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More greedy defining sets in Latin squares

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Abstract

A Greedy Defining Set is a set of entries in a Latin square with the property that when the square is systematically filled in with a greedy algorithm, the greedy algorithm succeeds. Let $g(n)$ be the smallest Greedy Defining Set for any Latin square of order n. We give theorems on the upper bounds of $g(n)$ and a table listing upper bounds of $g(n)$ for small values of n. For a circulant Latin square, we find that the size of the smallest Greedy Defining Set is $\lfloor\frac{(n(n-1)}{6}\rfloor$.

1 Introduction

The first paper on Greedy Defining Sets appeared in Zaker [2] and dealt with these sets in graphs. Also in this paper, Greedy Defining Sets in Latin squares were defined and attributed to Eric Mendelsohn. The first results on Greedy Defining Sets in Latin squares appeared in Zaker's thesis [4]. The first paper on Greedy Defining Sets in Latin squares appeared in Zaker [3]. Since we are only discussing Greedy Defining Sets in Latin squares, we define Latin squares and then Greedy Defining Sets in Latin squares.

A *Latin square* is an $n \times n$ array whose entries are single elements from some set, N, of n elements with the property that each element appears exactly once in each row and exactly once in each column. The integer n is the *order* of the Latin square. We will also talk about the order of the GDS which is the order of the Latin square containing the GDS. In this paper, the set N will be the integers from 1 to n, inclusive. We will also label the rows and columns from 1 to n. A Latin square can also be considered as a set of 3-tuples, (i, j, k) where i is the row label, j is the column label and k is the integer or element in the (i,j) position in the Latin square. The *conjugate of a Latin square* is the Latin square produced when the coordinate positions of the triples have been permuted. There are six conjugates but they need not all be different.

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Next we need to define a greedy algorithm for putting entries in the empty cells of an Latin square of order n. The entries will be the integers from 1 to n. The cells or (i,j) positions of the Latin square are filled in from left to right and from top to bottom. Hence (1,1) is filled in first, (1,2) next and so on until (n,n) is filled. To fill in a cell, (i, j), the least integer possible is used that does not violate a Latin square property, i.e. the element put into (i, j) must be distinct from any element already filled in that row or column. If at some cell, all the integers in N are ruled out, then the algorithm fails. Zaker [3] has shown that this algorithm is successful if and only if the order of a Latin square is a power of 2. If we want the greedy algorithm to succeed for orders that are not a power of 2, then we need to fill in some of the cells in the Latin square before the algorithm is invoked. So we define a *Greedy Defining Set* (GDS) of order n and size s to be a set of s triples of a Latin square of order n that will cause the greedy algorithm to successfully fill in the complete Latin square. It is understood that the algorithm skips over the cells that are part of the defining set. Of course, the entries in the rows (columns) of a GDS must also be unique. We denote by $g(n)$ the size of the minimum GDS of any Latin square of order n. We also define the *conjugates* of a GDS similar to the conjugates of a Latin square.

The following example is a Latin square and its GDS. The elements of the GDS are enclosed by brackets.

(5)	1	2	3	4
1	2	4	5	(3)
2	3	1	4	5
3	4	5	1	2
4	5	3	2	1

Let L_2 be the Latin square of order 2 with the first row in natural order and let $\bigoplus$ be the direct product operator. We are now ready to record Zaker's first proposition.

Proposition 1.1 *$g(n)$ = 0 if and only if n is a power of 2. The Latin square so defined is $L_2 \bigoplus L_2 \bigoplus \cdots \bigoplus L_2 = L_2^n$.*

Zaker [3] defined the notion of a descent in a Latin square. A *descent* in a Latin square is a set of three cells in a Latin square, $\{(a, b, e)$, (a, c, f) and $(d, b, f)\}$ where $a < d, b < c$ and $f < e$. The element (a,b,e), which has another element of the descent in its row and another element of the descent in its column, is called the *apex* of the descent. The element of the descent in the same row as the apex is called the *hand* of the descent and the remaining element is called the *foot* of the descent. In the example above {(1,1,5),(1,2,1),(2,1,1)} and {(2,3,4), (2,5,3) (5,3,3)} are both descents with (1,1,5) and (2,3,4) being the apexes of their descents, with (1,2,1) and (2,5,3) being the hands of their descents and with (2,1,1) and (5,3,3) being the feet of their descents. Clearly, for the greedy algorithm to put a 5 into position (1,1) requires that at least one element in the first descent intersects the Greedy Defining Set. The following was proved by Zaker [3].

Theorem 1.2 *A subset of a Latin square is a Greedy Defining Set if and only if the subset intersects every descent.*

The following is the first new result and it will be useful in our computer searches.

Lemma 1.3 *Conjugacy leaves descents invariant.*

Proof If $\{(a,b,e),\ (a,c,f)$ and $(d,b,f)\}$ where $a < d, b < c$ and $f < e$ is a descent in a Latin square L, then so is $\{(a,e,b),\ (a,f,c)$ and $(d,f,b)\}$ a descent in the corresponding conjugate of L. The other 4 conjugates can also be checked. □

This immediately gives us the following theorem.

Theorem 1.4 *Let α be a particular permutation of the symbols, rows and columns. If S is the GDS of a Latin square L, then the α-conjugate of S is a GDS of of the α-conjugate of L.*

Another important idea is that often an $m \times m$ subsquare in the larger Latin square can be filled in, under certain conditions, in isolation from the rest of the square. For instance an $m \times m$ subsquare based on the elements 1 to m in the top left corner of a larger Latin square can be completed in isolation from the rest of the Latin square if the GDS does not contain any 3-tuples like (a,b,c) where $a, c \leq m$ and $b > m$ or where $b, c \leq m$ and $a > m$ or where $a, b \leq m$ and $c > m$.

