 133 & 137 & 58 & 92 & 163 & 122 & 148 & 47 & 103 & 152 & \mid \\
& 34 & 71 & 124 & 26 & 64 & 56 & 49 & 146 & 4 & 86 & \mid \\
& 67 & 38 & 157 & 158 & 97 & 23 & 82 & 113 & 37 & 53 & \mid \; \hookleftarrow \; .
\end{array}$$

A further robust chaplet for $\mathbb{Z}_{165}$ is obtainable from Theorem 8.1 below.

6 Values of n with $\xi(n) = 6$

We now turn to odd values of n with $\xi(n) = 6$. In the range $5 < n < 300$ there are 10 of these, all covered by Theorem 6.4 below. Such values arise both when $n = pq$, where p and q are distinct primes congruent to 1 (mod 6), and when $n = 9p$, where the prime p is congruent to 1 (mod 6). The first of our theorems in this section is for the latter case only.

Theorem 6.1 *Let $n = 9p$ where p is a prime satisfying $p \equiv 1 \pmod 6$, whence $\xi(n) = 6$. Choose the value of x from $\{3p+1, 6p+1\}$, whence $x^3 \equiv 1 \pmod n$. Let z be a strong primitive λ-root of n such that $\{x, x^2\} \cap \langle z\rangle = \emptyset$ and $z - 1 \in \langle z, x\rangle$. Then*

$$\hookrightarrow \; z^0 \;\; z^1 \;\; \ldots \;\; z^{\lambda(n)-1} \mid xz^0 \;\; xz^1 \;\; \ldots \;\; xz^{\lambda(n)-1} \mid x^2z^0 \;\; x^2z^1 \;\; \ldots \;\; x^2z^{\lambda(n)-1} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: Immediate, once we note that $\{x, x^2\} = \{3p+1, 6p+1\}$ and thus that, for any fixed i, the i^{th} elements of any two segments are mutually congruent, modulo $3p$. □

Example 6.1: For $n = 63 = 9 \times 7$, take $z = 23$ and $x = 6p + 1 = 43$. Then we obtain the following robust chaplet for $\mathbb{Z}_{63}$:

$$\hookrightarrow \; 1 \;\; 23 \;\; 25 \;\; 8 \;\; 58 \;\; 11 \mid 43 \;\; 44 \;\; 4 \;\; 29 \;\; 37 \;\; 32 \mid 22 \;\; 2 \;\; 46 \;\; 50 \;\; 16 \;\; 53 \mid \hookleftarrow .$$

Employing the argument in Note 4.3, we can use this chaplet for $\mathbb{Z}_{63}$ to form a robust chaplet for $\mathbb{Z}_n$ with $n = 3 \times 63 = 189$. The first segment in this longer chaplet will contain the successive powers of $z + 3p = 44$, as $\text{ord}_{189}(44) = \text{ord}_{63}(23) = 6$. The subsequent segments, in any order, will be the first segment multiplied through successively by the successssive powers of $1 + 3p = 22$. As in Exs 3.6(a) and 4.3(b), we can arrange the segments so that the first element of each segment is twice the immediately preceding element, modulo n.

Note 6.1: In the range $5 < n < 300$, specimen sets of values satisfying the conditions of Theorem 6.1 are $(n, z) = (63, 2)$, $(117, 2)$, $(171, 5)$, $(279, 41)$.

Our next two theorems are closely related to Theorems 3.3 and 4.2 above, even though the conditions now have to be stated differently.

Theorem 6.2 *Let n be an odd integer such that $\xi(n) = 6$. Suppose that $\mathbb{Z}_n$ contains units v and y such that y is a strong primitive λ-root of n, whilst v is of order 3 $\pmod n$, $v \notin \langle y\rangle$ and $v \equiv y^{-1} - 1 \pmod n$. Then*

$$\hookrightarrow \; y^0 \;\; y^1 \;\; \ldots \;\; y^{\lambda(n)-1} \mid vy^0 \;\; vy^1 \;\; \ldots \;\; vy^{\lambda(n)-1} \mid v^2y^0 \;\; v^2y^1 \;\; \ldots \;\; v^2y^{\lambda(n)-1} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$. Now write $x = v + 1 = y^{-1}$ and $z = -v^{-1} = -(x-1)^{-1}$, so that $\text{ord}_n(z) = 6$ and $\text{ord}_n(x) = \lambda(n)$. Then

$$\hookrightarrow \; x^0z^0 \;\; x^0z^1 \;\; x^0z^2 \mid x^1z^0 \;\; x^1z^1 \;\; x^1z^2 \mid \cdots \mid x^{\lambda(n)-1}z^0 \;\; x^{\lambda(n)-1}z^1 \;\; x^{\lambda(n)-1}z^2 \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As for Theorem 3.3. □

