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On graphs with largest Laplacian eigenvalue at most 4

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Abstract

In this paper graphs with the largest Laplacian eigenvalue at most 4 are characterized. Using this we show that the graphs with the largest Laplacian eigenvalue less than 4 are determined by their Laplacian spectra. Moreover, we prove that ones with no isolated vertex are determined by their adjacency spectra.

1 Introduction

In this paper we are concerned with finite simple graphs. Let G be such a graph with n vertices, m edges and the adjacency matrix $A(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the adjacency and the Laplacian eigenvalues of G, respectively. The multiset of the eigenvalues of $A(G)$ and $L(G)$ are called the *adjacency spectrum* and *Laplacian spectrum* of G, respectively. The maximum eigenvalue of $A(G)$ is called the *index* of G. Two graphs are said to be *cospectral* with respect to the adjacency (Laplacian, respectively) matrix if they have the same adjacency (Laplacian, respectively) spectrum. A graph is said to be *determined* (DS for short) *by its adjacency spectrum or Laplacian spectrum* if there is no other non-isomorphic graph with the same spectrum with respect to the adjacency or Laplacian matrices, respectively.

There are some results on determining graphs with small number of Laplacian eigenvalues exceeding a given value. (See [2, 4, 7] and the references therein). All

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connected bipartite graphs whose third largest Laplacian eigenvalue is less than three have been characterized by Zhang [12]. Moreover in [11], graphs with fourth Laplacian eigenvalue less than two are identified. In this paper we characterize all graphs with largest Laplacian eigenvalue at most 4.

Since the problem of characterizing all DS graphs seems to be very difficult, finding any new infinite family of these graphs will be an interesting problem (see[9, 10]). Using the characterization of graphs with the largest Laplacian eigenvalue at most 4, we show that graphs with the largest Laplacian eigenvalue less than 4 are DS with respect to the Laplacian matrix. Moreover, we prove that ones with no isolated vertex are DS with respect to the adjacency matrix.

2 Graphs of index less than 2

In [8], all connected graphs of index at most 2 are identified. Among them all connected graphs of index 2 are well known. Using this we can determine all graphs with index less than 2.

Theorem 2.1 [8] *The list of all connected graphs of index at most* 2 *includes precisely the following graphs:*
i) $P_n, C_n, Z_n(n \geq 2), W_n(n \geq 2)$,
ii) $T(a, b, c)$ *for* $(a, b, c) \in \{(1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 3), (2, 2, 2)\}$,
iii) $K_{1,4}$.

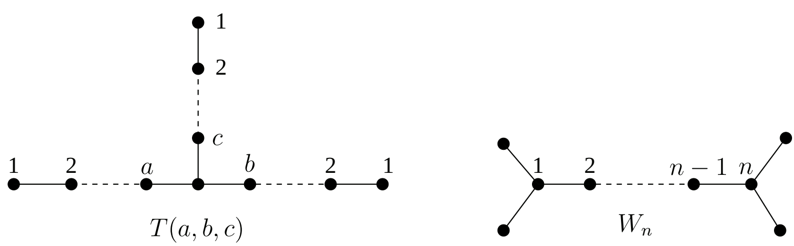

Figure 1

Notation The path and cycle with n vertices are denoted by P_n and C_n, respectively. For $a, b, c \geq 1$, we denote the graph shown in Fig. 1 (left) by $T(a, b, c)$. In particular, $Z_n(n \geq 2)$ stands for $T(1, n-1, 1)$. For $n \geq 2$, we denote the graph shown in Fig. 1 (right) by W_n. Again, we denote the graphs $T(1, 2, 2)$, $T(1, 2, 3)$, $T(1, 2, 4)$, $T(1, 2, 5)$, $T(1, 3, 3)$ and $T(2, 2, 2)$ by T_i for $i = 1, 2, \ldots, 6$, respectively.

Among graphs of index at most 2, the graphs C_n, $W_n(n \geq 2)$, $K_{1,4}$ and T_i for $i = 4, 5, 6$ have 2 as an eigenvalue.