In the next section we show some new minimal GDSs and prove some recursive constructions. In the following section we disprove a conjecture of Zaker's [3] and at the same time prove a conjecture of Zaker's [4] on GDSs for back-circulant Latin squares. In the last section, we print a table of best known upper bounds for GDS in Latin squares of small orders.

2 Bounds

The number of values known for $g(n)$, other than powers of 2 is quite small. Zaker computed $g(n)$ for $n \leq 6$. Using a backtrack program, we tried all possible GDSs starting at size 1 and then increasing the number of elements in our set until we find a GDS. We also give the number, $\#g(n)$, of inequivalent such minimal GDSs up to conjugacy when we know them. This table gives the intuition something to go on when contemplating GDSs.

n	1	2	3	4	5	6	7	8	9	10
$g(n)$	0	0	1	0	2	2	3	0	4	5
$\#g(n)$	1	1	4	1	30	3	4	1	30	?

Another backtrack approach by John A. Bate [1], computes the number of deflections in a Latin square. This is a position in the square that does not contain the element that normally the greedy algorithm would put there. Clearly a point of deflection must be the apex of a descent. Finding Latin squares with h (h small) apexes is straight forward — put in the apexes in all possible ways and see if the greedy algorithm completes. Of course, a Latin square of order n with the smallest number of apexes will also have a small but not necessarily smallest greedy set for Latin squares of that order. The size of the greedy set for the square constructed by Bate's algorithm would be less than or equal to h. We let the smallest number of apexes for a Latin square of order n be $h(n)$. In the following figure we add the column $g(n)$ for comparison.

n	1	2	3	4	5	6	7	8	9	10
$g(n)$	0	0	1	0	2	2	3	0	4	5
$h(n)$	0	0	1	0	2	2	4	0	5	6
# of $h(n)$	1	1	4	1	8	4	236	1	30	64

The following example is from Bate [1]. The Latin square has 4 apexes which is best possible. Its smallest greedy set has size 4 which is not best possible over all Latin squares.

1	2	3	4	5	6	7
2	1	5*	3	4	7	6
3	6*	1	2	7	4	5
7*	3	2	1	6	5	4
4	5	6	7	1	2	3
5	4	7	6	2	3*	1
6	7	4	5	3	1	2

The first recursive theorem is due to Zaker [3]. It handles the GDS for a direct product and the proof will not be given in this paper.

Theorem 2.1 *Let $n = rs$. Then $g(n) \leq r^2 g(s) + s^2 g(r) - g(s)g(r)$.*

The second recursive theorem, also due to Zaker, is now stated.

Theorem 2.2 *Let $n = 2^k - 1$ for some integer $k > 1$. Then $g(n) \leq n - k$.*

Zaker uses these two theorems to show that there are families of Latin squares in which the GDS grows at most linearly.

Next, we give a modified direct product construction when one of the subsquares is of order 2. The construction, for $n = 3$ gives Zaker's 6×6 example in his Figure 2 in [3].

Theorem 2.3 *If there exists a GDS, S, of order n and size f with a 3-tuple $(1, 1, n)$, then $g(2n) \leq 4f - 2$.*

Proof We construct G, the GDS of order $2n$ as follows: For every 3-tuple $(a, b, c) \in S$, G contains the 3-tuples (a, b, c), $(a, b + n, c + n)$, $(a + n, b, c + n)$ and $(a + n, b + n, c)$. So G defines the Latin square which is the direct product of the order n Latin square with the order 2 standard Latin square. This was proved in [3].

We will now produce G' from G as follows: delete from G, the 2×2 Latin square, L_2 with entries $(1, 1, n)$, $(1, n+1, 2n)$, $(n+1, 1, 2n)$ and $(n+1, n+1, n)$ and replace it with the entries $(1, 1, 2n)$ and $(n+1, n+1, 2n)$. The Latin square greedily defined by G', call it L', is the same as L except L_2 has been replaced by the other Latin square of order 2 on the same symbols. Now, $(1, n+1, n)$ can not be the apex of any descent as the entries to the right of it in row 1 of L' are all larger than n. Also, $(1, n+1, n)$ can not be the hand of any descent as the entries to the left of it in row 1 of L' are smaller than n, except for $(1, 1, 2n)$. Finally, $(1, n+1, n)$ can not be the foot of any descent as it is in the first row of the Latin square. Similarly, $(n+1, 1, n)$ does not intersect any descents in L' except for the one that has $(1, 1, 2n)$ as the apex. Since $(1, 1, 2n)$ and $(n+1, n+1, 2n)$ are in G', $(1, n+1, n)$ and $(n+1, 1, n)$ do not have to be in G'. The construction does produce some other descents but they all have $(1, 1, 2n)$ and $(n+1, n+1, 2n)$ as their apex. So the set G' intersects all descents in L'. □

To use Theorem 2.3 to get an improved upper bound on $g(2n)$ requires a greedy set of minimum size to contain the triple $(1, 1, n)$. This occurs for the GDSs of order 5, 6 and 10 which are listed in the Conclusion. The above theorem can be modified for a GDS of order n containing a three tuple $(a, 1, n)$ or $(1, a, n)$, where $a \neq 1$, but we only get 1 less rather than 2 less elements in G'. Examples of these GDS's are also found in the Conclusion.