Example 6.2: For $n = 63$ we can take $(v, y) = (43, 53)$ in Theorem 6.2 to obtain the following robust chaplets for $\mathbb{Z}_{63}$:

$$\hookrightarrow \ 1 \ \ 53 \ \ 37 \ \ 8 \ \ 46 \ \ 44 \mid 43 \ \ 11 \ \ 16 \ \ 29 \ \ 25 \ \ 2 \mid 22 \ \ 32 \ \ 58 \ \ 50 \ \ 4 \ \ 23 \mid \ \hookleftarrow$$

and

$$\hookrightarrow \ 1 \ \ 41 \ \ 43 \mid 44 \ \ 40 \ \ 2 \mid 46 \ \ 59 \ \ 25 \mid 8 \ \ 13 \ \ 29 \mid 37 \ \ 5 \ \ 16 \mid 53 \ \ 31 \ \ 11 \mid \ \hookleftarrow \ .$$

Note 6.2(a): If the condition of Theorem 6.2 are satisfied by $(v, y) = (v_1, y_1)$ then they are also satisfied by $(v, y) = (v_2, y_2)$ where $v_2 \equiv v_1^2$ and $y_2 \equiv y_1 v_1 \pmod{n}$.

Note 6.2(b): In the range $5 < n < 300$, specimen sets of values satisfying the conditions of Theorem 6.2 are $(n, v, y) = (63, 43, 53)$, $(91, 53, 59)$, $(117, 79, 98)$ and $(259, 149, 19)$. The theorem fails for $n = 133$, 171, 189, 217, 247 and 279. For $n = 133$, 171 and 189 the failure arises solely because the values 59, 59 and 38, respectively, are negating, not strong, primitive λ-roots of n.

Theorem 6.3 *Let n be an odd integer such that $\xi(n) = 6$. Suppose that $\phi(n) = 2\pi\omega$ where π and ω (not necessarily coprime) satisfy $3 < \pi < \lambda(n)$ and $3 < \omega < \lambda(n)$. Suppose further that $\mathbb{Z}_n$ contains non-negating units v and $y = (v+1)^{-1}$ such that $\mathrm{ord}_n(v) = \omega$ and $\mathrm{ord}_n(y) = \pi$, with $\langle v \rangle \cap \langle y \rangle = \{1\}$ and $-1 \notin \langle v, y \rangle$. Then*

$$\begin{array}{llllllllllll} \hookrightarrow & v^0y^0 & v^0y^1 & \ldots & v^0y^{\pi-1} & \mid & v^1y^0 & v^1y^1 & \ldots & v^1y^{\pi-1} & \mid & \cdots \ \mid \\ & & & & v^{\omega-1}y^0 & & v^{\omega-1}y^1 & & \ldots & v^{\omega-1}y^{\pi-1} & \mid & \hookleftarrow \pmod{n} \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Now, if ω is odd, write $x = v + 1 = y^{-1}$ and $z = -v^{-1} = -(x-1)^{-1}$, so that $\mathrm{ord}_n(x) = \pi$ and $\mathrm{ord}_n(z) = 2\omega$. Then

$$\begin{array}{llllllllllll} \hookrightarrow & x^0z^0 & x^0z^1 & \ldots & x^0z^{\omega-1} & \mid & x^1z^0 & x^1z^1 & \ldots & x^1z^{\omega-1} & \mid & \cdots \ \mid \\ & & & & x^{\pi-1}z^0 & & x^{\pi-1}z^1 & & \ldots & x^{\pi-1}z^{\omega-1} & \mid & \hookleftarrow \pmod{n} \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As for Theorem 3.3, save that the second part does not apply if ω is even. □

Note 6.3: In the range $5 < n < 300$, specimen parameter sets for robust chaplets that are obtainable from Theorem 6.3 are as follows, where ‡ indicates a solution that has $v \equiv 2y \pmod{n}$:

$(n, \pi, \omega, v, y, x, z) =$
$(133, 6, 9, 36, 18, 37, 48)^\ddagger$, $(171, 6, 9, 139, 11, 140, 155)$, $(189, 6, 9, 43, 116, 44, 167)^\ddagger$,
$(217, 6, 15, 4, 87, 5, 54)$, $(247, 12, 9, 196, 163, 197, 155)$ and $(279, 6, 15, 97, 242, 98, 23)$;

$(n, \pi, \omega, v, y) = (217, 15, 6, 99, 102)$ and $(259, 9, 12, 80, 16)$.

Our final theorem in this section of the paper extends the approach adopted in Theorem 5.2 above, and covers every n with $\xi(n) = 6$ that lies in the range $5 < n < 300$.