Corollary 2.1 *The list of all connected graphs of index less than* 2 *consist of precisely the following graphs:*
i) $P_n, Z_n(n \geq 2)$,
ii) T_i *for* $i = 1, 2, 3$.

3 Graphs with largest Laplacian eigenvalue at most 4

In this section we characterize all graphs with the largest Laplacian eigenvalue at most 4.

Lemma 3.1 [3] *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then $\Delta(G) + 1 \leq \mu_{\max} \leq \max\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v}, uv \in E(G)\}$ where $\Delta(G)$, $\mu_{\max}$ and m_v denote the maximum vertex degree of G, the largest Laplacian eigenvalue of G and the average of degrees of vertices adjacent to the vertex v in G, respectively.*

Lemma 3.2 [9] *Let T be a tree with n vertices and let $L(T)$ be its line graph. Then for $i = 1, \ldots, n-1$, $\mu_i(T) = \lambda_i(L(T)) + 2$.*

Lemma 3.3 [1] *Let G be a connected graph and let H be a proper subgraph of G. Then $\mu_1(H) \leq \mu_1(G)$.*

In regular graphs we can calculate the characteristic polynomial of the Laplacian matrix in terms of the characteristic polynomial of the adjacency matrix. Let G be a regular graph of degree r and let $P_L(\lambda)$ and $P_A(\lambda)$ be the characteristic polynomials of the Laplacian matrix and the adjacency matrix of G, respectively. Then $D = rI$, $L = rI - A$ and we have $P_L(\lambda) = (-1)^n P_A(r - \lambda)$. So λ is an eigenvalue of L if and only if $r - \lambda$ is an eigenvalue of A.

Lemma 3.4 *Let $n \geq 3$ be a natural number. Then C_n has 4 as a Laplacian eigenvalue if and only if n is even.*

Proof: Since C_n is a 2-regular graph, 4 is a Laplacian eigenvalue of C_n if and only if -2 is an adjacency eigenvalue of C_n. Moreover C_n is a graph of index 2. So -2 is an adjacency eigenvalue if and only if C_n is a bipartite graph. Hence C_n has 4 as a Laplacian eigenvalue if and only if n is even. □

Theorem 3.1 *The list of all connected graphs with the largest Laplacian eigenvalue at most 4 includes precisely the following graphs: $P_n, C_n (n \geq 3), K_{1,3}, K_4, H_1$ and H_2, where H_1 and H_2 are obtained from K_4 by deleting two adjacent edges and one edge, respectively.*

Proof: By Lemma 3.1, we can see that $\mu_{max}(P_n)$ and $\mu_{max}(C_n)$ are at most 4. Using the computer package newGRAPH [5], we have:

$$\mu_{max}(K_{1,3}) = \mu_{max}(K_4) = \mu_{max}(H_1) = \mu_{max}(H_2) = 4.$$

Now let G be a connected graph with the largest Laplacian eigenvalue at most 4 and let $\Delta(G)$ be the maximum degree of G. By Lemma 3.1, we have $\Delta(G) + 1 \leq \mu_{max}(G) \leq 4$. If $\Delta(G) \leq 1$, then G is either P_1 or P_2 and $\mu_{max}(G) \in \{0, 2\}$. If

$\Delta(G) = 2$, then G is either P_n or C_n for $n \geq 3$. Now Let $\Delta(G) = 3$. Using the computer package newGRAPH, we can see that $\mu_{max}(T(1,1,2)) > 4$. So by Lemma 3.3, the largest Laplacian eigenvalue of every connected graph on $n \geq 5$ vertices and maximum degree 3 is greater than 4. Since $\Delta(G) = 3$ and G is a connected graph with the largest Laplacian eigenvalue at most 4, G has exactly 4 vertices. Again by the computer package newGRAPH, we can see that G is one of the $K_{1,3}, K_4, H_1$ or H_2. □

Corollary 3.1 *The list of all connected graphs with the largest Laplacian eigenvalue less than* 4 *consist of precisely the following graphs:* P_n *and* C_{2n+1}, *for* $n \geq 1$.

