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Some results on 2-perfect cube decompositions

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Abstract

A cube decomposition $\mathcal{Q}$ of a graph G is said to be 2-perfect if for every edge $\{x, y\} \in E(G)$, x and y are connected by a path of length 1 in exactly one cube of $\mathcal{Q}$, and are also connected by a path of length 2 in exactly one (distinct) cube of $\mathcal{Q}$. For both K_v and $K_v - F$, we give constructions for half of the cases which satisfy the obvious necessary conditions, with a small number of exceptions. We also show that there does not exist a 2-perfect Q-decomposition of K_{16} and conjecture that the same is true for K_{25}.

1 Introduction

Let G and $H_1, H_2, \ldots, H_n$ be graphs. The set $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ is a *decomposition* of G if $E(H_1), E(H_2), \ldots, E(H_n)$ partitions $E(G)$. If H_i is isomorphic to H, for $1 \leq i \leq n$, then $\mathcal{H}$ is said to be an *H-decomposition* of G. Furthermore, an H-decomposition $\mathcal{H}$ is said to be 2-*perfect* if for every $\{x, y\} \in E(G)$, x and y are connected by a path of length 1 in exactly one copy of $H \in \mathcal{H}$, and are also connected by a path of length 2 in exactly one (distinct) copy of $H \in \mathcal{H}$.

Most commonly $G = K_v$, the complete graph on v vertices, or $K_v - F$, the complete graph on v vertices with the edges of a 1-factor removed.

An H-decomposition of K_v is said to be an H-design. For $k < v$, a K_k-design is said to be *resolvable* if we can partition the copies of K_k in the decomposition into *parallel classes* such that each class contains each vertex of K_v exactly once.

The problem of determining all values of v for which there exists an H-decomposition of K_v is called the *spectrum problem* for H. The spectrum problem for 2-perfect m-cycle decompositions has been considered by several authors: for example, see [1], [2], [10], [11], [12], [16].

An n-cube, denoted Q_n, is defined to be K_2 if $n = 1$ and, for $n \geq 2$, $Q_n = Q_{n-1} \times K_2$. Consequently, a 2-cube is simply a 4-cycle and a 3-cube is a 3-dimensional cube.

In 1979 in [8], Kotzig asked: “For which n and v does there exist an n-cube decomposition of K_v?” This remains an open problem today.

An n-cube decomposition of K_v is possible only if the number of edges in K_v is divisible by $n2^{n-1}$, the number of edges in an n-cube. Furthermore, since an n-cube is n-regular, the decomposition is possible only if the degree of a vertex in K_v is divisible by n; that is $n \mid v-1$.

Kotzig proves Theorems 1.1 and 1.2 in [9], thus establishing some necessary conditions for existence of an n-cube decomposition of K_v.

Theorem 1.1 *Let n be even. If K_v can be decomposed into n-cubes, then* $v \equiv 1 \pmod{n2^n}$.

Theorem 1.2 *Let n be odd. If K_v can be decomposed into n-cubes, then one of the two following conditions holds:*

1. $v \equiv 1 \pmod{n2^n}$; *or*
2. $v \equiv 0 \pmod{2^n}$ *and* $v \equiv 1 \pmod{n}$.

In the same paper Kotzig proves that the conditions of Theorem 1.1 and Part 1 of Theorem 1.2 are also sufficient. Maheo was the first to prove, in [15], that the necessary conditions of Theorem 1.2 are also sufficient when $n = 3$. The spectrum problem for 5-cube decompositions of K_v has been completed settled by Bryant *et al.*; see [4].

The obvious necessary conditions for existence of a Q_3-decomposition of $K_v - F$ have also been shown to be sufficient; see [18]. Thus we have the following theorem.

Theorem 1.3 *There exists a Q_3-decomposition of K_v or $K_v - F$ if, and only if,* $v \equiv 1, 16 \pmod{24}$ *or* $v \equiv 2 \pmod{6}$, *respectively.*

If we let the vertex set of K_v be denoted $\{u_1, u_2, \ldots, u_v\}$, then we can represent this graph by $(u_1, u_2, \ldots, u_v)$, or by any permutation of this. We let the 3-cube with vertex set $\{a, b, c, d, e, f, g, h\}$ and edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{e, f\}, \{f, g\}, \{g, h\}, \{h, e\}, \{a, e\}, \{b, f\}, \{c, g\}, \{d, h\}\}$ be denoted $[a, b, c, d \mid e, f, g, h]$; see Figure 1.