Theorem 6.4 *Let n be an odd integer such that $\xi(n) = 6$. Suppose that n has a non-negating (but not necessarily inward) primitive λ-root x such that $2 \notin \langle x \rangle$ and $x - 4 \in 4\langle x \rangle$. Then*

$$\hookrightarrow \ 1 \ \ 2 \ \ 4 \mid x \ \ 2x \ \ 4x \mid x^2 \ \ 2x^2 \ \ 4x^2 \mid \cdots \mid x^{\lambda(n)-1} \ \ 2x^{\lambda(n)-1} \ \ 4x^{\lambda(n)-1} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: Immediate, as $\langle x, 2\rangle$ must contain exactly half of the units of $\mathbb{Z}_n$, no two of these units being the negatives of one another, modulo n. □

Example 6.4: For $n = 63$, we may take x to be the outward primitive λ-root 29 to obtain the following chaplet for $\mathbb{Z}_{63}$;

$$1 \ \ 2 \ \ 4 \mid 29 \ \ 58 \ \ 53 \mid 22 \ \ 44 \ \ 25 \mid 8 \ \ 16 \ \ 32 \mid 43 \ \ 23 \ \ 46 \mid 50 \ \ 37 \ \ 11 \mid \hookleftarrow .$$

Note 6.4: In Theorem 6.4, the element 2 is not necessarily a primitive λ-root and is not necessarily non-negating. For n-values covered by Theorem 6.4 in the range $5 < n < 300$, specimen values of x are as follows: $(n, x) = (63, 29)$, $(91, 15)$, $(117, 47)$, $(133, 15)$, $(171, 62)$, $(189, 29)$, $(217, 22)$, $(247, 21)$, $(259, 15)$ and $(279, 71)$. For all of these n-values, 2 is a strong primitive λ-root, except that 2 is a negating primitive λ-root for $n = 171$, and is a non-negating element of order $\lambda(n)/2$ for $n = 217$. Further chaplets for $\mathbb{Z}_{91}$, $\mathbb{Z}_{133}$, $\mathbb{Z}_{217}$ and $\mathbb{Z}_{279}$ are obtained in §8 below.

7 Values of n with $\xi(n) \geq 8$

In our range $5 < n < 300$ there are four odd n-values with $\xi(n) \geq 8$, namely $195 = 3 \times 5 \times 13$, $255 = 3 \times 5 \times 17$, $273 = 3 \times 7 \times 13$ and $275 = 5^2 \times 11$, and we obtain robust chaplets for all of them. For these values, we have $\xi(n) = 8$, 8, 12 and 10 respectively. Accordingly we start by restricting ourselves to integers n with $n = 3pq$ and $\xi(n) = 4k$, $k > 1$, where 3, p and q are distinct odd primes. Theorem 8.4 of Cameron and Preece [7] shows that, for such values of n we can always find two units of $\mathbb{Z}_n$ that together generate half of the units of $\mathbb{Z}_n$, no two of the elements in this half being the negatives of one another. This enables us to re-use the methodology of Theorems 3.3, 4.2 and 6.2 above, with a sign change in the definition of x, to obtain the following theorem for our present needs. The sign change is needed as the roles of π and ω, as in Theorem 3.3, are now taken by $\lambda(n)$ and $2k$, both of which are even, whereas previously ω has been odd.

Theorem 7.1 *Let n be an integer of the form $n = 3pq$ where p and q are distinct integers greater than 3, and where $\xi(n) = 4k$ $(k > 1)$. Suppose that $\mathbb{Z}_n$ contains units v and y such that y is a strong primitive λ-root of n, whilst v is a non-negating unit of order $2k$ (mod n), $\{v, v^2, \ldots, v^{2k-1}\} \cap \langle y \rangle = \emptyset$ and $y \equiv (v + 1)^{-1}$ (mod n). Then*

$$\hookrightarrow \ v^0y^0 \ \ v^0y^1 \ \ \ldots \ v^0y^{\lambda(n)-1} \mid v^1y^0 \ \ v^1y^1 \ \ \ldots \ v^1y^{\lambda(n)-1} \mid$$
$$\cdots \mid v^{2k-1}y^0 \ \ v^{2k-1}y^1 \ \ \ldots \ v^{2k-1}y^{\lambda(n)-1} \mid \hookleftarrow$$

is a robust chaplet for $\mathbb{Z}_n$. *Now write* $x = -(v+1) = -y^{-1}$ *and* $z = -v^{-1} = +(x+1)^{-1}$. *Then*

