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Covering separating systems and an application to search theory

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Abstract

A Covering Separating System on a set X is a collection of blocks in which each element of X appears at least once, and for each pair of distinct points $a, b \in X$, there is a block containing a and not b, or vice versa. An introduction to Covering Separating Systems is given, constructions are described for a class of minimal Covering Separating Systems and an application to Search Theory is presented.

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1 Introduction

A Covering Separating System (KSS) is a type of combinatorial design which arises naturally in Search Theory in the context of non-adaptive group testing, to determine whether a defective element exists, and if so, to find it efficiently.

In Section 2 the concept of a minimal KSS is defined along with some fundamental results on KSSs. Section 3 presents some theorems on the sizes of minimal KSSs, some constructions for minimal KSSs, and shows that a simple lower bound is achieved for most KSSs in a particular class. Section 4 uses the notion of "near-symmetry" to extend the results of the previous section to a larger class of KSSs. The final section, Section 5, describes one of the contexts in which KSSs arise in Search Theory.

Throughout this paper $k < n$ will denote positive integers, $[n]$ the set $\{1, 2, \ldots, n\}$ and $\mathcal{K} \subseteq 2^{[n]}$. The **size** of a collection of blocks $\mathcal{K}$ is the cardinality $|\mathcal{K}|$ of $\mathcal{K}$ and its **volume** is $V(\mathcal{K}) = \sum_{A \in \mathcal{K}} |A|$.

A point $a \in [n]$ is said to be **separated** from $b \in [n]$ if there is a block in $\mathcal{K}$ which contains a but not b. A **separating system** on $[n]$ (an (n)SS or SS) is a collection of blocks of $[n]$ in which for any pair of distinct points $a, b \in [n]$, either a is separated from b or b is separated from a. Separating Systems were defined by Rényi [5].

Separating Systems will often be represented by arrays. The rows of the array represent the blocks of the SS. Hence the terms "rows" and "blocks" will be used interchangeably.

An (n)**covering separating system (or** (n)**KSS)** is an (n)SS in which the set $[n]$ is covered, meaning each point of $[n]$ occurs at least once. A KSS in which each block has cardinality k is called an (n, k)**covering separating system, or** (n, k)**KSS**.

2 Covering Separating Systems

The minimum size of a KSS is given by the function K with appropriate parameters: define

$K(n) = \min\{|\mathcal{K}| : \mathcal{K} \text{ is an } (n)KSS\}$ and $K(n, k) = \min\{|\mathcal{K}| : \mathcal{K} \text{ is an } (n, k)KSS\}$.

Theorem 1. *The size of a minimal (n)KSS is given by*

$$K(n) = \lfloor \log_2(2n) \rfloor.$$

Proof. A minimal (n)KSS is a minimal $(n + 1)$SS. If a minimal $(n + 1)$SS covers $[n + 1]$ then another minimal $(n + 1)$SS of the same size that does not cover $[n + 1]$ can be constructed by removing every occurrence of $n + 1$. Therefore $K(n)$ is the minimum size of an $(n + 1)$SS, which by Rényi [5] is $\lceil \log_2(n + 1) \rceil$. Hence $K(n) = \lceil \log_2(n + 1) \rceil = \lfloor \log_2(2n) \rfloor$. □

Let $\mathcal{K}$ be an (n)KSS of size r. A point $i \in [n]$ is a p**-point** of $\mathcal{K}$ if it occurs in exactly p blocks of $\mathcal{K}$. As $\mathcal{K}$ must cover the set $[n]$, every point is a p-point for some

$p \geq 1$. The KSS $\mathcal{K}$ can contain at most $\binom{r}{p}$ p-points, for otherwise there would be at least two p-points which were not separated.

For fixed r, to construct an (n)KSS of size r for the largest possible n, one must use as many p-points as possible for each p. That is, use $\binom{r}{p}$ p-points for $1 \leq p \leq r$. This gives an (n)KSS for $n = 2^r - 1$. Up to isomorphism this is the unique $(2^r - 1)$KSS with r blocks. It has volume $r2^{r-1}$ and is also a $(2^r - 1, 2^{r-1})$KSS. Define $\mathcal{F}_r$ to be this "full" KSS.

