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Finite Euclidean graphs over $\mathbb{Z}_{2^r}$ are non-Ramanujan

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Abstract

Graphs are attached to $\mathbb{Z}_{2^r}^d$ where $\mathbb{Z}_{2^r}$ is the ring with 2^r elements using an analogue of Euclidean distance. M. R. DeDeo (2003) showed that these graphs are non-Ramanujan for $r \geqslant 4$. In this paper, we will show that finite Euclidean graphs attached to $\mathbb{Z}_{2^r}^d$ are non Ramanujan for $r \geqslant 2$ except for $r = 2$ and $d = 2, 3$. Together with the results in Medrano et al. (1998), this implies that finite Euclidean graphs over $\mathbb{Z}_{p^r}$ for p prime are non-Ramanujan except for the smallest cases.

1 Introduction

Finite Euclidean graphs have been studied by many authors (see [10]). In [8], Medrano, Myers, Stark and Terras studied the spectra of the finite Euclidean graphs over finite fields and showed that these graphs are asymptotically Ramanujan graphs. In [2], Bannai, Shimabukuro and Tanaka showed that these graphs are again always asymptotically Ramanujan for more general setting (i.e. they replace the Euclidean distance by nondegenerate quadratic forms). The author recently applied these results in several interesting combinatorics problems, for example tough Ramsey graphs (with P. Dung) [12], Szemeredi-Trotter type theorem and sum-product estimate [14], the Erdös distance problem [13] and integral graphs (with S. Li) [5].

Medrano et al. [9] study a finite analogue of real symmetric spaces, namely the finite Euclidean space $\mathbb{Z}_{p^r}^d$ over the finite ring $\mathbb{Z}_{p^r}$ for odd prime p using a finite analogue of the usual Euclidean distance. It is shown that, for p an odd prime, Euclidean graphs over $\mathbb{Z}_{p^r}^d$ are non-Ramanujan for $r \geq 2$ unless $p = 3$ when $r = d = 2$ (see Medrano et al. [9], Theorem 2.5). The case $p = 2$ was avoided as this case is unwieldy. DeDeo [4] addresses the question of how graphs attached to $\mathbb{Z}_{2^r}^d$. She

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shows that Euclidean graphs over $\mathbb{Z}_{2^r}^d$ are non-Ramanujan for $r \geq 4$. Although the degree and the spectrum of the Euclidean graphs over $\mathbb{Z}_{p^r}$ can be related to the degree of the Euclidean graphs over $\mathbb{Z}_p$ for odd prime p, the degree and the spectrum of the Euclidean graphs over $\mathbb{Z}_{2^r}$ can only be related to the Euclidean graphs over $\mathbb{Z}_8$ by DeDeo's method. This bars her from proving that the Euclidean graphs over $\mathbb{Z}_{2^r}$ are non-Ramanujan for $r = 2$ and 3. Note that, a part of the motivation to study these graphs is to find finite analogues of real Euclidean space $\mathbb{R}^n$ and analogies to the harmonic analysis on $\mathbb{R}^n$ as discussed in Terras [11]. This has been of interest to physicists for some time in areas such as statistical theory of the energy levels of a complex physical system. The analogous properties, the distribution of eigenvalues and eigenfunctions associated to finite Euclidean graphs, are discussed in Medrano et al. [9], and DeDeo [4].

In this paper, we will show that the Euclidean graphs over $\mathbb{Z}_{2^r}$ for $r = 2$ and 3 are also non-Ramanujan except for the smallest cases, $r = 2$, $d = 2$ and 3. Together with Theorem 2.5 in Medrano et al. [9] and Theorem 11 in DeDeo [4], we have a complete picture of the Ramanujancy of finite Euclidean graphs over $\mathbb{Z}_{p^r}$ (Theorem 11).

The rest of this paper is organized as follows. In Section 2, we summary preliminary facts of finite Euclidean graphs over $\mathbb{Z}_{2^r}$ and study the spectrum of these graphs. We show that eigenvalues of the finite Euclidean graphs over $\mathbb{Z}_{2^r}$ also can be related to Kloosterman sums and Gauss sums (Theorem 7) as in the odd case (see Theorem 2.9 in [9]). We then prove that the finite Euclidean graphs over rings are non-Ramanujan except the smallest cases in Section 3.