Proof: Using the computer package newGRAPH, we have $\mu_{max}(K_{1,3}) = \mu_{max}(K_4) = \mu_{max}(H_1) = \mu_{max}(H_2) = 4$. On the other hand by Lemma 3.4, C_n has 4 as a Laplacian eigenvalue if and only if n is even. Moreover by Corollary 2.1 and Lemma 3.2, $\mu_{max}(P_n) < 4$. So by Theorem 3.1, all connected graphs with the largest Laplacian eigenvalue less than 4 are the following graphs: P_n and C_{2n+1}, for $n \geq 1$. □

4 New family of DS graphs with respect to the Laplacian matrix

In this section we prove that graphs with the largest Laplacian eigenvalue less than 4 can be determined by their Laplacian spectra. The Laplacian spectrum of the union of two graphs is obviously the union of their spectra (counting the multiplicities of the eigenvalues). The expressions $G_1 + G_2$ and $\hat{G_1} + \hat{G_2}$ will denote the union of the graphs G_1 and G_2 and the union of their Laplacian spectra (counting the multiplicities of the eigenvalues), respectively. The expressions kG and $k\hat{G}$ denote the union of k copies of G and $\hat{G}$, respectively.

Lemma 4.1 [9] *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum:*
i) *The number of vertices,*
ii) *The number of edges.*
For the adjacency matrix, the following follows from the spectrum:
iii) *The number of closed walks of any length.*
For the Laplacian matrix, the following follows from the spectrum:
iv) *The number of spanning trees,*
v) *The number of components,*
vi) *The sum of squares of degrees of vertices.*

Lemma 4.2 *Each connected graph with the largest Laplacian eigenvalue at most* 4 *can be determined by its Laplacian spectrum.*

Proof: Using Lemma 4.1, each cospectral graph to the given connected graph with the largest Laplacian eigenvalue at most 4 (with respect to the Laplacian matrix) is a connected graph with the largest Laplacian eigenvalue at most 4. The graphs P_n and C_n can be determined by their Laplacian spectra [9]. Using the computer package newGRAPH, we can see the graphs $K_{1,3}, K_4, H_1$ and H_2 have different Laplacian spectra. So these graphs can be determined by their Laplacian spectra. □

Theorem 4.1 [6] *Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r} + Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1T_1 + t_2T_2 + t_3T_3$ be a graph of index less than 2. Then G can be determined by its Laplacian spectrum.*

Corollary 4.1 *Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r}$ be a graph with the largest Laplacian eigenvalue less than 4. Then G can be determined by its Laplacian spectrum.*

Lemma 4.3 *Let $G = C_{2j_1+1} + C_{2j_2+1} + \cdots + C_{2j_k+1}$ be a graph with the largest Laplacian eigenvalue less than 4. Then G can be determined by its Laplacian spectrum.*

Proof: We give the proof by induction on the number of components of G. We know that C_n is DS with respect to the Laplacian matrix (see [9]). As we will see in Theorem 4.2, Laplacian eigenvalues of C_{2j+1} are known and we have:

$$\hat{C}_{2j+1} = \{2 - 2\cos\frac{2i\pi}{2j+1} \mid i = 1, 2, \ldots, 2j+1\}.$$

Now let $\bar{G}$ be cospectral to G with respect to the Laplacian matrix. So $\bar{G}$ is a graph with the largest Laplacian eigenvalue less than 4 and it can be represented as the following

$$\bar{G} = P_{\bar{i}_1} + P_{\bar{i}_2} + \cdots + P_{\bar{i}_{\bar{r}}} + C_{2\bar{j}_1+1} + C_{2\bar{j}_2+1} + \cdots + C_{2\bar{j}_{\bar{k}}+1}.$$

