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The transformation graph G^{++-}

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Abstract

The transformation graph G^{++-} of G is the graph with vertex set $V(G) \cup E(G)$ in which the vertices u and v are joined by an edge if one of the following conditions holds: (i) $u, v \in V(G)$ and they are adjacent in G, (ii) $u, v \in E(G)$ and they are adjacent in G, (iii) one of u and v is in $V(G)$ while the other is in $E(G)$, and they are not incident in G. In this paper, for a graph G, we determine the independence number of G^{++-} and give a lower bound for the connectivity of G^{++-}. Furthermore, we provide some simple sufficient conditions for G^{++-} to be hamiltonian.

1 Introduction

All graphs considered here are finite, undirected and simple. We refer to [2] for unexplained terminology and notation. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. For two vertices u and v of G, if there is an edge e joining them, we say u and v are *adjacent*. In this case, both u and v are end vertices of e, and u (or v) and e are said to be *incident*. Two edges e and f are also said to be adjacent if they have an end vertex in common.

For a vertex v of G, if there is no confusion, the degree $d_G(v)$ is simply denoted by $d(v)$. The symbols $\Delta(G), \delta(G), \kappa(G)$, $\alpha(G)$, $M(G)$ and $\omega(G)$ denote the maximum degree, the minimum degree, the connectivity, the independence number, the cardinality of a maximum matching and the number of components of G, respectively.

As usual, K_n is the complete graph of order n. For two positive integers r and s, $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices. In particular, $K_{1,s}$ is called a star. For $s \geqslant 2$, $K_{1,s}+e$ is the graph obtained from $K_{1,s}$ by adding a new edge which joins two vertices of degrees one. We say two graphs G and H are disjoint if they have no vertex in common, and denote their union by $G + H$; it is called the disjoint union of G and H. The disjoint union of k copies of G is written as kG.

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The line graph $L(G)$ of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G. The *total graph* $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices are adjacent if and only if they are adjacent or incident in G. The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex set as G, but where two vertices are adjacent if and only if they are not adjacent in G. For simplicity, if a graph G is isomorphic to H, we write $G \cong H$, and if it is not, $G \ncong H$. For a graph G and a set A of graphs, we denote by $G \in A$ the fact that G is isomorphic to a graph in A, and $G \notin A$, otherwise.

Wu and Meng [8] generalized the concept of total graphs to a total transformation graph G^{xyz} with $x, y, z \in \{-, +\}$, where G^{+++} is precisely the total graph of G, and G^{---} is the complement of G^{+++}. Each of these eight kinds of transformation graph G^{xyz} appears to have some nice properties; for instance, their diameters are small in most cases [8], and their edge connectivities are equal to their minimum degree etc. [3, 12].

Fleischner and Hobbs [5] showed that G^{+++} is hamiltonian if and only if G contains an EPS-subgraph. Ma and Wu [7] showed that for a graph G of order $n \geqslant 6$, G^{---} is hamiltonian if and only if $G \notin \{K_{1,n-1}, K_{1,n-1}+e, K_{1,n-2}+K_1\}$. Wu, Zhang and Zhang [10] proved that for any graph G of order n, G^{-++} is hamiltonian if and only if $n \geqslant 3$. For the hamiltonicity of line graphs and their complements, see [6] and [9].

We shall investigate the transformation graph G^{++-} of a graph G. G^{++-} is the graph with $V(G^{++-}) = V(G) \cup E(G)$, in which two vertices u and v are joined by an edge in G^{++-} if one of the following conditions holds: (i) $u, v \in V(G)$ and they are adjacent in G, (ii) $u, v \in E(G)$ and they are adjacent in G, (iii) one of u and v is in $V(G)$ while the other is in $E(G)$, and they are not incident in G.

In this note, for a graph G, we determine the independence number of G^{++-} and give a lower bound for the connectivity of G^{++-}. Furthermore, we provide a simple sufficient condition for G^{++-} to be hamiltonian.