$$\begin{array}{l} \hookrightarrow\ x^0z^0\ \ x^0z^1\ \ \ldots\ \ x^0z^{2k-1}\ |\ x^1z^0\ \ x^1z^1\ \ \ldots\ \ x^1z^{2k-1}\ | \\ \qquad\qquad \ldots\ |\ x^{\lambda(n)-1}z^0\ \ x^{\lambda(n)-1}z^1\ \ \ldots\ \ x^{\lambda(n)-1}z^{2k-1}\ |\ \hookleftarrow \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: As y is a strong primitive λ-root of $3pq$, we must have $y \equiv 2$ and thus $v \equiv 1 \pmod 3$. Thus $z \equiv 2$ and $x \equiv 1 \pmod 3$. Also the congruence $v \equiv y^{-1} - 1 \pmod n$ implies $x \equiv z^{-1} - 1 \pmod n$. Therefore the proof for the second chaplet in the statement of the theorem exactly imitates the proof for the first, which in turn follows the same lines as for Theorems 3.3, 4.2 and 6.2. □

Example 7.1: For $n = 195 = 3 \times 5 \times 13$, the possible values of v in Theorem 7.1 are 118, 118^{-1} (= 157), 148, 148^{-1} (= 112), 31 and 31^{-1} (= 151). These give, respectively, $(v, y, x, z) = (118, 59, 76, 38)$, $(157, 137, 37, 77)$, $(148, 89, 46, 83)$, $(112, 107, 82, 47)$, $(31, 128, 163, 44)$ and $(151, 68, 43, 164)$. For the first of these possibilities the robust chaplets for $\mathbb{Z}_{195}$ are as follows:

$$\begin{array}{rrrrrrrrrrrrrl} \hookrightarrow & 1 & 59 & 166 & 44 & 61 & 89 & 181 & 149 & 16 & 164 & 121 & 119 & | \\ & 118 & 137 & 88 & 122 & 178 & 167 & 103 & 32 & 133 & 47 & 43 & 2 & | \\ & 79 & 176 & 49 & 161 & 139 & 11 & 64 & 71 & 94 & 86 & 4 & 41 & | \\ & 157 & 98 & 127 & 83 & 22 & 128 & 142 & 188 & 172 & 8 & 82 & 158 & | \ \hookleftarrow \end{array}$$

and

$$\begin{array}{rrrrrrrrrrrrrl} \hookrightarrow & 1 & 38 & 79 & 77 & | & 76 & 158 & 154 & 2 & | & 121 & 113 & 4 & 152 & | \\ & 31 & 8 & 109 & 47 & | & 16 & 23 & 94 & 62 & | & 46 & 188 & 124 & 32 & | \\ & 181 & 53 & 64 & 92 & | & 106 & 128 & 184 & 167 & | & 61 & 173 & 139 & 17 & | \\ & 151 & 83 & 34 & 122 & | & 166 & 68 & 49 & 107 & | & 136 & 98 & 19 & 137 & | \ \hookleftarrow . \end{array}$$

Note 7.1: For $n = 255 = 3 \times 5 \times 17$, the possible values of v in Theorem 7.1 are 38, 38^{-1} (= 47), 98 and 98^{-1} (= 242). For $n = 273 = 3 \times 7 \times 13$, the possible values of v are 40, 40^{-1} (= 157), 166 and 166^{-1} (= 199).

We now proceed to a useful analogue of Theorems 5.2 and 6.4.

Theorem 7.2 *Let* n *be an odd integer such that* $\xi(n) = 8$. *Suppose that* n *has a non-negating (but not necessarily inward) primitive* λ*-root* x *such that* $\{1, 2, 4, 8\} \cap \langle x \rangle = \{1\}$ *and* $x - 8 \in 8\langle x \rangle$. *Then*

$$\begin{array}{l} \hookrightarrow\ 1\ \ 2\ \ 4\ \ 8\ |\ x\ \ 2x\ \ 4x\ \ 8x\ |\ x^2\ \ 2x^2\ \ 4x^2\ \ 8x^2\ | \\ \qquad \ldots\ |\ x^{\lambda(n)-1}\ \ 2x^{\lambda(n)-1}\ \ 4x^{\lambda(n)-1}\ \ 8x^{\lambda(n)-1}\ |\ \hookleftarrow \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: Immediate, as $\langle x, 2\rangle$ must contain half of the units of $\mathbb{Z}_n$, no two of these units being the negatives of one another, modulo n. □

Example 7.2: For $n = 195$, with $\xi(n) = 8$, the primitive λ-root x in Theorem 7.2 can be any of the values 46, 76, 106, 136 (all congruent to 16, mod 30) and 127. For the last of these, we give the robust chaplet for $\mathbb{Z}_{195}$ in full, with the value $x - 8$ marked with an asterisk, so that readers can readily check that the differences are as required:

$$\begin{array}{lrrrrrrrrrrrrl}
\hookrightarrow & 1 & 2 & 4 & 8 & 127 & 59 & 118 & 41 & 139 & 83 & 166 & 137 & \\
 & 103 & 11 & 22 & 44 & 16 & 32 & 64 & 128 & 82 & 164 & 133 & 71 & \\
 & 79 & 158 & 121 & 47 & 88 & 176 & 157 & 119^{*} & 61 & 122 & 49 & 98 & \\
 & 142 & 89 & 178 & 161 & 94 & 188 & 181 & 167 & 43 & 86 & 172 & 149 & \hookleftarrow .
\end{array}$$

Another robust chaplet for $\mathbb{Z}_{195}$ is obtained in §8 below.