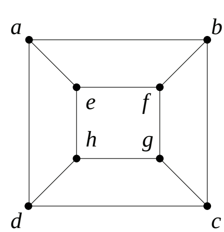

Figure 1. The 3-cube denoted by $[a, b, c, d \mid e, f, g, h]$.

We define the *distance* between two vertices in a connected graph to be the length of the *shortest path* which connects them. For example, the distances between a and b, a and c, and a and g in the cube $[a, b, c, d \mid e, f, g, h]$ are one, two and three respectively. If the distance between two vertices x and y is k, then x and y are said to be *separated by distance* k.

Given a cube $A = [a, b, c, d \mid e, f, g, h]$, we can form two copies of K_4, namely (a, c, f, h) and (b, d, e, g), by connecting all vertices that are separated by distance two. Let $A(2) = \{(a, c, f, h), (b, d, e, g)\}$. We refer to $A(2)$ as the *distance* 2 *graph of* A.

Let $\mathcal{Q}$ be a Q_3-decomposition of a graph G. Let $\mathcal{Q}(2) = \bigcup_{A \in \mathcal{Q}} A(2)$. Then $\mathcal{Q}$ is 2-perfect if $\mathcal{Q}(2)$ forms a K_4-decomposition of G.

Results on K_4-decompositions of K_v and $K_v - F$ can be found in [7] and [3] respectively.

Theorem 1.4 *There exists a K_4-decomposition of K_v or $K_v - F$ if, and only if, $v \equiv 1, 4 \pmod{12}$ or $v \equiv 2 \pmod 6$, $v \neq 8$, respectively.*

For the remainder of the paper we denote a 3-cube by simply Q.

We investigate the problem of determining for which v there exists a 2-perfect Q-decomposition of K_v or $K_v - F$. For such decompositions to exist the conditions of Theorems 1.3 and 1.4 must be satisfied. That is, the obvious necessary conditions for existence of a 2-perfect Q-decomposition of K_v or $K_v - F$ are $v \equiv 1, 16 \pmod{24}$ or $v \equiv 2 \pmod 6$, $v \neq 8$, respectively.

We obtain partial results for this problem, most notably:

- For all $v \equiv 2 \pmod{12}$ there exists a 2-perfect Q-decomposition of $K_v - F$;
- There does not exist a 2-perfect Q-decomposition of K_{16}; and
- If $v \equiv 1 \pmod{24}$, $v \notin \{25, 96, 120, 144, 168, 216\}$, then there exists a 2-perfect Q-decomposition of K_v.

For convenience we now introduce some more notation and terminology to be used throughout this paper.

We denote by $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$ the complete multipartite graph with p_i partite sets of size n_i, for $1 \leq i \leq x$. The complete equipartite graph with p partite sets each containing n vertices is denoted $K(n^p)$.

We denote by $G - H$ the graph with $V(G - H) = V(G)$ and $E(G - H) = E(G) \setminus E(H)$.

A *nested* K_3 is an ordered pair of graphs (H_1, H_2), such that H_1 and H_2 are copies of K_3 and K_4 respectively, and H_1 is a subgraph of H_2. A *nested K_3-decomposition of a graph G* is a set $\mathcal{H}$ of nested K_3s such that $\{H_1 \mid (H_1, H_2) \in \mathcal{H}\}$ is a K_3-decomposition of G, and $\{H_2 - H_1 \mid (H_1, H_2) \in \mathcal{H}\}$ is a $(K_4 - K_3)$-decomposition of G.

If no confusion is likely to arise, then we will use uv to denote the edge $\{u, v\}$.

In Section 2 we consider cases which depend upon the existence of nested K_3-decompositions; this section includes some results on decompositions of $K_v - F$ and multipartite graphs. In Section 3 we consider decompositions of K_v. This includes the non-existence proof for K_{16}, and a method for constructing 2-perfect Q-decompositions for almost all $v \equiv 1 \pmod{24}$.