2 Main results

We start with some simple observations. Let G be a graph of order n and size m. Then the order of G^{++-} is $n+m$, $d_{G^{++-}}(x) = m$ for any $x \in V(G)$ and $d_{G^{++-}}(e) = n - 4 + d(u) + d(v)$ for any $e = uv \in E(G)$. So

$$\delta(G^{++-}) = \min\{m, n - 4 + \min_{uv \in E(G)} \{d(u) + d(v)\}\}.$$

Theorem 2.1. *For any graph G,*

$$\alpha(G^{++-}) = \begin{cases} 2 & \text{if } G \cong K_2 \quad \text{or} \quad K_3, \\ \max\{\alpha(G), M(G)\} & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that if $G \in \{K_1, K_2, K_3\}$, the result holds. So we treat the remaining case. It is clear that $\alpha(G^{++-}) \geqslant \max\{\alpha(G), M(G)\}$.

To complete the proof, we will show that $\alpha(G^{++-}) \leqslant \max\{\alpha(G), M(G)\}$. Let S be a maximum independent set of G^{++-} and $S = S_1 \cup S_2$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$. Let us consider three cases.

Case 1. $|S_1| \geqslant 2$.

We show that $S_2 = \emptyset$. Otherwise, we can take a vertex $e \in S_2$. Then each vertex of S_1 is incident with e in G, which implies $|S_1| \leqslant 2$. Thus together with the assumption $|S_1| \geqslant 2$, we have $|S_1| = 2$. Namely, the two elements of S_1 are exactly the two end vertices of e in G. But, since S is an independent set of G^{++-}, they are not adjacent in G^{++-}, and so in G, a contradiction. Thus $|S| = |S_1| \leqslant \alpha(G) \leqslant \max\{\alpha(G), M(G)\}$.

Case 2. $|S_2| \geqslant 2$.

We show that $S_1 = \emptyset$. Otherwise, we can take a vertex u, say, from S_1. Then all elements of S_2 are edges incident with u in G. But, on the other hand, S_2 are a matching of G, and thus the elements of S_2 are not pairwise adjacent in G, a contradiction. Thus $|S| = |S_2| \leqslant M(G) \leqslant \max\{\alpha(G), M(G)\}$.

Case 3. $\max\{|S_1|, |S_2|\} \leqslant 1$.

Then $|S| = |S_1| + |S_2| \leqslant 2$. It remains to show that $\max\{\alpha(G), M(G)\} \geqslant 2$. If $\alpha(G) \geqslant 2$, we are done, and otherwise $\alpha(G) = 1$ then G is a complete graph of order at least 4, and thus $M(G) \geq 2$. Thus $\max\{\alpha(G), M(G)\} \geqslant 2$.

The proof is complete. □

Wu and Meng [8] proved that G^{++-} is connected if and only if $G \ncong 2K_2$ and G has at least two edges, and furthermore, that $\mathrm{diam}(G^{++-}) \leqslant 4$ when G^{++-} is connected.

Theorem 2.2. *For a graph G of order $n \geqslant 6$ and size $m \geqslant 3$, $\kappa(G^{++-}) \geqslant \min\{m-1, n+\kappa(L(G))-1\}$.*

Proof. Let S be a minimum cut of G^{++-} with $|S| < \delta(G^{++-})$. Thus each component of $G^{++-} - S$ has at least two vertices. We say that a component H of $G^{++-} - S$ is of type-1 (respectively, type-2, or type-3) if $V(H) \subseteq V(G)$ (respectively, $V(H) \subseteq E(G)$, or $V(H) \cap V(G) \neq \emptyset$ and $V(H) \cap E(G) \neq \emptyset$).

Claim 1. Components of type-1 and type-2 do not appear in $G^{++-} - S$ at the same time.

Proof of Claim 1. If it is not true, we can take two vertices x and y from a component of type-1 and two vertices e and e' from a component of type-2. By the definition of G^{++-}, both e and e' must be incident with both x and y in G. Therefore, e and e' are parallel edges in G, which contradicts the fact that G is a simple graph. □

Claim 2. All components cannot be of type-1.

Proof of Claim 2. If all components of $G^{++-} - S$ are of type-1 then $E(G) \subseteq S$ and thus $|S| \geq m$, which contradicts $|S| < \delta(G^{++-}) \leqslant m$. □

Claim 3. If $G^{++-} - S$ contain a component of type-1 then $|S| = m-1 = \delta(G^{++-}) - 1$.