Theorem 2.4 *If there exists a GDS, S, of order n and size f with a 3-tuple $(a, 1, n)$ or $(1, a, n)$, where $1 < a \leq n$, then $g(2n) \leq 4f - 1$.*

Proof We will prove the $(a, 1, n)$, $a \neq 1$ case. The proof is almost the same as the proof in Theorem 2.3 except in 2 places. The first place is that (a, i, n) is not in row 1. However, since there is only one n in the upper right quadrant, $(a, n+1, n), a \neq 1$ can not be the foot of a descent. The second place the proof diverges is that it does not work for $(a+n, 1, n), a \neq 1$ at all, so $(a+n, 1, n)$ must be in G'. □

The resulting GDS, G' in the proof of Theorem 2.3 contains $(1, 1, 2n)$ and so G' can also be used as an input to the construction. Iterating this gives the following corollary.

Corollary 2.5 *If there is a GDS of order n and size f which contains the tuple $(1, 1, n)$, then for positive k, $g(n2^k) \leq f4^k - \frac{2(4^k-1)}{3}$.*

The resulting GDS in Theorem 2.4 contains $(a, 1, 2n)$ (or $(1, a, n)$), $a \neq 1$, and the GDS can also be used as an input to the construction. Iterating this gives the following corollary.

Corollary 2.6 *If there is a GDS of order n and size f which contains the tuple $(a, 1, n)$ (or $(1, a, n)$) where $1 < a \leq n$, then for positive k, $g(n2^k) \leq f4^k - \frac{(4^k-1)}{3}$.*

We note that a conjugate of the GDS constructed in the previous results have at least 2 entries in the last row (or column) of the Latin square it defines.

We have just given a doubling construction to produce GDS's of even order so now we give a doubling construction to produce GDS's of odd order. Conceptually, we divide the big Latin square into 4 nearly equal pieces. The construction requires a GDS, G_1 of order n, a GDS, G_2, of order $n+1$ with as many elements in the last row as possible and a GDS, G_3 of order $n+1$ with a constant diagonal. Roughly speaking, G_1 goes in the top left, G_2 goes in the top right,G_2^t goes in the bottom left and a modified G_3 goes in the bottom right. More precisely, we give you the theorem and proof.

Theorem 2.7 *If there is a GDS, G_2, of order $n+1$ and size p with r elements of the GDS in the last row and there is a GDS, G_3, of order $n+1$ and size q which defines a Latin square with a constant main diagonal and if there is a GDS, G_1, of order n and size s then $g(2n+1) \leq s + 2(p-r) + n + q$.*

Proof Let G be the GDS of order $2n+1$ that we are constructing. Let L be the Latin square that is produced by G. If $a \neq n+1$, and $(a, b, c) \in G_2$, let G contain the 3-tuples $(a, b+n, c+n)$ and $(b+n, a, c+n)$. If $(n+1, b, c)$ is a 3-tuple of the last row of the Latin square defined by G_2, then G should also contain the 3-tuples $(b+n, b+n, c+n)$ if $b \neq n+1$. Finally, consider the GDS, G_3. Delete any 3-tuples (a, b, c) from G_2 that have $a = b$. Call this new set H. Let m be the constant diagonal element in the Latin square defined by G_3. If any third element in a 3-tuple of H is larger than m then subtract 1 from it. Let G contain the modified H. Now G obeys the Latin property and has the right number of elements but does it define a Latin square.

Since G does not contain 3-tuples like (a, b, c) where $a, c \leq n$ and $b > n$ or where $b, c \leq n$ and $a > n$ or where $a, b \leq n$ and $c > n$, L's top left $n \times n$ corner is filled in with a Latin square of order n. The top right corner is filled in like the first n rows of a $n+1 \times n+1$ Latin square based on the symbols $\{n+1, n+2, \ldots, 2n+1\}$. It is true that the elements, if any, of the GDS on the last row have been projected down onto the main diagonal but they still do the same defining role as they did for the first n rows of a $n+1 \times n+1$ Latin square. By symmetry, the bottom left corner is also filled in as the first n columns of a $n+1 \times n+1$ Latin square based on the symbols $\{n+1, n+2, \ldots, 2n+1\}$. Finally, the bottom right corner is filled in like the Latin square based on the symbols $\{1, 2, \ldots, n\}$ defined by the modified H, except for the diagonal. Since the diagonal (except for $(2n+2, 2n+2)$) is in G and is larger than n, this corner completes also. □

It is possible that in order to get the best result possible from this theorem, G_2 and G_3 are not minimal GDSs. If we use a minimum G_1 in the above theorem, then we can state the following corollary.

Corollary 2.8 *If there is a GDS, G_2, of order $n+1$ and size p with r elements of the GDS in the last row and there is a GDS, G_3, of order $n+1$ and size q which defines a Latin square with a constant main diagonal then $g(2n+1) \leq g(n) + 2(p-r) + n + q$.*

In Theorem 2.7, if $n+1$ is a power of 2, and G_1, G_2 and G_3 are minimal defining sets of L_2^n, then we get Theorem 2.2.

3 GDS's and Circulant Latin Squares

The first theorem of this section gives an upper bound on the size of a GDS of any order. We will do this for the GDS that defines the circulant Latin square CL $= \{(i, j, (j-i+1)(\text{mod} * n))\}$ where $(\text{mod} *n)$ is $(\text{mod } n)$ except that 0 is interpreted as n. These Latin squares are defined for all orders. In [3], Zaker conjectured that the minimum GDS for this Latin square is $\lfloor \frac{(n-1)^2}{4} \rfloor$. Unfortunately this is only true for $n \leq 4$. However, in [4], Zaker did construct a GDS of size $\lfloor \frac{n(n-1)}{6} \rfloor$ in the back-circulant Latin square and conjectured that it was the minimum for that square. Although he knew that circulant Latin squares were conjugates, he did not have the theorem that conjugacy leaves descents invariant. The following theorem is a direct consequence of Zaker's Theorem and Theorem 1.4, but since Zaker's thesis is hard to get and in Persian, the complete proof is given. All arithmetic in the proof of the following theorem is done mod $*n$.