Our remaining task in this section is to use methodology similar to that already presented in this paper, in order to produce robust chaplets for $\mathbb{Z}_{275}$. This is most readily achieved when Theorem 6.2 is generalised by replacing the requirement $\xi(n) = 6$ by $\xi(n) = 4k + 2$ $(k \geq 1)$, so that v is of order $2k + 1$ instead of 3 (mod n). For $n = 275$ we have $k = 2$, and can, for example, let v take any of the values 81^i $(i = 1, 2, 3$ or $4)$, *i.e.* the values 81, 236, 141 and 146 respectively. If we take $v = 81^4 = 146$ we have $y = 147^{-1} = 58$, giving us the robust chaplet

$$\begin{array}{lrrrrrrrrrrrrrl}
\hookrightarrow & 1 & 58 & 64 & \ldots & 147 & 146 & 218 & 269 & \ldots & 12 & 141 & 203 & 224 & \ldots & 102 & \\
 & & & & & & 236 & 213 & 254 & \ldots & 42 & 81 & 23 & 234 & \ldots & 82 & \hookleftarrow
\end{array}$$

for $\mathbb{Z}_{275}$, with 20 elements in each segment. Then we have $x = 147$ along with $z = -146^{-1} = 194$, giving us the robust chaplet

$$\begin{array}{lrrrrrrrrrrl}
\hookrightarrow & 1 & 194 & 236 & 134 & 146 & 147 & 193 & 42 & 173 & 12 & \\
 & 159 & 46 & 124 & 131 & 114 & \cdots & 58 & 252 & 213 & 72 & 218 & \hookleftarrow
\end{array}$$

for $\mathbb{Z}_{275}$, with 20 segments in total. We give a further type of robust chaplet for $\mathbb{Z}_{275}$ in §8 below.

8 Chaplets with three generators

Most of the chaplets obtained so far in this paper are obtained by using two "generators", these sometimes being written as v and y, sometimes as x and z. However, just as daisy chains with three generators are available for certain values of n (see [16]), so can three generators be used to produce chaplets for certain n-values. We

denote these generators u, v and y, with their orders, modulo n, denoted ψ, ω and π respectively, where $\psi\omega\pi = \phi(n)/2$. We introduce this approach by considering how it works for $n = 275 = 11 \times 5^2$, which has $\xi(n) = 10$. In this particular case, *mirabile dictu*, we twice find ourselves involved with two generators that differ by 1, but this is not an essential feature of the approach.

There are 200 units in $\mathbb{Z}_{275}$, and $\mathbb{U}_{275} = \langle 226 \rangle \times \langle 227 \rangle$ where the units 226 and 227 have orders 10 and 20 respectively. Thus, as $226^2 \equiv 201 \pmod{275}$, the unit 201 has order 5, and there are 100 units in $\langle 201 \rangle \times \langle 227 \rangle$. No two of these are the negatives of one another, and we use these 100 elements for our chaplet. The unit 227 is a primitive λ-root of 275, and $\langle 227 \rangle = \langle 31 \rangle \times \langle 32 \rangle$ where the units 31 and 32 have orders 5 and 4 respectively. Thus, as $32^{-1} \equiv 43 \pmod{275}$, the units for our chaplet are the members of $\langle 201 \rangle \times \langle 31 \rangle \times \langle 43 \rangle$. Write $(u, v, y) = (201, 31, 43)$, so that $(\psi, \omega, \pi) = (5, 5, 4)$. As y is a strong unit here, so, by analogy with the first result in Theorem 3.3, the cycle

$$\begin{aligned} \hookrightarrow v^0y^0 \quad v^0y^1 \quad \ldots \quad v^0y^{\pi-1} \mid v^1y^0 \quad v^1y^1 \quad \ldots \quad v^1y^{\pi-1} \mid \cdots \mid \\ v^{\omega-1}\,y^0 \quad v^{\omega-1}y^1 \quad \ldots \quad v^{\omega-1}y^{\pi-1} \mid \hookleftarrow \end{aligned}$$

is such that the differences are the negatives of the entries in the cycle. The difference between the beginning and the end of the cycle as printed is

$$1 - v^4y^3 \equiv 1 - v^{-1}y^{-1} \equiv v^{-1}(v - y^{-1}) \equiv -y^{-1} \equiv -v^4 \pmod{275}\ .$$