2 Results which use nested K_3-decompositions

Theorem 2.1 *If there exists a nested K_3-decomposition of $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$, then there exists a 2-perfect Q-decomposition of $K(2n_1^{p_1}, 2n_2^{p_2}, \ldots, 2n_x^{p_x})$.*

Proof. Let $E = E(K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x}))$. Let $\mathcal{H}$ be a nested K_3-decomposition of $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$.

Let $B_i = ((a, b, c), (a, b, c, d)) \in \mathcal{H}$ be the i^{th} nested K_3 from the decomposition of $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$. Let $E_1^i = \{ab, ac, bc\}$ and $E_2^i = \{ad, bd, cd\}$. By definition of a nested K_3-decomposition, we have

$$\bigcup_i E_j^i = E, \quad \text{for } j \in \{1, 2\}.$$

Given B_i, let $A_i = [a_1, b_2, c_1, d_1 \mid c_2, d_2, a_2, b_1]$. Let D_1^i be the edge set of A_i, and let D_2^i be the edge set of the distance 2 graph of A_i. Then

$$D_1^i = E(A_i) = \{a_1b_2,\ a_2b_1,\ a_1c_2,\ a_2c_1,\ b_1c_2,\ b_2c_1,\ a_1d_1,\ b_1d_1,\ c_1d_1,\ a_2d_2,\ b_2d_2,\ c_2d_2\},$$

and

$$D_2^i = E(A_i(2)) = \{a_1b_1,\ a_1c_1,\ b_1c_1,\ a_2b_2,\ a_2c_2,\ b_2c_2, a_1d_2,\ b_1d_2,\ c_1d_2,\ a_2d_1,\ b_2d_1,\ c_2d_1\}.$$

We can rewrite these sets as follows:

$$\begin{aligned} D_1^i &= \{u_1v_2,\ u_2v_1 \mid uv \in E_1^i\} \cup \{u_1v_1,\ u_2v_2 \mid uv \in E_2^i\}, \text{ and} \\ D_2^i &= \{u_1v_1,\ u_2v_2 \mid uv \in E_1^i\} \cup \{u_1v_2,\ u_2v_1 \mid uv \in E_2^i\}. \end{aligned}$$

Note that $D_j^{i_1} \cap D_j^{i_2} = \emptyset$, for all $i_1 \neq i_2$, where $j = 1, 2$. Hence

$$\begin{aligned} \cup_i D_1^i &= \{u_1v_2,\ u_2v_1 \mid uv \in \cup_i E_1^i\} \cup \{u_1v_1,\ u_2v_2 \mid uv \in \cup_i E_2^i\}, \\ &= \{u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2 \mid uv \in E\}, \text{ and} \\ \cup_i D_2^i &= \{u_1v_1,\ u_2v_2 \mid uv \in \cup_i E_1^i\} \cup \{u_1v_2,\ u_2v_1 \mid uv \in \cup_i E_2^i\} \\ &= \{u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2 \mid uv \in E\}. \end{aligned}$$

Since E is the edge set of $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$, then $\{u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2 \mid uv \in E\}$ is the edge set of $K(2n_1^{p_1}, 2n_2^{p_2}, \ldots, 2n_x^{p_x})$.

Hence $\{A_i\}$ and $\{A_i(2)\}$ are Q- and K_4-decompositions of $K(2n_1^{p_1}, 2n_2^{p_2}, \ldots, 2n_x^{p_x})$, respectively. Thus $\mathcal{Q} = \{A_i\}$ is a 2-perfect Q-decomposition of $K(2n_1^{p_1}, 2n_2^{p_2}, \ldots, 2n_x^{p_x})$.

Corollary 2.2 *If there exists a nested K_3-decomposition of K_v, then there exists a 2-perfect Q-decomposition of $K_{2v} - F$.*

Theorem 2.3 [5, 13, 17] *There exists a nested K_3-decomposition of K_v for all $v \equiv 1 \pmod 6$, $v \geq 7$.*

Theorem 2.4 *There exists a 2-perfect Q-decomposition of $K_v - F$ for all $v \equiv 2 \pmod{12}$, $v \geq 14$.*

Proof. The result follows from Corollary 2.2 and Theorem 2.3.

3 Decompositions of K_v

We begin by proving that there does not exist a 2-perfect Q-decomposition of K_{16}. We then give further results on decompositions of multipartite graphs which we combine with decompositions of K_{49} and K_{73} to construct an infinite family of 2-perfect Q-decompositions.