Using Lemma 4.1, the two graphs G and $\bar{G}$ have the same number of components, vertices and edges. So $\bar{r} = 0$ and $k = \bar{k}$. Without any loss of generality we can assume $j_1 \geq j_2 \geq \cdots \geq j_k$ and $\bar{j}_1 \geq \bar{j}_2 \geq \cdots \geq \bar{j}_k$. Since G and $\bar{G}$ have the same largest eigenvalues, we have $\mu_1(G) = \mu_1(\bar{G}) = 2 - 2\cos\frac{2j_1\pi}{2j_1+1} = 2 - 2\cos\frac{2\bar{j}_1\pi}{2\bar{j}_1+1}$ and so $j_1 = \bar{j}_1$. By deleting the same components from G and $\bar{G}$ and using induction on the number of components of G the proof is complete. □

Theorem 4.2 *Any graph with the largest Laplacian eigenvalue less than 4 can be determined by its Laplacian spectrum.*

Proof: The adjacency spectrum of P_n and C_n are known [1] and we have:

$$Spec_A(P_n) = \{2\cos\frac{j\pi}{n+1} \mid j = 1, 2, \ldots, n\},$$

$$Spec_A(C_n) = \{2\cos\frac{2j\pi}{n} \mid j = 1, 2, \ldots, n\}.$$

So Laplacian eigenvalues of C_n are known and we have:

$$\hat{C}_n = \{2 - 2\cos\frac{2j\pi}{n} \mid j = 1, 2, \ldots, n\}.$$

On the other hand since $L(P_i) = P_{i-1}$, by Lemma 3.2,

$$\hat{P}_n = \{2 + 2\cos\frac{j\pi}{n} \mid j = 1, 2, \ldots, n-1\} + \{0\}.$$

Using the previous facts we have $\mu_{2n}(C_{2n+1}) = 2 - 2\cos\frac{2\pi}{2n+1}$ and $\mu_{n-1}(P_n) = 2 - 2\cos\frac{\pi}{n}$. Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r} + C_{2j_1+1} + C_{2j_2+1} + \cdots + C_{2j_k+1}$ be a graph with the largest Laplacian eigenvalue less than 4. We give the proof by induction on the number of components of G. By Corollary 4.1 and Lemma 4.3 for $r = 0$ or $k = 0$ the graph G is DS. Now let $rk > 0$ and let $\bar{G}$ be cospectral to G with respect to the Laplacian matrix. So $\bar{G}$ is a graph with the largest Laplacian eigenvalue less than 4 and it can be represented as a linear combination of the form

$$\bar{G} = P_{\bar{i}_1} + P_{\bar{i}_2} + \cdots + P_{\bar{i}_{\bar{r}}} + C_{2\bar{j}_1+1} + C_{2\bar{j}_2+1} + \cdots + C_{2\bar{j}_{\bar{k}}+1}.$$

Using Lemma 4.1, G and $\bar{G}$ have the same number of components, vertices and edges. So $r = \bar{r}$ and $k = \bar{k}$. Without any loss of generality we can assume $i_1 \geq i_2 \geq \cdots \geq i_r$, $\bar{i}_1 \geq \bar{i}_2 \geq \cdots \geq \bar{i}_r$, $j_1 \geq j_2 \geq \cdots \geq j_k$ and $\bar{j}_1 \geq \bar{j}_2 \geq \cdots \geq \bar{j}_k$. We denote the least non-zero Laplacian eigenvalues of G and $\bar{G}$ by μ and $\bar{\mu}$, respectively. It is clear that $\mu \in \{2 - 2\cos\frac{\pi}{i_1}, 2 - 2\cos\frac{2\pi}{2j_1+1}\}$ and $\bar{\mu} \in \{2 - 2\cos\frac{\pi}{\bar{i}_1}, 2 - 2\cos\frac{2\pi}{2\bar{j}_1+1}\}$. It is clear that $\mu = \bar{\mu}$ and so either $i_1 = \bar{i}_1$ or $j_1 = \bar{j}_1$. By deleting the same components from G and $\bar{G}$ and using induction on the number of components of G the proof is complete. $\square$

Remark There are some non-isomorphic cospectral graphs with largest Laplacian eigenvalue 4. For instance the Laplacian spectrum of each of $K_{1,3} + C_3$ or $H_1 + P_3$ is $\{0^2, 1^2, 3^2, 4\}$.