Theorem 3.1 *The GDS for the Latin square CL of order n has size less than or equal to $\lfloor \frac{n(n-1)}{6} \rfloor$.*

Proof The following is a GDS for the Latin square CL. It consists of three triangles of entries within the Latin square which we will call the *top*, *side* and *bottom* triangles for obvious reasons. The top triangle consists of the tuples: $\{(i, j, j-i+1) \mid i = 2, 3, \ldots, \lfloor \frac{n+3}{3} \rfloor; j = 1, 2, \ldots i-1\}$. The side triangle consists of tuples: $\{(i, j, j-i+1) \mid i = \lfloor \frac{n+6}{3} \rfloor, \ldots, \lfloor \frac{2n+1}{3} \rfloor; j = n, n-1, \ldots n - \lfloor \frac{2n+1}{3} \rfloor + i\}$. The bottom triangle consists of the tuples: $\{(i, j, j-i+1) \mid i = \lfloor \frac{2n+7}{3} \rfloor, \ldots, n; j = 1, 2, \ldots i - \lfloor \frac{2n+4}{3} \rfloor)\}$. Since the triangles do not have any elements in common, the number of elements in the triangles is $\frac{(\lfloor \frac{n}{3} \rfloor+1)\lfloor \frac{n}{3} \rfloor}{2} + \frac{(\lfloor \frac{n-1}{3} \rfloor+1)\lfloor \frac{n-1}{3} \rfloor}{2} + \frac{(\lfloor \frac{n-2}{3} \rfloor+1)\lfloor \frac{n-2}{3} \rfloor}{2} = \lfloor \frac{n(n-1)}{6} \rfloor$.

If the numbers to the left of the 1's in the rows of CL are forced correctly then the 1's and the numbers to the right of the 1s in the rows will clearly also be forced correctly. So the first $\lfloor \frac{n+3}{3} \rfloor$ rows are obviously filled in correctly. So consider the greedy algorithm when it fills the position $(\lfloor \frac{n+3}{3} \rfloor + 1, 1)$. There are $\lfloor \frac{n}{3} \rfloor$ defining set elements in that column from the top triangle, $\lfloor \frac{n-1}{3} \rfloor$ defining set elements in that row from the side triangle $\lfloor \frac{n-2}{3} \rfloor$ defining set elements in that column from the bottom triangle and there is a 1 already forced in that column in position $(1, 1)$. All these elements are distinct. That leaves only $n - \lfloor \frac{n}{3} \rfloor - \lfloor \frac{n-1}{3} \rfloor - \lfloor \frac{n-2}{3} \rfloor - 1 = 1$ element left and hence that element, $\lfloor \frac{2n+2}{3} \rfloor$, must go in that position.

When the algorithm fills in the next position in that row, $(\lfloor \frac{n+3}{3} \rfloor + 1, 2)$, the set of elements in that row and column coming from the Greedy Defining Set or already placed by the greedy algorithm is almost the same as when the algorithm filled in the previous position, $(\lfloor \frac{n+3}{3} \rfloor + 1, 1)$. The elements in column 2 of the bottom triangle are the same as the elements in column 1 of the bottom triangle except that there is an element, 2, in column 1 but not in column 2 of the bottom triangle. But 2 has already been placed by the algorithm in position $(1, 2)$. Also the elements in column 2 of the top triangle are almost the same as the elements in column 1 of the upper triangle except that there is one element in column 1 that is not in column 2 of the top triangle, i.e., element $\lfloor\frac{2n+2}{3}\rfloor + 1$ in position $(\lfloor\frac{n+3}{3}\rfloor, 1)$. In the row $\lfloor\frac{n+3}{3}\rfloor + 1$, the only difference in the elements between the two cases is that the algorithm has added the element $\lfloor\frac{2n+2}{3}\rfloor$ in position $(\lfloor\frac{n+3}{3}\rfloor + 1, 1)$. Hence there is exactly one element missing from row $\lfloor\frac{n+3}{3}\rfloor + 1$ and column 2 and the element $\lfloor\frac{2n+2}{3}\rfloor + 1$ must be filled in by the algorithm in position $(\lfloor\frac{n+3}{3}\rfloor + 1, 2)$. This same process continues to fill in the row correctly as the greedy algorithm fills in row $\lfloor\frac{n+3}{3}\rfloor + 1$ from left to right until it fills in the 1 and from there the rest of the row is obviously filled in correctly.