Proof of Claim 3. By Claims 1 and 2, $G^{++-} - S$ must contain a component of type-3. First we show $\omega(G^{++-} - S) = 2$. By contradiction, suppose $\omega(G^{++-} - S) > 2$. We take a vertex $e \in V(G^{++-} - S) \cap E(G)$, from a component of type-3 and two vertices $u_1, u_2 \in V(G^{++-} - S) \cap V(G)$ from two other different components. Then by the definition of G^{++-}, $e = u_1u_2$ while u_1 and u_2 are not adjacent in G since u_1 and u_2 are not adjacent in G^{++-}. So G consists of exactly two components, one of which is H_1, say, of type-1 and the other is H_3, say, of type-3. By the same argument as in the proof of Claim 1, one can deduce that $|V(H_3) \cap E(G)| = 1$. Let $V(H_3) \cap E(G) = \{e\}$. Again by the definition of G^{++-} and the fact that H_1 has order at least two, $V(H_1) = \{u, v\}$, where $uv = e$. Thus $H_1 \cong K_2$. It follows that $|S| \geqslant \max\{m - 1 + \kappa(G), m - 1 + d(u) - 1 + d(v) - 1\} \geqslant m - 1$. Moreover, since $|S| < \delta(G^{++-}) \leqslant m$, we have $|S| = m - 1$ and $\delta(G^{++-}) = m$. This proves the claim. □

Claim 4. If $G^{++-} - S$ has a component of type-2 then $|S| \geqslant n - 1 + \kappa(L(G))$.

Proof of Claim 4. By Claim 1, $G^{++-} - S$ does not contain any component of type-1. If all components of $G^{++-} - S$ are of type-2 then $V(G) \subseteq S$ and $L(G) - S$ is not connected, thus $|S| \geqslant n + \kappa(L(G))$. So assume that $G^{++-} - S$ contains a component of type-3. Next we see that $\omega(G^{++-} - S) = 2$. By contradiction, suppose $\omega(G^{++-} - S) > 2$. We take a vertex $v \in V(G^{++-} - S) \cap V(G)$ from a component of type-3 and two vertices $e_1, e_2 \in V(G^{++-} - S) \cap E(G)$ from other two different components. Then by the definition of G^{++-}, e_1 and e_2 are not adjacent in G, while they have v as their common end vertex in G, a contradiction. If H_3 is the component of type-3 then by the similar argument as in the proof of Claim 1, $|V(H_3) \cap V(G)| = 1$. So $|S| \geqslant n - 1 + \kappa(L(G))$. □

Claim 5. If all components of $G^{++-} - S$ are of type-3 then $|S| = m - 1 = \delta(G^{++-}) - 1$.

Proof of Claim 5. Suppose $\omega(G^{++-} - S) > 2$, and let $u_1, u_2, u_3 \in V(G)$ be taken from three different components of $G^{++-} - S$. Take a vertex $e_1 \in E(G)$ from the component of $G^{++-} - S$, which contains u_1. Then by the definition of G^{++-}, $e_1 = u_2u_3$, and so u_2 and u_3 are adjacent. But, on the other hand, since u_2 and u_3 are in different components of $G^{++-} - S$, u_2 and u_3 are not adjacent in G, a contradiction. This proves $\omega(G^{++-} - S) = 2$. By the adjacency relation between vertices of G^{++-}, $|V(H_i) \cap V(G)| \leqslant 2$ for each $i = 1$ and 2, since otherwise one can find an edge of G from $V(H_i)$ which will have three end vertices coming from $V(H_j) \cap V(G)$, where $\{i, j\} = \{1, 2\}$, a contradiction.