3.1 $v = 16$

It is possible to decompose K_{16} into copies of K_4 or Q. However, despite all obvious necessary conditions being satisfied, there does not exist a 2-perfect Q-decomposition of K_{16}.

Lemma 3.1 *Every K_4-decomposition of K_{16} is resolvable, and two copies of K_4 are vertex disjoint if, and only if, they are in the same parallel class.*

Proof. This follows from the fact that a K_4-decomposition of K_{16} is an affine plane of order four.

Lemma 3.2 *There exist no 2-perfect Q-decompositions of K_{16}.*

Proof. Let $V = \{a_b \mid a, b \in \{1,2,3,4\}\}$, and suppose that there exists a 2-perfect Q-decomposition of K_{16} on the vertex set V, given by $\mathcal{Q} = \{A_1, A_2, \ldots, A_{10}\}$. We now seek a contradiction.

Note that each cube contains twelve pairs of vertices separated by distance 1, twelve pairs of vertices separated by distance 2, and four pairs of vertices separated by distance 3.

Let X be the set of triples $\{\{x,y\}, A, B\}$, where $x, y \in V$, $A, B \in \mathcal{Q}$, and x and y are separated by distance k in A, and by distance l in B, where $k, l \in \{1, 3\}$. Our argument is based on the size of X and the maximum number of triples in X containing a given pair $\{x, y\}$.

We begin by considering $\mathcal{Q}(2) = A_1(2) \cup A_2(2) \cup \ldots \cup A_{10}(2)$, the K_4-decomposition of K_{16} on V, formed by the distance 2 graphs of the cubes in $\mathcal{Q}$.

By Lemma 3.1, we can assume without loss of generality that $\{(1_1, 2_1, 3_1, 4_1), (1_2, 2_2, 3_2, 4_2), (1_3, 2_3, 3_3, 4_3), (1_4, 2_4, 3_4, 4_4)\}$ forms a parallel class, with $A_1(2) = \{(1_1, 2_1, 3_1, 4_1), (1_2, 2_2, 3_2, 4_2)\}$ and $A_2(2) = \{(1_3, 2_3, 3_3, 4_3), (1_4, 2_4, 3_4, 4_4)\}$.

In general, we see that $A_1, A_2, \ldots, A_{10}$ occur in vertex-disjoint pairs (parallel classes). A corollary is that a given pair of vertices, x and y say, may be separated by distance 1, 2 or 3 in at most five distinct cubes of $\mathcal{Q}$.

Since the other copies of K_4 must intersect each of the K_4s in $A_1(2) \cup A_2(2)$, we can assume without loss of generality that $A_3(2) = \{(1_1, 1_2, 1_3, 1_4), (2_1, 2_2, 2_3, 2_4)\}$.

Thus we have $\{\{1_1, 2_2\}, A_1, A_3\}, \{\{1_2, 2_1\}, A_1, A_3\} \in X$. In fact for each A_x, where $3 \leq x \leq 10$, we have exactly two triples in X containing A_1 and A_x. Thus there are sixteen triples in X which contain A_1. The same is true for every other cube in $\mathcal{Q}$, so accounting for repetition we have $|X| = 80$.

A pair of distinct vertices x and y must be separated by distance 2 in exactly one cube, meaning that they are separated by distance 1 or 3 in m cubes, where $1 \leq m \leq 4$. Now x and y must be separated by distance 1 in exactly one of these cubes, and so must be separated by distance 3 in $m-1$ different cubes.

The pair $\{x, y\}$ will occur in $\binom{m}{2}$ triples in X. That is, if $m = 1$, 2, 3 or 4 respectively, then x and y will be separated by distance 3 in 0, 1, 2 or 3 different cubes respectively, and will occur respectively in 0, 1, 3 or 6 triples of X. But there are only four pairs of vertices separated by distance 3 in each of the ten cubes; since $3 \nmid 40$, this is insufficient to give the 80 triples in X (a contradiction).

3.2 $v \equiv 1 \pmod{24}$

We take this opportunity to present some results on 2-perfect Q-decompositions of equipartite graphs which can be obtained directly from nested K_3-decompositions. Of particular importance is the existence result for $K(8^4)$ since it allows us to construct decompositions of $K(48^p)$ and $K(72^1, 48^p)$, where $p \geq 4$. (As an aside, there do not exist any 2-perfect Q-decompositions of multipartite graphs with two or three partite sets, and if the graph has four partite sets, then it is a necessary condition that the partite sets have the same size.)