2 Spectrum of finite Euclidean graphs

Let $\mathbb{Z}_{2^r}$ be the ring $\mathbb{Z}/2^r\mathbb{Z}$, $r \geqslant 2$. We identify $\mathbb{Z}_{2^r}$ with $\{0, 1, \ldots, 2^r - 1\}$ and regard $\mathbb{Z}_{2^{r-1}}$ as a subset of $\mathbb{Z}_{2^r}$. The finite Euclidean space $\mathbb{Z}_{2^r}^d$ consists of column vectors x with jth entry x_j in $\mathbb{Z}_{2^r}$. Define the distance between x and y in $\mathbb{Z}_{2^r}^d$ by

$$\Delta(x, y) = {}^t(x - y).(x - y) = \sum_{i=1}^{d} (x_i - y_i)^2. \tag{2.1}$$

This distance has values in $\mathbb{Z}_{2^r}$ and it is point-pair invariant.

Given a in $\mathbb{Z}_{2^r}$ define the finite Euclidean graph $E(2^r, d, a)$ with vertices the vectors in $\mathbb{Z}_{2^r}^d$ and two vectors being adjacent if and only if $\Delta(x, y) = a$.

Let Γ be an additive group. For $S \subseteq \Gamma$, $0 \notin S$ and $S^{-1} = \{-s : s \in S\} = S$, the Cayley graph $G = C(\Gamma, S)$ for additive group Γ and edge set S is the undirected graph having vertices the elements of Γ and edges between x and $y = x + s$ for $x, y \in \Gamma$, $s \in S$. The Cayley graph $G = C(\Gamma, S)$ is regular of degree $|S|$.

Let

$$S_{2^r,d}(a) = \{x \in \mathbb{Z}_{2^r}^d \mid \Delta(x, 0) = a\}. \tag{2.2}$$

Then the finite Euclidean graph $E(2^r, d, a)$ is a Cayley graph $C(\mathbb{Z}_{2^r}^d, S_{2^r,d}(a))$.

Recall that the eigenvalues of Cayley graphs of abelian groups can be computed easily in terms of the characters of the group. This result, described in, for example, [6], implies that the eigenvalues of the finite Euclidean graph $E(2^r, d, a)$ are all the numbers

$$\lambda_a^{(r)}(b) = \sum_{x \in S_{d,2^r}(a)} e_{2^r}({}^t b.x), \tag{2.3}$$

where $b \in \mathbb{Z}_{2^r}^d$ and the exponential $e_n(x) = \exp\{2\pi i x/n\}$. In this section, we will study these eigenvalues $\lambda_a^{(r)}(b)$'s.

We define the following conventions which will be used throughout the paper.

- The notation $\delta(\mathcal{P})$ where $\mathcal{P}$ is some property, means that it's 1 if $\mathcal{P}$ is satisfied, 0 otherwise.
- For any $k \in \mathbb{Z}_{2^r}^*$. We define $I_{2^r}(k)$ the unique element $I_{2^r}(k) \in \mathbb{Z}_{2^r}^*$ satisfying
$$k I_{2^r}(k) \equiv 1 \mod 2^r.$$
- For any integer k and $x = (x_1, \ldots, x_d)^t \in \mathbb{Z}_{2^r}^d$, $k \mid x$ if and only if $k \mid x_i$ for all i in $\{1, \ldots, d\}$.
- For any integer k and $x = (x_1, \ldots, x_d)^t \in \mathbb{Z}_{2^r}^d$, $k \nmid x$ if and only if $k \nmid x_i$ for some i in $\{1, \ldots, d\}$.