When the greedy algorithm starts the next row in position $(\lfloor\frac{n+3}{3}\rfloor + 2, 1)$, the elements in column 1 and row $\lfloor\frac{n+3}{3}\rfloor + 2$ that are in the GDS or already placed by the algorithm is almost the same as when the algorithm was about to fill $(\lfloor\frac{n+3}{3}\rfloor+1, 1)$. The only difference is that the element $\lfloor\frac{2n+2}{3}\rfloor$ in $(\lfloor\frac{n+3}{3}\rfloor + 1, 1)$ has been added and an element, $\lfloor\frac{2n+2}{3}\rfloor - 1$, from the side triangle in position $(\lfloor\frac{n+3}{3}\rfloor + 1, n)$ has been dropped. The two elements are different from each other and different from the other elements in the top or bottom triangle in column 1 and side triangle in row $\lfloor\frac{n+3}{3}\rfloor + 2$. Hence again there is only one choice for the element to go in position $(\lfloor\frac{n+3}{3}\rfloor + 2, 1)$, i.e., $\lfloor\frac{2n+2}{3}\rfloor - 1$. The analysis of the algorithm correctly placing the rest of the elements in row $\lfloor\frac{n}{3}\rfloor + 2$ is similar to the previous row. This continues until the last row is filled in. There is a slight perturbation in the algorithm due to the bottom triangle is already defined but this causes no problems. □

In order to change the inequality to an equality in the above theorem we need to study the descents in CL.

First, we will count the number of descents through each position of the Latin square and put these numbers in an order n matrix, called the *counting matrix*. As a default let every integer in the counting array be initialized to 0. We will fill in the non-zero entries in the columns of the counting square a pair of positions at a time. In column i, for $i \leq \lfloor(n-1)/2\rfloor$, positions ($i+1$,i) and ($n$,i) have integer $n-2i$. For a while these entries decrease by 1 as we fill in the pairs in the column moving towards the center. This happens $i-1$ times, i.e., for column 1 it does not happen, that is, column 1 is a constant $n-2$ except for $(1, 1)$ which is 0: for column 2 it happens once, that is, $(3, 2)$ and $(n, 2)$ have symbol $n-4$, $(4, 2)$ and $(n-1, 2)$ have symbol $n-5$ and the rest of the positions in between these positions in column 2 have symbol $n-5$ also. The partial columns containing these non-zero integers are called the southwest corner of the counting matrix. We will also use the term southwest corner for the same partial columns in the Latin square and for the GDS. More particular, for $1 \leq i \leq \lfloor(n-1)/2\rfloor$, columns i from row $i+1$ to row n inclusive form the southwest corner of a square. See the figure below for an example.

There is another region of non-zeroes in the counting square which we call the northeast corner of the counting matrix. In row i, $2 \leq i \leq n-1$ the non-zero entries are from column $i+1$ to column n inclusive. The first integer on the left in a row is always 1. The entries from left to right in a row then increase by 1 until they are equal to $i-1$ at which point they stay constant. Of course, the bottom rows never reach the point where the entries stay constant. Again check the figure for guidance. Note that the southwest and northeast corners do not overlap.

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

Latin square CL(7)

0	0	0	0	0	0	0
5	0	1	1	1	1	1
5	3	0	1	2	2	2
5	2	1	0	1	2	3
5	2	0	0	0	1	2
5	2	0	0	0	0	1
5	3	1	0	0	0	0

Counting Matrix for CL(7)

We first count the number of descents in the Latin square. Since any descent has a hand in the northeast corner and since the northeast corner contains only the hands of descents, we just need to add up the numbers in the northeast corner to get the number of descents in the Latin square. For order n, the column sums of the northeast corner are $1, 2, 4, 6, 9, \ldots, \lfloor \frac{(n-1)^2}{4} \rfloor$. Hence the number of descents in the Latin square is:

$$\sum_{i=1}^{n-1} \lfloor \frac{i^2}{4} \rfloor = \sum_{i=1}^{n-1} \frac{i^2}{4} - \frac{\lfloor \frac{n}{2} \rfloor}{4} = \lceil \frac{n(n-2)(2n+1)}{24} \rceil.$$

We record this as a proposition.

Proposition 3.2 *The number of descents in CL of order n is:* $\lceil \frac{n(n-2)(2n+1)}{24} \rceil$.

So we want to find the minimum sized GDS that intersects each one of the $\lceil \frac{n(n-2)(2n+1)}{24} \rceil$ descents. Before we can do that, we need to maximize the number of distinct descents that a set of positions in a column in the southwest corner of CL can intersect. We find that the best way to do this is to greedily pick positions with the largest values in that column of the counting matrix of the southwest corner of CL. To prove that this selection in the row is optimal we show that any better solution leads to a contradiction.

Lemma 3.3 *Any i elements confined to column j of the southwest corner of the Latin square, CL, of order n can intersect at most $(n-2j)+(n-2j-1)+\cdots+(n-2j-i+1) = h$ distinct descents.*

Proof If we choose a elements from the top a rows of column j and $a-i$ elements from the bottom $a-i$ rows of column j, we find these i elements intersect h distinct descents. To prove this let us examine column j in the counting matrix. From the top, column j has j 0 entries followed by entries $n-2j$, $n-2j-1$, $n-2j-2$, and so on until the minimum of $n-3j+1$ or 0 is reached. From the bottom column j has entries $n-2j$, $n-2j-1$, $n-2j-2$, and so on until the minimum of $n-3j+1$ or 0 is reached. All other entries in the middle of column j have the value min($n-3j+1$,0).

The cell $(j+1, j)$ in the southwest corner of CL intersects $n-2j$ descents in the apex which are those descents that are also intersected in the foot by the bottom $n-2$ elements in column j. If $(j+1, j)$ is chosen as one of the i elements from column j, then this choice contributes $n-2$ distinct descents to the sum of the distinct descents intersected by the i elements from column j. Since we do not want to count descents twice,, we should delete from the values in the counting matrix the descents already intersected by the top non-zero element in column j. When we do this, column j becomes, starting at row $j+1$: $0, n-2j-1, n-2j-2, \ldots, n-3j, n-3j, \ldots, n-3j, n-3j+1, \ldots, n-2j-1$. The only change is that there is one less non-zero at the top of the column and the bottom $n-2j$ entries are reduced by 1. This has the same form as what we started with except when we now pick the remaining top non-zero integer in column j, the number of descents is one less. The same is true if we consider the bottom entry of column j. Continuing in this fashion picking from the top or bottom non-zero from column j, we see that if the set of positions consists of the top a elements and bottom $i-a$ elements of column j, then h distinct descents can be intersected.