5 New family of DS graphs with respect to the adjacency matrix

In this section we show that graphs with no isolated vertex and the largest Laplacian eigenvalue less than 4 can be determined by their adjacency spectra.

Lemma 5.1 *Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r}$ be a graph of index less than 2 where $i_1 \geq i_2 \geq \cdots \geq i_r > 1$. Then G can be determined by its adjacency spectrum.*

Proof: We give the proof by induction on the number of components of G. We know that P_n is DS with respect to the adjacency matrix (see [9]). It is clear that the largest adjacency eigenvalue of P_n is $2\cos\frac{\pi}{n+1}$. Now let $\bar{G}$ be cospectral to G with respect to the adjacency matrix. So $\bar{G}$ is a graph with index less than 2 and by Corollary 2.1, it can be represented in as a linear combination of the form

$$\bar{G} = P_{\bar{i}_1} + P_{\bar{i}_2} + \cdots + P_{\bar{i}_l} + Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1T_1 + t_2T_2 + t_3T_3.$$

Using Lemma 4.1, G and $\bar{G}$ have the same number of vertices, edges and closed walks of length 4 (two times the number of edges plus four times the number of pathes of length three). We have $l = r$ and $t_1 = t_2 = t_3 = k = 0$. Without any loss of generality we assume $\bar{i}_1 \geq \bar{i}_2 \geq \cdots \geq \bar{i}_r > 1$. So we have $\lambda_1(G) = \lambda_1(\bar{G}) = 2\cos\frac{\pi}{i_1+1} = 2\cos\frac{\pi}{\bar{i}_1+1}$. Hence $i_1 = \bar{i}_1$ and by deleting the same components from G and $\bar{G}$ and using induction on number of components of G the proof is complete. □

Lemma 5.2 *Let $G = C_{i_1} + C_{i_2} + \cdots + C_{i_r}$ be a graph of index at most 2 where $i_1 \geq i_2 \geq \cdots \geq i_r > 2$. Then G can be determined by its adjacency spectrum.*

Proof: We give the proof by induction on the number of components of G. We know that C_n is DS (see [9]). It is easy to see that the second largest adjacency eigenvalue of C_n is $2\cos\frac{2\pi}{n}$. Now let $\bar{G}$ be cospectral to G with respect to the adjacency matrix. So $\bar{G}$ is a graph with index at most 2 and by Theorem 2.1, it can be represented as a linear combination of the form

$$\bar{G} = W_{s_1} + W_{s_2} + \cdots + W_{s_f} + C_{\bar{i}_1} + C_{\bar{i}_2} + \cdots + C_{\bar{i}_l} + P_{l_1} + P_{l_2} + \cdots + P_{l_s}$$
$$+ Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1T_1 + t_2T_2 + t_3T_3 + t_4T_4 + t_5T_5 + t_6T_6 + hK_{1,4}.$$

Using Lemma 4.1, G and $\bar{G}$ have the same number of vertices and edges. So we have $l = r$ and $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = f = h = k = s = 0$. Without any loss of generality we assume $\bar{i}_1 \geq \bar{i}_2 \geq \cdots \geq \bar{i}_r > 1$. Let $\lambda < 2$ be the second largest eigenvalue of G. So we have $\lambda = 2\cos\frac{2\pi}{i_1} = 2\cos\frac{2\pi}{\bar{i}_1}$. Therefore $i_1 = \bar{i}_1$ and by deleting the same components from G and $\bar{G}$ and using induction on number of components of G the proof is complete. □

Theorem 5.1 *Each graph with no isolated vertex and the largest Laplacian eigenvalue less than 4 can be determined by its adjacency spectrum.*