Now regard the 5 segments, in the order given, as a super-segment $S(v, y)$, and consider the super-cycle

$$\hookrightarrow\ u^0S(v,y)\ \mid\ u^1S(v,y)\ \mid\ \cdots\ \mid\ u^{\psi-1}S(v,y)\ \mid\ \hookleftarrow$$

where the super-segment $u^iS(v, y)$ is obtained by multiplying every element in $S(v, y)$ by u^i. All that is needed to confirm that the super-cycle is a robust chaplet for $\mathbb{Z}_{275}$ is to check that the differences at the fences between the super-segments compensate for the differences missing within the super-segments. In fact, the difference at the third fence between the super-segments is $u^3 - u^2v^4y^3$, which turns out to be congruent to $-v^4$, modulo 275, which as we have seen is the difference between the two ends of the first super-segment. The rest of the checking follows at once. The robust chaplet in its entirety is as follows, where double fences separate the super-segments:

$$\begin{array}{l}
\hookrightarrow\ 1\ \ 43\ 199\ \ 32\ \mid\ 31\ 233\ 119\ 167\ \mid 136\ \ 73\ 114\ 227\ \mid\ 91\ \ 63\ 234\ 162\ \mid\ 71\ \ 28\ 104\ \ 72\ \|\\
201\ 118\ 124\ 107\ \mid 181\ \ 83\ 269\ \ 17\ \mid 111\ \ 98\ \ 89\ 252\ \mid 141\ \ 13\ \ \ 9\ 112\ \mid 246\ 128\ \ \ 4\ 172\ \|\\
251\ \ 68\ 174\ \ 57\ \mid\ 81\ 183\ 169\ 117\ \mid\ 36\ 173\ \ 14\ \ 52\ \mid\ 16\ 138\ 159\ 237\ \mid 221\ 153\ 254\ 197\ \|\\
126\ 193\ \ 49\ 182\ \mid\ 56\ 208\ 144\ 142\ \mid\ 86\ 123\ \ 64\ \ \ 2\ \mid 191\ 238\ \ 59\ \ 62\ \mid 146\ 228\ 179\ 272\ \|\\
\ 26\ \ 18\ 224\ \ \ 7\ \mid 256\ \ \ 8\ \ 69\ 217\ \mid 236\ 248\ 214\ 127\ \mid 166\ 263\ \ 34\ \ 87\ \mid 196\ 178\ 229\ 222\ \|\ \hookleftarrow\ .
\end{array}$$

In this particular case, a further robust chaplet is obtained by replacing u and v in the construction by v^4 and u^2 respectively.

In the range $5 < n < 300$, robust chaplets with three generators, as just described, have been found as follows:

n	$\xi(n)$	ψ	ω	π	u	v	y
$91 = 7 \times 13$	6	3	4	3	79	83	9
$133 = 7 \times 19$	6	3	2	9	58	20	23
$195 = 3 \times 5 \times 13$	8	4	3	4	31	16	8^*
$217 = 7 \times 31$	6	5	6	3	190	94	25
		6	5	3	94	190	$25^\dagger$
$225 = 3^2 \times 5^2$	2	3	5	4	151	91	143^*
		5	3	4	181	76	143^*
$261 = 3^2 \times 29$	2	3	7	4	88	226	215^*
		7	3	4	190	175	215^*
$275 = 5^2 \times 11$	10	5	5	4	201	31	$43^\dagger$
$279 = 3^2 \times 31$	6	3	5	6	187	190	242
		5	3	6	94	187	242^*

Here an asterisk * marks a chaplet with $(u+1)vy \equiv 1 \pmod{n}$, so that the difference at the end of the first super-segment is -1. Likewise a dagger † marks a chaplet with $(v+1)y \equiv 1 \pmod{n}$, so that the difference at the end of the very first segment is -1.

A more intricate construction is provided by the following theorem.

Theorem 8.1 *Suppose that n is an odd integer such that*

$$\mathbb{U}_n = \langle -1 \rangle \times \langle u \rangle \times \langle v \rangle \times \langle y \rangle$$

where u, v and y are elements from $\mathbb{U}_n$ with $\mathrm{ord}_n(u) = 2$, $\mathrm{ord}_n(v) = \omega$ and $\mathrm{ord}_n(y) = \pi$ (the integers ω and π not necessarily being either prime or coprime). Suppose also that there is a unit c in $\mathbb{Z}_n$ such that $c \in u\langle v, y \rangle$, $c - 1 \in \langle v, y \rangle$, $y - c \in u\langle v, y \rangle$ and $vy - c \in (y - c)\langle v \rangle$. To avoid degeneracy, suppose further that $c^2 \not\equiv y \pmod{n}$. Then