Theorem 3.3 [14] *There exists a nested K_3-decomposition of $K(n^p)$ for each combination of n and p given in Table* 1.

Lemma 3.4 *There exists a 2-perfect Q-decomposition of $K(2n^p)$ for each combination of n and p given in Table* 1.

Proof. This follows from Theorems 2.1 and 3.3.

Corollary 3.5 *There exists a 2-perfect Q-decomposition of $K(8^4)$.*

n	p
$n \equiv 0 \pmod 2$	$p \equiv 1, 4 \pmod{12}$, $p \geq 4$
$n \equiv 0 \pmod 4$	7
$n \equiv 0 \pmod 6$	$p \equiv 0, 1 \pmod 4$, $p \geq 5$
$n \equiv 0 \pmod{24}$	$p \equiv 0, 1 \pmod 5$, $p \geq 5$

Table 1: Values of n and p for which there exists a nested K_3-decomposition of $K(n^p)$.

The next result permits the construction of many other 2-perfect Q-decompositions.

Lemma 3.6 *If there exists a 2-perfect Q-decomposition of $K(a^b)$ and a K_b-decomposition of $K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$, then there exists a 2-perfect Q-decomposition of $K(an_1^{p_1}, an_2^{p_2}, \ldots, an_x^{p_x})$.*

Proof. Let $G = K(n_1^{p_1}, n_2^{p_2}, \ldots, n_x^{p_x})$. Assume that there exists a K_b-decomposition of G. Replace each vertex of G by a new vertices to obtain a $K(a^b)$-decomposition of $K(an_1^{p_1}, an_2^{p_2}, \ldots, an_x^{p_x})$. The result follows by our second assumption.

Theorem 3.7 [3] *There exists a K_4-decomposition of $K(n^p)$ if, and only if, $n(p-1) \equiv 0 \pmod 3$, $n^2p(p-1) \equiv 0 \pmod{12}$, $p \geq 4$ or $p = 1$, excluding the cases where $(p, n) \in \{(4, 2), (4, 6)\}$.*

Corollary 3.8 *Let $p \geq 4$. Then there exists a 2-perfect Q-decomposition of $K(48^p)$.*

Proof. If $p = 4$, then the result follows from Lemma 3.4. If $p \geq 5$, then by Theorem 3.7 there exists a K_4-decomposition of $K(6^p)$. The result then follows by Corollary 3.5 and Lemma 3.6.

Theorem 3.9 [6] *There exists a K_4-decomposition of $K(6^p, n^1)$ for all $p \geq 4$ and $n \equiv 0 \pmod 3$ such that $0 \leq n \leq 3p-3$, excluding the case where $(p, n) = (4, 0)$, and possibly excluding the cases where $(p, n) \in \{(7, 15), (11, 21), (11, 24), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.*

Corollary 3.10 *For $p \geq 4$ there exists a 2-perfect Q-decomposition of $K(72^1, 48^p)$.*

Proof. Let $p \geq 4$. By Theorem 3.9 there exists a K_4-decomposition of $K(9^1, 6^p)$. The result then follows by Corollary 3.5 and Lemma 3.6.

We now give some existence results for 2-perfect Q-decompositions of K_v for relatively small values of v. In each case the decomposition is cyclic; the decomposition of K_{49} uses two starter cubes and the decomposition of K_{73} uses three starter cubes. It should be noted that while there exists a Q-decomposition of K_{25}, an exhaustive computational search did not find a cyclic 2-perfect Q-decomposition using a single starter cube (developed modulo 25).

Lemma 3.11 *There exists a 2-perfect Q-decomposition of K_{49}.*

Proof. Let the vertex set of K_{49} be $\mathbb{Z}_{49}$. A suitable decomposition is given by developing the following two starter cubes modulo 49:

$$[0, 1, 3, 35 \mid 37, 44, 28, 24], \quad [0, 3, 11, 29 \mid 15, 43, 33, 10].$$

Lemma 3.12 *There exists a 2-perfect Q-decomposition of K_{73}.*

Proof. Let the vertex set of K_{73} be $\mathbb{Z}_{73}$. A suitable decomposition is given by developing the following three starter cubes modulo 73:

$$[0, 1, 3, 6 \mid 5, 28, 63, 46], \quad [0, 4, 19, 53 \mid 21, 52, 30, 39], \quad [0, 7, 31, 19 \mid 29, 37, 21, 66].$$

Combining the results of this section, we obtain the following theorem.