Before beginning the discussion, we need to consider the Gauss sums over rings. For $c \in \mathbb{Z}_n^*$ define the Gauss sum

$$G_n(c) = \sum_{k \in \mathbb{Z}_n} e_n(ck^2). \tag{2.4}$$

If $(c, n) = 1$ and $n \equiv 2 \mod 4$, then (cf. Theorem 1.5.1 in [3])

$$G_n(c) = 0. \tag{2.5}$$

If $(c, n) = 1$ and $n \equiv 0 \mod 4$, then (cf. Theorem 1.5.4 in [3])

$$G_n(c) = \left(\frac{n}{c}\right)(1 + i^c)\sqrt{n}, \tag{2.6}$$

here the Jacobi symbol $\left(\frac{n}{c}\right)$ is defined as follows:

- If $c = p$ is prime then
$$\left(\frac{n}{p}\right) = \begin{cases} 1 & p \nmid n, \quad n \text{ is square mod } p \\ -1 & p \nmid n, \quad n \text{ is nonsquare mod } p \\ 0 & p \mid n. \end{cases}$$
- If $c = p_1^{r_1} \ldots p_t^{r_t}$ is the prime decomposition of c then
$$\left(\frac{n}{c}\right) = \left(\frac{n}{p_1}\right)^{r_1} \cdots \left(\frac{n}{p_t}\right)^{r_t}.$$

For more information about the Gauss sums, we refer the reader to Section 1.5 in [3].

For any $a \in \mathbb{Z}_{2^r}$ and $b \in \mathbb{Z}_{2^r}^d$ we define

$$f_{2^r}(a,b) = \sum_{v \in \mathbb{Z}_{2^r}} e_{2^r}(-va) \sum_{x \in \mathbb{Z}_{2^r}^d} e_{2^r}({}^t b.x + v {}^t x.x), \tag{2.7}$$

and

$$g_{2^r}(a,b) = \sum_{v \in \mathbb{Z}_{2^r}^*} e_{2^r}(-va) \sum_{x \in \mathbb{Z}_{2^r}^d} e_{2^r}({}^t b.x + v {}^t x.x) \tag{2.8}$$

If $r = 1$ then $f_2(a,b)$ is easy to compute.

Lemma 1 *For any $a \in \mathbb{Z}_2^*$ and $b \in \mathbb{Z}_2^d$ then*

$$f_2(a,b) = \begin{cases} 2^d & b = (0,\ldots,0) \\ -2^d & b = (1,\ldots,1) \\ 0 & \textit{otherwise.} \end{cases} \tag{2.9}$$

Proof We have

$$\begin{aligned} f_2(a,b) &= \sum_{x \in \mathbb{Z}_2^d} e_2({}^t b.x) + e_2(a) \sum_{x \in \mathbb{Z}_2^d} e_2({}^t b.x +^t x.x) \\ &= \prod_{i=1}^d (e_2(0) + e_2(b_i)) - \prod_{i=1}^d (e_2(0) + e_2(b_i + 1)) \\ &= 2^d \prod_{i=1}^d \delta(b_i = 0) - 2^d \prod_{i=1}^d \delta(b_i = 1) \\ &= \begin{cases} 2^d & b = (0,\ldots,0) \\ -2^d & b = (1,\ldots,1) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This conclude the proof of the lemma. □

The following lemmas relate $\lambda_a^{(r)}(b)$ with $f_{2^r}(a,b)$ and $g_{2^r}(a,b)$.

Lemma 2 *For any $a \in \mathbb{Z}_{2^r}$ and $b \in \mathbb{Z}_{2^r}^d$ then*

$$\lambda_a^{(r)}(b) = \frac{1}{2^r} f_{2^r}(a,b). \tag{2.10}$$

Proof We have

$$\begin{aligned} \lambda_a^{(r)}(b) &= \sum_{x \in S_{d,2^r}(a)} e_{2^r}({}^t b.x) \\ &= \sum_{x \in \mathbb{Z}_{2^r}^d} \frac{1}{2^r} \sum_{v \in \mathbb{Z}_{2^r}} e_{2^r}({}^t b.x + v({}^t x.x - a)) \\ &= \frac{1}{2^r} \sum_{v \in \mathbb{Z}_{2^r}} e_{2^r}(-va) \sum_{x \in \mathbb{Z}_{2^r}^d} e_{2^r}({}^t b.x + v {}^t x.x) \\ &= \frac{1}{2^r} f_{2^r}(a,b), \end{aligned}$$

completing the lemma. □

Lemma 3 *For any $a \in \mathbb{Z}_{2^r}$ and $b \in \mathbb{Z}_{2^r}^d$ then*

$$f_{2^r}(a,b) = g_{2^r}(a,b) + \delta(2|b)2^d f_{2^{r-1}}(a,c), \tag{2.11}$$

where $c \in \mathbb{Z}_{2^{r-1}}^d$ satisfies $b = 2c$ given that $2|b$.