$$\begin{array}{c} \hookrightarrow \; v^0y^0 \quad cv^0y^0 \quad v^0y^1 \quad cv^0y^1 \quad \ldots \quad v^0y^{\pi-1} \quad cv^0y^{\pi-1} \; | \\ v^1y^0 \quad cv^1y^0 \quad v^1y^1 \quad cv^1y^1 \quad \ldots \quad v^1y^{\pi-1} \quad cv^1y^{\pi-1} \; | \; \cdots \; | \\ v^{\omega-1}y^0 \quad cv^{\omega-1}y^0 \quad v^{\omega-1}y^1 \quad cv^{\omega-1}y^1 \quad \ldots \quad v^{\omega-1}y^{\pi-1} \quad cv^{\omega-1}y^{\pi-1} \; | \; \hookleftarrow \end{array}$$

is a robust chaplet for $\mathbb{Z}_n$.

Proof: The checking of differences is straightforward. □

Example 8.1: For $n = 77$, we can take $u = 43$ in Theorem 8.1, along with $v = 71$ and $y = 67$, where $\mathrm{ord}_n(v) = 5$ and $\mathrm{ord}_n(y) = 3$. We thus obtain the following robust chaplet for $\mathbb{Z}_{77}$:

$$\begin{array}{rcccccc|c} \hookrightarrow & 1 & 72 & 67 & 50 & 23 & 39 & | \\ & 71 & 30 & 60 & 8 & 16 & 74 & | \\ & 36 & 51 & 25 & 29 & 58 & 18 & | \\ & 15 & 2 & 4 & 57 & 37 & 46 & | \\ & 64 & 65 & 53 & 43 & 9 & 32 & | \ \hookleftarrow . \end{array}$$

Note 8.1: In the range $5 < n < 300$, specimen parameter sets for robust chaplets obtainable from Theorem 8.1 are as follows:

n	77		93	99	129	147	161		165	201
$\xi(n)$	2		2	2	2	2	2		4	2
ω	3	5	5	3	7	7	3	11	4	11
π	5	3	3	5	3	3	11	3	5	3
u	34	43	32	89	44	50	139	22	89	68
v	67	71	4	34	64	64	116	29	133	82
y	64	67	25	64	79	67	141	116	31	163
c	5	72	20	5	11	2	82	30	59	107

n	207	209		213		217	237	253		297
$\xi(n)$	2	2		2		6	2	2		2
ω	3	5	9	5	7	15	13	5	11	9
π	11	9	5	7	5	3	3	11	5	5
u	116	153	56	143	143	125	80	208	45	188
v	70	115	199	199	37	121	10	70	12	133
y	64	199	20	172	199	211	55	144	185	82
c	2	17	48	5	20	164	50	2	5	59

9 Chaplets, terraces and graph decompositions

We now briefly illustrate how chaplets for $\mathbb{Z}_n$ can be used in the construction of terraces for $\mathbb{Z}_n$, and we mention a fertile link with graph theory.

Let **a** be a **linear** arrangement $(a_1, a_2, \ldots, a_n)$ of **all** the elements (including the zero element) of $\mathbb{Z}_n$, and let **b** be the sequence $(b_1, b_2, \ldots, b_{n-1})$ given by $b_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, n-1)$, modulo n. Then [5] **a** is a *terrace* for $\mathbb{Z}_n$ (in short, a $\mathbb{Z}_n$ terrace) if the sequences **b** and $-$**b** together contain exactly 2 occurrences of each element from $\mathbb{Z}_n \setminus \{0\}$. (A terrace for $\mathbb{Z}_n$ provides a partition of the edges of $2K_n$ into Hamiltonian paths, invariant under the group $\mathbb{Z}_n$ acting regularly.) If $n = 2m + 1$

where m is a positive integer, then a terrace for $\mathbb{Z}_n$ is *narcissistic* [1] if $b_i = b_{n-i}$ for all $i = 1, 2, \ldots, m$. If we have a narcissistic terrace **a** for $\mathbb{Z}_n$ where $a_{m+1} \equiv 0 \pmod{n}$, then $a_i = -a_{n+1-i}$ for all $i = 1, 2, \ldots, m$. Thus a narcissistic terrace for $\mathbb{Z}_n$ is completely specified by its first $m + 1$ elements.

As with chaplets, we write **displayed** terraces without commas and brackets.

Consider first the narcissistic $\mathbb{Z}_{49}$ terrace where the first 25 elements are as follows:

1 30 18 | 15 9 25 | 29 37 32 | 43 16 39 | 8 44 46 | 22 23 4 | 36 2 11 | 21 42 35 | 0 | .