But is h the maximum number of distinct descents that can be intersected by i elements from column j in the southwest corner of CL? Let us assume that there is a maximum configuration of i elements in column j that intersects more than h distinct descents. Let us pick one, C, that has a elements from the top a rows in column j and b elements from the bottom b rows of column j such that $c=a+b$ is as large as possible but $a+b<i$. Let us call those c entries in column j the *start* of the configuration. The size of the start will be denoted by c. We will show that there is a configuration with the same maximum value for the number of descents intersected but has a larger start, larger by 1. This contradiction will prove the lemma.

Consider the elements of column j in the southwest corner of the Latin square, and all the elements of the descents that they intersect. The elements making up these descents are column j along with a subset of the northeast corner of the matrix which are all hands of the descent whose apex is in column j. We give an example to make this clear. Here, $n=15$ and $j=3$.

15				4	5	6	7	8	9	10	11	12
14					4	5	6	7	8	9	10	11
13						4	5	6	7	8	9	10
12							4	5	6	7	8	9
11								4	5	6	7	8
10									4	5	6	7
9										4	5	6
8											4	5
7												4
6												
5												
4												

Part of CL of order 15

9
8
7
7
7
7
7
7
7
7
8
9

column 3 of counting matrix

Consider an element like 10 in column $j = 3$. It intersects 4 descents for which it is the apex and whose hands are the other 4 elements in the row of the matrix, i.e., $4, 5, 6, 7$. But it also intersects 3 descents for which it is the foot, i.e., the other element 10's on a diagonal in the northeast corner of the matrix. are the corresponding hands of the descents. So as to not overcount the descents that intersect C, we will count the intersections by attaching an intersected descent to an element in column j in C. If the descent is intersected by elements in column j in both the apex and foot then the descent is attached to the apex element. So, in the example, let C contain the following elements in column three, 15, 10 and 6. There would be 9 descents attached to element 15 in column 3 as 15 is the apex for 9 descents), there would be 6 descents attached to element 10 in column 3 as 10 is the apex for 4 descents and is the foot for 3-1=2 unattached descents and there would be 5 descents attached to element 6 in column 3 as 5 is the foot for $7 - 2 = 5$ unattached descents. So C intersects $9 + 6 + 5 = 20$ distinct descents.

Consider C in full generality again. Let the elements $n, n-1, \ldots, n-a+1$ in column j and the elements $j+1, j+2, \ldots, j+b$ in column j be the start of C. The start may be null. The elements $n-a$ and $j+b+1$ in column j are not in C. Let element d in row i and column j be the element of C with the smallest value of column j such that d is not in the start of C. Then $j+b+1 < d < n-a-1$. The idea is to form a new set C' from C by replacing the element d with the element $n-a$. If this can be done without decreasing the intersection count we have a contradiction.

We will count the descents by attached descents to avoid double counting. Divide column j into 3 parts depending on the counting matrix. The part of column j where the counting matrix decreases we call the *downslope*, the part where it stays constant we call the *flat* and the part where it increases we call the *upslope*. In the figure holding the example $n = 15$ and $j = 3$, the downslope consists of the first 3 positions shown in column 3, The upslope consists of the last 3 positions in column 3. The flat consists of the positions between the upslope and the downslope. There are 4 cases to consider depending on which part of column j the element d and its replacement reside in.

Case 1: Assume that d is in the upslope. Then so is $n-a$. Since d and $n-q$ are in the upslope, neither one is the foot of a descent. So if d is replaced by $n-a$ to form C', the number of attached descents goes up from $d-2j$ to $n-a-2j$ for an increase of $n-a-d$. Note that if d was the apex for a descent whose foot was $x \in C$, then $n-a$ is the apex for a descent whose foot is the same x. So this does not affect the count of intersected distinct descents. But it is also possible that $n-a$ is also the apex for some extra descents whose feet are elements of C but these descents do not intersect d in column j. This will affect the count when C' is formed. But there are at most $(n-a)-d = n-a-d$ of them. In order to avoid double-counting when counting intersections for C, we must subtract this number from $n-a-d$. But this still means that the number of distinct descents intersected by C does not go down when d is replaced by $n-a$ to form C'.

Case 2: If d is in the downslope, then we consider d'. Let d' be the element of C in row i and column j with the largest value of i such that d is not in the start of C. We replace d' with $b+j+1$. This case is now symmetric to case 1.

Case 3: Let d be in the flat and let $n-a$ in the upslope. Recall that C has no elements between $n-a$ and d. The element d intersects $d-2j$ descents in the apex and $n-j+1-d$ descents in the foot. However some of these latter descents may be attached to other elements in C and so may not be attached to d. This number is $\min(n-j+1-d, a)$. So the number of descents attached to d is $d-2j+n-j+1-d-\min(n-j+1-d,a) = n-3j+1-\min(n-j+1-d,a)$. The element $n-a$ intersects $n-a-2j$ descents in the apex and 0 descents in the foot. So the number of descents attached to $n-a$ is $n-a-2j$. So when d is replaced by $n-a$, the number of attached descents increases by $n-a-2j-(n-3j+1-\min(n-j+1-d,a)) = \min(j-1, n-a-d)$. But some of the descents attached to $n-a$ in C' may originally have been attached to elements lower in column j in C, i.e., intersected in the foot. There are $n-a-d$ more elements in the row (possible hands) containing $n-a$ in column j, then in the row containing d in column j. But these elements must have a value between $d-1$ and $d-j+1$, so there are at most $j-1$ of them. So when we count the number the distinct descents intersected by elements in column j in C', we must subtract $\min(j-1, n-a-d)$ from the previous number, $\min(j-1, n-a-d)$. The net affect is always that the count for C' stays the same as the count for C.