Theorem 3.13 *If $v \equiv 1 \pmod{24}$, $v \notin \{25, 96, 120, 144, 168, 216\}$, then there exists a 2-perfect Q-decomposition of K_v.*

Proof. When $v = 1$ the result is trivial, and the required decompositions of K_{49} and K_{73} are given in Lemmas 3.11 and 3.12, respectively.

For larger values of v we take a 2-perfect Q-decomposition of either $K(48^p)$ (Corollary 3.8) or $K(72^1, 48^p)$ (Corollary 3.10), where $p \geq 4$. We then add a new vertex, ∞, and place a 2-perfect Q-decomposition of either K_{49} or K_{73} on each partite set together with the vertex ∞.

4 Concluding remarks

For both K_v and $K_v - F$, we have given constructions for half of the cases which satisfy the obvious necessary conditions, with a small number of exceptions. We have shown that there does not exist a 2-perfect Q-decomposition of K_{16} and conjecture that the same is true for K_{25}. We have also given results which allow for the construction of 2-perfect Q-decompositions of many complete multipartite graphs, which could be used with small existence results to develop constructions for the remaining K_v and $K_v - F$ cases.

References

[1] P. Adams and E. J. Billington, Completing some spectra for 2-perfect cycle systems, *Australas. J. Combin.* **7** (1993), 175–187.

[2] P. Adams and E. J. Billington, The spectrum for 2-perfect 8-cycle systems, *Ars Combin.* **36** (1993), 47–56.

[3] A. E. Brouwer, A. Schrijver and H. Hanani, Group divisible designs with block-size four, *Discrete Math.* **20** (1977) no. 1, 1–10.

[4] D. Bryant, S. I. El-Zanati, B. Maenhaut and C. Vanden Eynden, Decomposition of complete graphs into 5-cubes, *J. Combin. Des.* **14** (2006) no. 2, 159–166.

[5] C. J. Colbourn and M. J. Colbourn, Nested triple systems, *Ars Combin.* **16** (1983), 27–34.

[6] G. Ge and R. Rees, On group-divisible designs with block size four and group-type $6^u m^1$, *Discrete Math.* **279** (2004) no. 1-3, 247–265.

[7] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[8] A. Kotzig, Selected open problems in graph theory, *Graph Theory and Related Topics*, Academic Press, New York (1979), 358–367.

[9] A. Kotzig, Decompositions of complete graphs into isomorphic cubes, *J. Combin. Theory Ser. B* **31** (1981) no. 3, 292–296.

[10] C. C. Lindner, K. T. Phelps and C. A. Rodger, The spectrum for 2-perfect 6-cycle systems, *J. Combin. Theory Ser. A* **57** (1991) no. 1, 76–85.

[11] C. C. Lindner and C. A. Rodger, 2-perfect m-cycle systems, *Discrete Math.* **104** (1992) no. 1, 83–90.

[12] C. C. Lindner and D. R. Stinson, Steiner pentagon systems, *Discrete Math.* **52** (1984) no. 1, 67–74.

[13] C. C. Lindner and D. R. Stinson, The spectrum for the conjugate invariant subgroups of perpendicular arrays, *Ars Combin.* **18** (1984), 51–60.

[14] J. Q. Longyear, Nested group divisible designs and small nested designs, *J. Statist. Plann. Inference* **13** (1986) no. 1, 81–87.

[15] M. Maheo, Strongly graceful graphs, *Discrete Math.* **29** (1980) no. 1, 39–46.

[16] E. Manduchi, Steiner heptagon systems, *Ars Combin.* **31** (1991), 105–115.

[17] D. R. Stinson, The spectrum of nested Steiner triple systems, *Graphs Combin.* **1** (1985) no. 2, 189–191.

[18] M. Waterhouse, Some equitably 2 and 3-colourable cube decompositions, *Australas. J. Combin.* **32** (2005), 137–145.

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