Proof: Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r} + C_{2j_1+1} + C_{2j_2+1} + \cdots + C_{2j_k+1}$ be a graph with the largest Laplacian eigenvalue less than 4 where $i_1 \geq i_2 \geq \cdots \geq i_r > 2$. Again we give the proof by induction on the number of components of G. If $rk = 0$ the assertion hold by Lemmas 5.2 and 5.1. Now let $\bar{G}$ be cospectral to G with respect to the adjacency matrix. Since any odd cycle does not have -2 as an adjacency eigenvalue (any add cycle is not a bipartite graph), $\bar{G}$ does not have any bipartite graph of index 2 as a component. On the other hand 2 is an adjacency eigenvalue with multiplicity k of G. So $\bar{G}$ has 2 as an eigenvalue with multiplicity k and we have

$$\bar{G} = P_{\bar{i}_1} + P_{\bar{i}_2} + \cdots + P_{\bar{i}_l} + C_{2\bar{j}_1+1} + C_{2\bar{j}_2+1} + \cdots + C_{2\bar{j}_k+1} + Z_{l_1} + Z_{l_2} + \cdots + Z_{l_t} + t_1T_1 + t_2T_2 + t_3T_3.$$

Again by Lemma 4.1, the two graphs G and $\bar{G}$ have the same number of vertices, edges and closed walks of length 4 (two times the number of edges plus four times the number of pathes of length three). So we have $l = r$ and $t_1 = t_2 = t_3 = t = 0$. Without any loss of generality we assume $\bar{i}_1 \geq \bar{i}_2 \geq \cdots \geq \bar{i}_r$, $\bar{j}_1 \geq \bar{j}_2 \geq \cdots \geq \bar{j}_k$ and $j_1 \geq j_2 \geq \cdots \geq j_k$. Let $\lambda < 2$ be the second largest adjacency eigenvalue of G. So we have $\lambda \in \{2\cos\frac{\pi}{i_1+1}, 2\cos\frac{2\pi}{2j_1+1}\}$. Since λ is the second largest adjacency eigenvalue of $\bar{G}$, we have $\lambda \in \{2\cos\frac{\pi}{\bar{i}_1+1}, 2\cos\frac{2\pi}{2\bar{j}_1+1}\}$. Hence we have $i_1 = \bar{i}_1$ or $j_1 = \bar{j}_1$ and by deleting similar components from G and $\bar{G}$ and using induction on the number of components of G the proof is complete. □

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References

[1] D. M. Cvetkovic, M. Doob and H. Sachs, *Spectra of graphs*, Third edition, Johann Abrosius Barth Verlag, 1995.

[2] R. Grone, R. Merris and V. S. Sunder, The Laplacian spectrum of a graph, *SIAM J. Matrix Anal. Appl.* **11** (1990), 218–238.

[3] J. S. Li and X. D. Zhang, On the Laplacian eigenvalues of a graph, *Linear Algebra Appl.* **285** (1998), 305–307.

[4] R. Merris, The number of eigenvalues greater than two in the Laplacian spectrum of a graph, *Portugal. Math.* **48** (1991), 345–349.

[5] newGRAPH. Available from: `http://www.mi.sanu.ac.yu/newgraph`.

[6] G. R. Omidi, On a Laplacian spectral characterization of graphs of index less than 2, *Linear Algebra Appl.* **429** (2008), 2724–2731.

[7] M. Petrovic, I. Gutman, M. Lepovic and B. Milekic, On bipartite graphs with small number of Laplacian eigenvalues greater than two and three, *Linear Multilin. Algebra* **47** (2000), 205–215.

[8] J. H. Smith, Some properties of the spectrum of a graph, in *Combinatorial Structures and their Applications*, Gordon and Breach, New York, 1970, 403–406.

[9] E. R. Van Dam and W. H. Haemers, Which graphs are determined by their spectrum?, *Linear Algebra Appl.* **373** (2003), 241–272.

[10] E. R. Van Dam and W. H. Haemers, Developments on spectral characterizations of graphs, *Discrete Math.* (to appear).

[11] X. D. Zhang, Graphs with fourth Laplacian eigenvalue less than two, *European J. Combin.* **24** (2003), 617–630.

[12] X. D. Zhang, Bipartite graphs with small third Laplacian eigenvalue, *Discrete Math.* **278** (2004), 241–253.

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