Case 4: Let d and $n-a$ both be in the flat. The only difference between this case and the previous case is that $n-a$ is the foot of several descents but they do not contribute to $n-a$'s count as they have been attributed to elements above $n-a$ in column j. The rest of the argument is the same.

So in each case, we can modify C to make a larger start without decreasing the number of distinct descents intersected by elements in column j of C. This is a contradiction. □

So now we know that for i elements in column j of the southwest corner of the Latin square, the maximum number of intersected distinct descents is $(n-2j)+(n-2j-1)+\cdots+(n-2j-i+1)$. Let the last(smallest) summand in column j be s_j. So if there are $i \geq 0$ elements in column j of the southwest corner then $s_j = (n-2j-i+1)$. But how many elements should we choose for a particular column in order to maximize the number of distinct descents intersected by elements of the GDS only from the southwest corner given that the number of elements in the southwest corner of the GDS is fixed?

First we note, that each column, j of an optimal southwest corner of the GDS should intersect $(n-2j)+(n-2j-1)+\cdots+(n-2j-i+1)$ distinct descents. Suppose this was not true. Then, since elements from different columns of the southwest corner of the GDS intersect distinct descents, we could replace the non-optimal configuration of column j with an optimal configuration. Then we would have the southwest corner of the new GDS intersect more distinct descents than the optimal configuration. This is a contradiction. So we may assume that an optimal southwest corner of the GDS has each column intersecting $(n-2j)+(n-2j-1)+\cdots+(n-2j-i+1)$ distinct descents. So $s_j = (n-2j-i+1)$ is defined for each column of an optimal southwest corner of a GDS. Then the problem becomes how many elements of the southwest corner of fixed size of the GDS should be in each column of the southwest corner. The next lemma answers this obliquely. Note that optimal refers to a set that intersects the maximum number of distinct descents.

Lemma 3.4 *Let a GDS have a fixed number of elements in the southwest corner of the Latin square, CL. Any optimal southwest corner of a GDS has the property that for j a column of the southwest part of the Latin square containing at least one element of the GDS and for $j\prime$ any column of the southwest corner of the Latin square, $s_{j\prime} - s_j \leq 1$.*

Proof Consider an optimal southwest corner of a GDS that does not obey $s_{j\prime} - s_j \leq 1$ for j a column of the southwest part of the Latin square containing at least one element of the GDS and for $j\prime$ any column of the southwest corner of the Latin square. Then there are two columns from the southwest part of the non-optimal GDS where $s_{j\prime} - s_j \geq 2$ with j a column containing at least one element of the GDS. But then if the entry in column j of the southwest part of the GDS corresponding to s_j is replaced by an entry in column $j\prime$ of the southwest corner of the optimal GDS. According to Lemma 3.3, the largest number of distinct descents that the new entry in column $j\prime$ could intersect is $s_{j\prime} - 1$. So the total number of descents that the southwest corner of the GDS intersects (after the switch) is greater than or equal to $s_{j\prime} - s_j - 1 \geq 1$. This is a contradiction. □

So we know what an optimal southwest corner of a GDS looks like. But what about the northeast corner. If we consider only the northeast corner (forgetting for the moment the southwest corner) then this is an easy question.

Lemma 3.5 *Let a GDS have a fixed number, t of elements in the northeast corner of the Latin square, CL. Then the number of distinct descents intersected by elements in the northeast corner of the GDS can be maximized by choosing the entries that correspond to the t largest numbers in the northeast part of the counting matrix for CL.*

Proof Since we know that the descents intersected by elements in the northeast corner of the GDS are distinct then we can maximize the number of distinct descents intersected by a fixed number, say t, of elements of the northeast corner by choosing elements which correspond to the t largest elements from the northeast corner of the counting matrix. □

Let r_t be the t^{th} largest number in the northeast corner of the counting matrix for CL.

Although we know what the optimal GDS (assuming no descent intersects both a northeast and southwest element of the GDS) looks like in the southwest and northeast corners assuming that the numbers in each corner are fixed, we do not know how many elements in the GDS should be in the southwest corner and how many should be in the northeast corner to maximize the number of distinct descents intersected by a GDS of fixed size. If the number of elements in the GDS is fixed, then some elements of the GDS from one corner can be moved to the other corner in order to get more distinct descent intersected by the GDS. Let us examine the following particular situation.

Assume all the s_j associated with the southwest corner are equal to x and r_t is equal to x or $x-1$. Then if m elements of the GDS are moved from the northeast corner to the southwest corner, we lose at least $m(x-1)$ descents that intersect elements of the GDS from the northeast corner and gain at most $m(x-1)$ descents intersecting elements from the southeast corner. Hence the number of distinct intersected descents is the same or decreases. If m elements of the GDS are moved from the southwest corner to the northeast corner, we lose at least mx descents that intersect elements of the GDS from the southwest corner and gain at most mx descents intersecting elements from the northeast corner. Again the number of distinct descents is the same or decreases. Clearly, the original configuration maximizes the number of distinct descents intersected by a GDS of a certain size if we ignore that some descents may be intersected by elements of the GDS from both the northeast and southwest corners. So if we have a GDS which meets the criterion in the beginning of the paragraph and the descents intersected by the elements of the GDS in the northeast corner are distinct from the descents intersected by the elements from the southwest corner then that GDS is optimal for its size.