Proof We have

$$f_{2^r}(a,b) = g_{2^r}(a,b) + \sum_{w \in \mathbb{Z}_{2^{r-1}}, x \in \mathbb{Z}_{2^r}^d} e_{2^{r-1}}(-wa)e_{2^r}({}^t b.x + 2w {}^t x.x). \tag{2.12}$$

We can write $x = 2^{r-1}x^1 + x^2$ uniquely where $x^1 \in \mathbb{Z}_2^d$ and $x^2 \in \mathbb{Z}_{2^{r-1}}^d$. Substitute into (2.12), we have

$$\begin{aligned}
f_{2^r}(a,b) &= g_{2^r}(a,b) + \sum_{w \in \mathbb{Z}_{2^{r-1}}, x^1 \in \mathbb{Z}_2^d, x^2 \in \mathbb{Z}_{2^{r-1}}^d} e_{2^{r-1}}(-wa)e_2({}^t b.x^1)e_{2^r}({}^t b.x^2 + 2w {}^t x^2.x^2) \\
&= g_{2^r}(a,b) + \left(\sum_{x^1 \in \mathbb{Z}_2^d} e_2({}^t b.x^1) \right) \left(\sum_{\substack{w \in \mathbb{Z}_{2^{r-1}} \\ x^2 \in \mathbb{Z}_{2^{r-1}}^d}} e_{2^{r-1}}(-wa)e_{2^r}({}^t b.x^2 + 2w {}^t x^2.x^2) \right) \\
&= g_{2^r}(a,b) + \delta(2|b)2^d \sum_{w \in \mathbb{Z}_{2^{r-1}}, x^2 \in \mathbb{Z}_{2^{r-1}}^d} e_{2^{r-1}}(-wa)e_{2^{r-1}}({}^t c.x^2 + w {}^t x^2.x^2) \\
&= g_{2^r}(a,b) + \delta(2|b)2^d f_{2^{r-1}}(a,c).
\end{aligned}$$

This concludes the proof of the lemma. □

From Lemma 1 and Lemma 2, we have the following formula for the spectrum of $E(2,d,a)$.

Theorem 4 *For any a odd and $b \in \mathbb{Z}_2^d$ then*

$$\lambda_a^{(1)}(b) = \begin{cases} 2^{d-1} & b = (0,\ldots,0) \\ -2^{d-1} & b = (1,\ldots,1) \\ 0 & \textit{otherwise.} \end{cases} \tag{2.13}$$

Lemma 5 *For any $v, b \in \mathbb{Z}_{2^r}^*$ and $r \geq 2$ then*

$$\sum_{x \in \mathbb{Z}_{2^r}} e_{2^r}(bx + vx^2) = 0. \tag{2.14}$$

Proof We have

$$\begin{aligned}\sum_{x\in\mathbb{Z}_{2^r}} e_{2^r}(bx+vx^2) &= \sum_{x\in\mathbb{Z}_{2^r}} e_{2^{r+2}}(v(2x+bI_{2^{r+2}}(v))^2 - I_{2^{r+2}}(v)b^2)\\ &= e_{2^{r+2}}(-I_{2^{r+2}}(v)b^2)\sum_{x\in\mathbb{Z}_{2^r}} e_{2^{r+2}}(v(2x+bI_{2^{r+2}}(v))^2)\\ &= \frac{1}{2}e_{2^{r+2}}(-I_{2^{r+2}}(v)b^2)\sum_{x\in\mathbb{Z}_{2^{r+1}}} e_{2^{r+2}}(v(2x+bI_{2^{r+2}}(v))^2),\end{aligned}$$