The first 7 segments are obtained by constructing a chaplet for $\mathbb{Z}_{49}$ as in Ex. 3.6(a), save that the value $u = 15$ is used instead of $u = 36$. The cycle is then broken by *not* joining the final element 11 to the initial element 1. Immediately after the element 11 the sequence is instead continued with the element 21, chosen so that the missing chaplet difference $1 - 11 = -10$ is compensated for by the terrace difference $21 - 11 = +10$. The segment | 21 42 35 | is obtained by multiplying the $\mathbb{Z}_7$ chaplet $\hookrightarrow$ 1 2 4 $\hookleftarrow$ throughout by 21 and breaking it at the end. The now missing difference $21 - 35 = -14$ is compensated for by the terrace difference $0 - 35 \equiv +14 \pmod{49}$.

Now consider the narcissistic $\mathbb{Z}_{33}$ terrace where the first 17 elements are as follows:

15 12 3 9 27 | 14 31 5 4 23 25 20 16 26 1 | 22 | 0 .

The first segment is obtained by multiplying the $\mathbb{Z}_{11}$ chaplet $\hookrightarrow$ 1 3 9 5 4 $\hookleftarrow$ by 15. Failure to join the ends of the outcome causes loss of the difference $15 - 27 = -12$, but this is compensated for at the second fence of the terrace, where the difference is $22 - 1 = 21 \equiv -12 \pmod{33}$. The elements of the second segment of the terrace, in reverse order, come straight from the $\mathbb{Z}_{33}$ chaplet in Ex. 4.1. The missing difference $14 - 1 = +13$ is compensated for at the first fence, where the difference is $14 - 27 = -13$, the change in sign being immaterial.

The approach used in the two examples just given can be used much more widely and very productively.

In the vocabulary of graph theory, the construction of the above $\mathbb{Z}_{33}$ terrace shows that, if we take the chaplet $[14, 31, 5, 4, 23, 25, 20, 16, 26, 1]$ simultaneously with the (linear) sequences $(1, 22, 0)$ and $(15, 12, 3, 9, 27)$, we can decompose K_{33} into copies of the disconnected graph $N_{10,3} \cup P_5$ on which $\mathbb{Z}_{33}$ acts regularly, where $N_{i,j}$ is a graph C_i with a graph P_j attached at their common vertex. Many similar, but simpler, decompositions are available, *e.g.*:

• If we take the $\mathbb{Z}_{33}$ chaplet along with the sequence $(3, 6, 12, 24, 15, 30, 19)$ we can decompose K_{33} into copies of $C_{10} \cup P_7$.

• If we take the chaplet along with the sequence $(1, 12, 24, 15, 30, 27, 21)$ we can decompose K_{33} into copies of $N_{10,7}$.

• If we take the chaplet along with the sequence $(5, 29, 17, 11, 22, 19, 4)$ we can decompose K_{33} into copies of a C_{10} with a P_7 attached as a "detour" between two adjacent vertices of the C_{10}.

• If we take the chaplet along with the sequence $(14, 23, 20, 26, 4, 16, 1)$ we can decompose K_{33} into copies of a connected graph with a mere 10 vertices.

Once again, these examples represent wide classes of possibilities.

10 Existence of chaplets

This paper leaves unanswered many questions about the existence of chaplets in general, and of robust chaplets in particular.

Within the range $5 < n < 300$, §3 above provides a robust chaplet for every prime n except $n = 17$, 193 and 257. The first and third of these exceptions invite the question whether there is any prime n satsifying $n - 1 = 2^i$ for which a robust chaplet exists. (Rees's **strong** chaplet for $n = 17$ appears in §1.) Whether a robust chaplet exists for any particular prime n satisfying $n - 1 = 3 \cdot 2^i$ (*e.g.* $n = 193$ — see Note 3.4) seems likely to be much harder to determine.

As chaplets do not exist for $n = 5$ and we have failed to produce a robust chaplet for $n = 17$, §3 fails to provide robust chaplets for any of the values $n = 5^2$, 5^3 and 17^2, all of which lie in the interval $5 < n < 300$. **Strong** chaplets for $n = 17^2 = 289$ can however be obtained by applying Theorem 3.6 to Rees's strong chaplet for $n = 17$, and strong chaplets for $n = 5^3 = 125$ can be obtained by applying Theorem 3.7 to the strong chaplet for $n = 25 = 5^2$ that is given in §1.

Within the range $5 < n < 300$, §§4–7 provide a robust chaplet for every odd composite n. However, possibilities within this range are very restricted, as it contains (a) no n-value with $\xi(n) > 12$, and (b) no n-value having more than three distinct prime factors.

Does a chaplet exist for every odd n with $n > 5$, excluding the powers of 3? Unless some further particular n-value can readily be shown not to have a chaplet, the fact that a composite n can have indefinitely many prime factors may make this question very hard to answer.

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