Next we show that $\{|V(H_1) \cap V(G)|, |V(H_2) \cap V(G)|\} = \{1, 2\}$. If it is not so, $|V(H_1) \cap V(G)| = |V(H_2) \cap V(G)| = 1$ or $|V(H_1) \cap V(G)| = |V(H_2) \cap V(G)| = 2$. If $|V(H_i) \cap V(G)| = 2$ for $i = 1, 2$ then by the definition of G^{++-}, $|V(H_i) \cap E(G)| = 1$ for each $i = 1, 2$. Thus $|S| = n + m - 6 \geqslant m$, a contradiction. Now we consider $|V(H_1) \cap V(G)| = |V(H_2) \cap V(G)| = 1$, and let $u_i \in V(H_i) \cap V(G)$ for $i = 1, 2$. Then $V(H_i)$ consists of u_i and some edges which are incident with u_j in G, where $\{i, j\} = \{1, 2\}$. Moreover, u_1 and u_2 are not adjacent in G, and each element of $V(H_1) \setminus \{u_1\}$ has no common end vertex with any element of $V(H_2) \setminus \{u_2\}$ in G, because they are not in the same component of $G^{++-}-S$. It implies that $|V(H_1)|-1+|V(H_2)|-1 \leq n-2$. But, on the other hand, by $|S| = n-2+m-(|V(H_1)|-1+|V(H_2)|-1)$, and by the assumption $|S| < m$, we have $|V(H_1)|-1+|V(H_2)|-1 > n-2$, a contradiction.

So, let $V(H_1) \cap V(G) = \{u_1, u_2\}$ and $V(H_2) \cap V(G) = \{v\}$. Then each element of $V(H_1) \cap E(G)$ is incident with v in G, and $V(H_2) \cap E(G) = \{u_1u_2\}$. Hence

$$|S| \geqslant n-3+m-(d(v)+1). \qquad (*)$$

Combining this with $|S| < m$, we have $d(v) > n-4$. On the other hand, v and u_i are in two different components of $G^{++-}-S$, v is not adjacent to u_i in G, which means that $d(v) \leq n-3$. Thus $d(v) = n-3$. By taking this into $(*)$ and observing $|S| < m$, we have $|S| = m-1$. □

This completes the proof. □

We use the following classical theorem due to Chvátal and Erdös [4].

Theorem 2.3. *If $\alpha(G) \leqslant \kappa(G)$ for a graph G of order at least three, then G is hamiltonian.*

Theorem 2.4. *Let G be a graph of order $n \geqslant 6$ and size m. If $m \geqslant \alpha(G)+1$, G^{++-} is hamiltonian.*

Proof. Let G be a graph as given in the hypothesis. To show G^{++-} is hamiltonian, by Theorem 2.3, it suffices to show that $\kappa(G^{++-}) \geqslant \alpha(G^{++-})$. We have seen from Theorems 2.1 and 2.2 that $\alpha(G^{++-}) = \max\{\alpha(G), M(G)\}$ and $\kappa(G^{++-}) \geqslant \min\{m-1, n+\kappa(L(G))-1\}$. Note that $M(G) \leqslant \frac{n}{2}$ and $\alpha(G) \leqslant n-1$ since G is not empty. Thus $n-1+\kappa(L(G)) \geqslant \max\{M(G), \alpha(G)\}$. So, by the assumption $m \geqslant \alpha(G)+1$, it suffices to show that $m-1 \geqslant M(G)$. Suppose on the contrary that $m-1 < M(G)$. Then $m = M(G)$ since $m \geqslant M(G)$. It follows that $G \cong mK_2 + (n-2m)K_1$, and thus $\alpha(G) \geqslant m$, which contradicts $m-1 \geqslant \alpha(G)$. □

Corollary 2.5. *Let G be a graph of order $n \geqslant 6$ and size m. If $m \geqslant n$, G^{++-} is hamiltonian.*

Proof. Since $m \geqslant 1$, G is not empty and thus $n-1 \geqslant \alpha(G)$. Combining with the assumption that $m \geqslant n$, we have $m \geqslant \alpha(G)+1$. By Theorem 2.4, G^{++-} is hamiltonian. □

3 Concluding remarks

Xu and Wu [11] recently established a simple sufficient and necessary condition for G^{-+-} to be hamiltonian. So, by the result of [5, 7, 10], we already know respectively, a sufficient and necessary condition for each of G^{+++}, G^{---} and G^{-++} each to be hamiltonian. In this note, for a graph G, we have obtained a simple sufficient condition for G^{++-} to be hamiltonian. It seems there does not exist a simple sufficient and necessary condition for G^{xyz} to be hamiltonian, when $xyz \in \{++-, +--, +-+, --+\}$. It is also interesting to investigate the chromatic number of G^{xyz}.

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