since $(2x+bI_{2^{r+2}}(v))^2 \equiv (2(x+2^r)+bI_{2^{r+2}}(v))^2 \bmod 2^{r+2}$. Let $a = bI_{2^{r+2}}(v)$ then a is odd as b and v are so. Thus, we have

$$\begin{aligned}\sum_{x\in\mathbb{Z}_{2^{r+1}}} e_{2^{r+2}}(v(2x+a)^2) &= \sum_{x\in\mathbb{Z}_{2^{r+2}}} e_{2^{r+2}}(vx^2) - \sum_{x\in 2\mathbb{Z}_{2^{r+1}}} e_{2^{r+2}}(vx^2)\\ &= G_{2^{r+2}}(v) - \sum_{x\in\mathbb{Z}_{2^{r+1}}} e_{2^r}(vx^2)\\ &= G_{2^{r+2}}(v) - 2G_{2^r}(v) = 0,\end{aligned}$$

where the last line follows from (2.6). This completes the proof of the lemma.

□

From Lemma 5, we have the following formula for $g_{2^r}(a,b)$.

Lemma 6 *For any $a \in \mathbb{Z}_{2^r}$ and $b \in \mathbb{Z}_{2^r}^d$ then*

$$g_{2^r}(a,b) = \begin{cases} 0 & 2 \nmid b\\ \sum_{v\in Z_{2^r}^*} e_{2^r}(-va - I_{2^r}(v)^t c.c)(G_{2^r}(v))^d & 2 \mid b, \end{cases} \tag{2.15}$$

where $c \in \mathbb{Z}_{2^{r-1}}^d$ satisfies $b = 2c$ given that $2 \mid b$.

Proof We have

$$g_{2^r}(a,b) = \sum_{v\in\mathbb{Z}_{2^r}^*} e_{2^r}(-va)\prod_{i=1}^{d}\left(\sum_{x\in\mathbb{Z}_{2^r}} e_{2^r}(b_i x + vx^2)\right).$$

From Lemma 5, if $2 \nmid b$ then $g_{2^r}(a,b) = 0$. Now, we suppose that $2|b$. We write $b = 2c$ for some $c \in \mathbb{Z}_{2^{r-1}}^d \subset \mathbb{Z}_{2^r}^d$. Then

$$\begin{aligned}g_{2^r}(a,b) &= \sum_{v\in Z_{2^r}^*} e_{2^r}(-va)\sum_{x\in\mathbb{Z}_{2^r}^d} e_{2^r}(2^t c.x + v^t x.x)\\ &= \sum_{v\in\mathbb{Z}_{2^r}^*} e_{2^r}(-va)\sum_{x\in\mathbb{Z}_{2^r}^d} e_{2^r}(v^t(x+I_{2^r}(v)c).(x+I_{2^r}(v)c) - I_{2^r}(v)^t c.c)\\ &= \sum_{v\in\mathbb{Z}_{2^r}^*} e_{2^r}(-va - I_{2^r}(v)^t c.c)\sum_{x\in\mathbb{Z}_{2^r}^d} e_{2^r}(v^t x.x)\\ &= \sum_{v\in Z_{2^r}^*} e_{2^r}(-va - I_{2^r}(v)^t c.c)(G_{2^r}(v))^d.\end{aligned}$$

This conclues the proof of the lemma. □

Putting Lemmas 2, 3 and 6 together, we derive the following simple formula for eigenvalues $\lambda_a^{(r)}(b)$'s of $E(2^r, d, a)$ in the spirit of formula (11) of Medrano et al. [8] and Theorem 2.9 of Medrano et al. [9].