If we examine the GDS constructed in Theorem 3.1, each column in the southwest corner is optimized. Further the whole southwest corner is optimized as the s_j associated from columns in the southwest corner are equal. It is also true that $s_j = r_t$ or $s_j = r_t + 1$ so that the whole GDS is optimized (ignoring that fact that some descent may be intersected by an element from each corner). Further, in this instance, the descents intersected by the elements of the GDS in the northeast are distinct from the descents intersected by the elements from the southwest. So the number of distinct intersected descents are truly maximized. The number of distinct descents that are intersected by the elements of this GDS is $\lfloor \frac{n(n-1)}{6} \rfloor$. If we consider a smaller GDS, then its elements would intersect fewer distinct descents and hence would not be a GDS. Since the GDS constructed in Theorem 3.1 has size $\lfloor \frac{n(n-1)}{6} \rfloor$, then $\lfloor \frac{n(n-1)}{6} \rfloor$ is optimal. Let us state this in the next theorem.

Theorem 3.6 *The minimum size for a GDS in the Latin square CL of order n is* $\lfloor \frac{n(n-1)}{6} \rfloor$.

Now the proof of this theorem could be done as a typical proof of the optimality of a typical greedy algorithm. The algorithm would always choose the next element to go in the greedy GDS as the one that intersects the most descents that are not already intersected by an element from the same corner, i.e.: southwest corner. There are many ties usually which lead to many optimal GDS's including the one from Theorem 3.1. The GDS from Theorem 3.1 does not have any descents that intersect elements from the two corners of the GDS so it really is optimal and some of the other GDS's do and are not really optimal.

4 Conclusion

We now produce a table of upper bounds for $g(n)$ for $n \leq 30$. We list the order, the best known upper bound of $g(n)$, a reference to a construction or program, the specialized input GDSs needed in the constructions and the specialized output GDSs produced by the constructions. In column 4, the first number is the order of the GDS. ig means that the GDS contains i triples that are in the last column of the Latin square defined by the GDS, a or $a*$ means that the GDS of order n contains a tuple $(1, 1, n)$ or $(j, 1, n)$, d means that the GDS defines a Latin square with a constant main diagonal and ri (or ci) means that that GDS defines a Latin square which has the i^{th} row (or column) in natural order. This means that the GDS has a conjugate GDS which defines a Latin square with a constant main diagonal. A number in brackets denotes a GDS's size if it is not the known minimal size. If column 5, the entry in brackets following a GDS is a conjugate of that GDS. If the entry in the second column has a * beside it, it is the value of $g(n)$.

We now give the GDS's of small order that the above table requires. In order to save space we will give them as a set of triples.
GDS: 5-1gr1c1 :{(2,2,5),(3,5,1)}
GDS: 5-1gar2 :{(1,1,5),(3,5,3)}
GDS: 6-2gr1c1 :{(2,6,1),(4,6,4)}
GDS: 6-ar2c2 :{(1,1,6),(4,4,6)}
GDS: 6-2gc1(3):{(1,6,2),(4,4,2),(5,6,4)}
GDS: 7-a*r1 :{(2,1,6),(3,1,7),(5,5,4)}
GDS: 9-1gdc1 :{(1,9,3),(3,7,4),(4,8,4),(6,5,6)}
GDS: 10-3ga :{(1,1,10),(2,10,3),(3,10,1),(6,6,7),(6,10,6)}

(Table on next page.)

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n	$\|GDS\|$	Authority	input	output
4	0*	Prop.1.1		dr1c1
5	2*	program		1gr1c1, 1gar2
6	2*	program		2gr1c1(2gd), ar2
6	3	program		2gc1
7	3*	program		a*r1(1gd)
8	0*	Prop.1.1		dr1c1
9	4*	program		1gdc1(a*r1)
10	5	program		3ga
10	6	Th.2.3	5-1gar2	2gar2(2gd)
11	9	Th.2.7	5-a, 6-2g, 6-2gd	2ga
11	11	Th.2.7	5c1-, 6-2gc1(3), 6-2gd	2gc1(2gd)
12	6	Th.2.3	6-ar2	ar2(4gd)
13	15	Th.2.7	6-a, 7-1g, 7-1gd	1ga
13	17	Th.2.7	6-r1, 7-r1,7-d	r1(d)
14	11	Th.2.4	7-a*r1	a*r1(2gd)
15	10	Th.2.7	7-a*r1,8-r1,8-d	a*r1(1gd)
16	0*	Prop.1.1		
17	18	Th.2.7	8-, 9-1g, 9-1d	
18	15	Th.2.4	9-a*	
19	23	Th.2.7	9-, 10-3g, 10-d(6)	
20	18	TH.2.3	10-a	
21	40	Th. 2.7	10-, 11-2g, 11-d(11)	
22	34	TH.2.3	11-a	
23	30	Th.2.7	11-, 12-4g, 12-d	
24	22	Th.2.3	12-a	
25	63	Th.2.7	12-, 13-1g, 13-d(17)	
26	58	Th.2.3	13-a	
27	57	Th.2.7	13-, 14-2g, 14-d	
28	43	Th.2.4	14a*	
29	53	Th.2.7	14-, 15-1g, 15-d	
30	39	Th.2.4	15-a*	

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