Theorem 7 *Let $a \in \mathbb{Z}_{2^r}$, $b \in \mathbb{Z}_{2^r}^d$ and $r, d \geq 2$. We have two cases.*

1. *Suppose that $2 \nmid b$ then $\lambda_a^{(r)}(b) = 0$.*
2. *Suppose that $2 \mid b$. Let $c \in \mathbb{Z}_{2^{r-1}}^d$ satisfies $b = 2c$, then*

$$\lambda_a^{(r)}(b) = \frac{1}{2^r} \sum_{v \in \mathbb{Z}_{2^r}^*} e_{2^r}(-va - I_{2^r}(v)^t c.c)(G_{2^r}(v))^d + 2^{d-1}\lambda_a^{(r-1)}(c). \qquad (2.16)$$

We also remark that most of results in DeDeo [4] can be derived from Theorem 7 and the properties of Kloosterman sums over ring $\mathbb{Z}_{2^r}$.

3 Ramanujancy of finite Euclidean graphs

Now we will study the Ramanujancy of finite Euclidean graphs $E(4, d, a)$ and $E(8, d, a)$. We are only interested in the case a odd as these graphs are disconnected for even a.

Theorem 8 *Let a be an odd integer and $d \geq 4$ then the Euclidean graph $E(4, d, a)$ is not Ramanujan.*

Proof Note that $G_4(1) = 2+2i$, $G_4(3) = 2-2i$ and if $v \in \mathbb{Z}_4^*$ then $I_4(v) = v$. Thus, equation (2.16) becomes

$$\lambda_a^{(2)}(2c) = \frac{1}{4} \sum_{v \in \mathbb{Z}_4^*} e_4(-v(a +^t c.c))(G_4(v))^d + 2^{d-1}\lambda_a^{(1)}(c). \qquad (3.1)$$

Substitute $b = (0, \ldots, 0)^t$ into (2.16), we obtain the degree of $E(4, d, a)$

$$\begin{aligned} |S_{4,d}(a)| = \lambda_a^{(2)}(0) &= \frac{1}{4}\left\{e_4(-a)(G_4(1))^d + e_4(-3a)(G_4(3))^d\right\} + 2^{d-1}\lambda_a^{(1)}(0) \\ &= 2^{d-2}(e_4(-a)(1+i)^d + e_4(-3a)(1-i)^d) + 2^{2d-2}, \end{aligned} \qquad (3.2)$$

where the last line follows from Theorem 4. We have four cases.

Case 1. Suppose that $d = 4k$ for some $k \geq 1$. Then equation (3.2) becomes

$$|S_{4,4k}(a)| = 2^{8k-2}. \qquad (3.3)$$

We choose $c = (1, 1, 1, 0, \ldots, 0)^t$ or $c = (1, 0 \ldots, 0)^t$ for $a \equiv 1$ or $3 \mod 4$, respectively. From Theorem 4, $\lambda_a^{(1)}(c) = 0$. Thus, equation (3.1) becomes

$$\lambda_a^{(2)}(2c) = \frac{1}{4}((2+2i)^d + (2-2i)^d) = (-1)^k 2^{6k-1}. \qquad (3.4)$$

But $|\lambda_a^{(2)}(2c)| > 2\sqrt{|S_{4,4k}(a)|}$, so $E(4,4k,a)$ is not Ramanujan.

Case 2. Suppose that $d = 4k+2$ for some $k \geq 1$. Then equation (3.2) becomes

$$|S_{4,4k+2}(a)| = \begin{cases} 2^{8k+2} + (-1)^k 2^{6k+2} & a \equiv 1 \mod 4 \\ 2^{8k+2} - (-1)^k 2^{6k+2} & a \equiv 3 \mod 4. \end{cases} \tag{3.5}$$

We choose $c = (1,1,0,\ldots,0)^t$ then equation (3.1) becomes

$$\lambda_a^{(2)}(2c) = \frac{1}{4}(\pm i(2+2i)^d \mp i(2-2i)^d) = \pm(-1)^k 2^{6k+2}. \tag{3.6}$$

So $E(4,4k+2,a)$ is not Ramanujan.

Case 3. Suppose that $d = 4k+1$ for some $k \geq 1$. Then equation (3.2) becomes

$$|S_{4,4k+1}(a)| = \begin{cases} 2^{8k} + (-1)^k 2^{6k} & a \equiv 1 \mod 4 \\ 2^{8k} - (-1)^k 2^{6k} & a \equiv 3 \mod 4. \end{cases} \tag{3.7}$$

We choose $c = (1,1,1,0,\ldots,0)^t$ or $c = (1,0\ldots,0)^t$ for $a \equiv 1$ or $3 \mod 4$, respectively then equation (3.1) becomes

$$\lambda_a^{(2)}(2c) = \frac{1}{4}((2+2i)^d + (2-2i)^d) = (-1)^k 2^{6k}. \tag{3.8}$$

So $E(4,4k+1,a)$ is not Ramanujan.

Case 4. Suppose that $d = 4k+3$ for some $k \geq 1$. Then equation (3.2) becomes

$$|S_{4,4k+3}(a)| = \begin{cases} 2^{8k+4} + (-1)^k 2^{6k+3} & a \equiv 1 \mod 4 \\ 2^{8k+4} - (-1)^k 2^{6k+3} & a \equiv 3 \mod 4. \end{cases} \tag{3.9}$$

We choose $c = (1,1,0,\ldots,0)^t$ then equation (3.1) becomes

$$\lambda_a^{(2)}(2c) = \frac{1}{4}(\pm i(2+2i)^d \mp i(2-2i)^d) = \pm(-1)^k 2^{6k+3}. \tag{3.10}$$

So $E(4,4k+3,a)$ is not Ramanujan.

This completes the proof of the theorem. □

Remark 9 *For the smallest cases, $d = 2$ and 3, finite Euclidean graphs $E(4,2,a)$ and $E(4,3,a)$ are Ramanujan. This can be checked easily by computer (see [4], p. 54).*

Theorem 10 *Let a be an odd integer and $d \geq 2$ then the Euclidean graph $E(8,d,a)$ is not Ramanujan.*

Proof Note that $G_8(1) = (1+i)2^{3/2}$, $G_8(3) = -(1-i)2^{3/2}$, $G_8(5) = -(1+i)2^{3/2}$, $G_8(7) = (1-i)2^{3/2}$ and $I_8(v) = v$ if $v \in \mathbb{Z}_8^*$. Thus, equation (2.16) becomes

$$\lambda_a^{(3)}(2c) = \frac{1}{8}\sum_{v \in \mathbb{Z}_8^*} e_8(-v(a +^t c.c))(G_8(v))^d + 2^{d-1}\lambda_a^{(2)}(c). \tag{3.11}$$

Substitute $b = (0, \ldots, 0)^t$ into (3.11), we obtain the degree of $E(8, d, a)$

$$|S_{8,d}(a)| = \lambda_a^{(2)}(0) = \frac{1}{8}\sum_{v\in\mathbb{Z}_8^*} e_8(-va)(G_8(v))^d + 2^{d-1}|S_{4,d}(a)|. \tag{3.12}$$

We have four cases.

Case 1. Suppose that $d = 4k$ for some $k \geq 1$. Substitute (3.3) into (3.12), we have

$$|S_{8,4k}(a)| = 2^{d-1}|S_{4,d}(a)| = 2^{12k-3}, \tag{3.13}$$

where we use the fact that $\sum_{v\in\mathbb{Z}_8^*} e_8(-va) = 0$ if a is odd. We choose $c = (1, 1, 1, 0, \ldots, 0)^t$ or $c = (1, 0 \ldots, 0)^t$ for $a \equiv 1$ or $3 \mod 4$, respectively (to make $a +^t c.c \equiv 0 \mod 4$). From Theorem 7, $\lambda_a^{(2)}(c) = 0$. Thus, equation (3.11) becomes

$$\lambda_a^{(3)}(2c) = \pm\frac{1}{8}\sum_{v\in\mathbb{Z}_8^*}(G_8(v))^d = \pm 2^{8k-1}. \tag{3.14}$$

But $|\lambda_a^{(3)}(2c)| > 2\sqrt{|S_{8,4k}(a)|}$, so $E(8, 4k, a)$ is not Ramanujan.

Case 2. Suppose that $d = 4k + 2$ for some $k \geq 0$. Substitute (3.5) into (3.12), we have

$$\begin{aligned} |S_{8,4k+2}(a)| &= 2^{12k+3} \pm 2^{10k+3} \pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-va)(G_8(v))^2 \\ &= 2^{12k+3} \pm 2^{10k+3}, \end{aligned} \tag{3.15}$$

where the last line follows from $e_8(-va) = -e_8(-(v+4)a)$ and $G_8(v) = -G_8(v+4)$. If $d = 2$ then from (3.15), we only need to consider the case $a \equiv 1 \mod 4$ (otherwise, $|S_{8,2}(a)| = 0$ and the graph is isolated points). So we can choose $c = (1, 0 \ldots, 0)^t$ or $c = (1, 1, 1, 0, \ldots, 0)^t$ for $a \equiv 1$ or $3 \mod 4$, respectively (to make $a + {}^t c.c \equiv 2 \mod 4$). Equation (3.11) becomes

$$\begin{aligned} \lambda_a^{(3)}(2c) &= \pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-v(4u+2))(G_8(v))^2 \\ &= \pm 2^{8k+1}(i(1+i)^2 - i(1-i)^2) = \pm 2^{8k+3}. \end{aligned}$$

So $E(8, 4k+2, a)$ is not Ramanujan.

Case 3. Suppose that $d = 4k + 1$ for some $k \geq 1$. Substitute (3.7) into (3.12), we have

$$\begin{aligned} |S_{8,4k+1}(a)| &= 2^{12k} \pm 2^{10k} \pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-va)G_8(v) \\ &= 2^{12k} \pm 2^{10k} \pm \begin{cases} 2^{8k+1} & a \equiv 1 \mod 4 \\ 0 & a \equiv 3 \mod 4. \end{cases} \end{aligned} \tag{3.16}$$

We choose $c = (1, 1, 1, 1, 0 \ldots, 0)^t$ or $c = (1, 1, 0, \ldots, 0)^t$ for $a \equiv 1$ or $3 \mod 4$, respectively, to make $a + {}^t c.c \equiv 1 \mod 4$. Equation (3.11) becomes

$$\begin{aligned} \lambda_a^{(3)}(2c) &= \pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-v(4u+1))G_8(v) \\ &= \pm 2^{8k+1}. \end{aligned}$$

So $E(8,4k+1,a)$ is not Ramanujan.

Case 4. Suppose that $d=4k+3$ for some $k\geq 0$. Substitute (3.9) into (3.12), we have

$$\begin{aligned}|S_{8,4k+3}(a)| &= 2^{12k+6}\pm 2^{10k+5}\pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-va)(G_8(v))^3\\ &= 2^{12k+6}\pm 2^{10k+5}\pm\begin{cases}0 & a\equiv 1 \mod 4\\ 2^{8k+5} & a\equiv 3 \mod 4.\end{cases}\end{aligned} \tag{3.17}$$

If $k=0$ and $a\equiv 3 \mod 4$ then DeDeo [4] showed that the graph is disconnected and non-Ramanujan. In other cases, we choose $c=(1,1,0,\ldots,0)^t$ or $c=(1,1,1,1,0\ldots,0)^t$ for $a\equiv 1$ or $3 \mod 4$, respectively, to make $a+{}^tc.c\equiv 3 \mod 4$. Equation (3.11) becomes

$$\begin{aligned}\lambda_a^{(3)}(2c) &= \pm 2^{8k-3}\sum_{v\in\mathbb{Z}_8^*} e_8(-v(4u+3))(G_8(v))^3\\ &= \pm 2^{8k+5}.\end{aligned}$$

So $E(8,4k+3,a)$ is not Ramanujan.

This completes the proof of the theorem. □

Putting Theorem 2.5 in [9], Theorem 11 in [4], Theorem 8 and Theorem 10 in this paper together, we have the following theorem.

Theorem 11 *Let $r,d\geq 2$, $a\in\mathbb{Z}$ and p is prime such that $p\nmid a$, then the Euclidean graph $E(p^r,d,a)$ is not Ramanujan except the smallest cases: $p=2,r=2$, $d=2$ and 3, and $p=3,r=d=2$.*

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