For example, $\mathcal{F}_4$ is the (n)KSS with largest n of size 4:

$$\begin{array}{cccccccc} 1 & 5 & 6 & 7 & 11 & 12 & 13 & 15 \\ 2 & 5 & 8 & 9 & 11 & 12 & 14 & 15 \\ 3 & 6 & 8 & 10 & 11 & 13 & 14 & 15 \\ 4 & 7 & 9 & 10 & 12 & 13 & 14 & 15 \end{array} \tag{1}$$

For $2^{r-1} \leq n \leq 2^r - 1$, an (n)KSS can be constructed using r blocks by systematically arranging $\binom{r}{1}$ 1-points, $\binom{r}{2}$ 2-points, and so on, until all of the points of $[n]$ have been used.

The discussion above, and the next lemma, help to determine some values of $K(n, k)$ in Section 3.

Lemma 2. *Let $\mathcal{K}$ be an (n)KSS with $|\mathcal{K}| = r$. Then*
(i) *there is at most one 1-point in each block of $\mathcal{K}$,*
(ii) *there are at most $(r - 1)$ 2-points in each block of $\mathcal{K}$,*
(iii) $V(\mathcal{K}) \geq 2n - |\mathcal{K}|$.

Proof. If part (i) did not hold then $\mathcal{K}$ would not separate $[n]$. For part (ii), the 2-points of a block must be separated by the remaining $r - 1$ blocks. As they appear as 1-points in these $r - 1$ blocks, by part (i) there can be at most $r - 1$ such points.

Part (iii) follows directly from part (i) and the fact that each of the other points must appear in at least 2 blocks. □

3 Minimum size (n, k)KSSs

This section considers minimum size (n, k)KSSs, beginning with the derivation of some bounds on the size of an (n, k)KSS.

Lemma 3. *The size $K(n, k)$ of a minimal (n, k)KSS satisfies:*
(i) $K(n, k) \geq \left\lceil \dfrac{2n}{k+1} \right\rceil$,
(ii) $K(n, k) \geq \lfloor \log_2(2n) \rfloor$,
(iii) $K(n, k) < n$ *whenever* $k \geq 2$.

Proof. (i) The volume of an (n, k)KSS with r blocks is rk. An upper bound on n is obtained by taking a 1-point in each block and assuming that the other $r(k - 1)$ places are taken by 2-points. Thus

$$n \leq r + \frac{r(k-1)}{2}.$$

Rearranging shows that $\frac{2n}{k+1} \leq r$. Hence

$$K(n,k) \geq \left\lceil \frac{2n}{k+1} \right\rceil .$$

(*ii*) This follows from Theorem 1 because $K(n,k) \geq K(n)$.
(*iii*) The array shown is an (n,k)KSS with $n-1$ blocks.

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k \\ 2 & 3 & 4 & \dots & k+1 \\ 3 & 4 & 5 & \dots & k+2 \\ \vdots & & & & \vdots \\ n-1 & n & 1 & \dots & k-2 \end{array} \qquad (2)$$

Therefore $K(n,k) < n$. ☐

A Completely Separating System on $[n]$ (an (n)CSS) is an (n)KSS in which for all distinct $a, b \in [n]$, a is separated from b and b is separated from a. The block sizes of a CSS can be restricted in the same way as for a KSS, giving an (n,k)CSS. The minimum size of an (n,k)CSS is denoted $R(n,k)$. There are many results about minimal Completely Separating Systems, see for example [4], [6] and [7].

The following theorem provides a useful relationship between the size of a minimal (n,k)KSS and that of a minimal $(n,k)CSS$.

Theorem 4. *For* $2 \leq k < n$,

$$K(n,k) \leq R(n,k) - 1.$$

Proof. Let $\mathcal{C}$ be a minimal $(n,k)CSS$ and for some $A \in \mathcal{C}$ let $\mathcal{K} = \mathcal{C} \setminus A$. Since each point of $[n]$ occurs at least twice in any (n,k)CSS, $k > 1$ (see [4]), $\mathcal{K}$ covers $[n]$. For distinct $a, b \in [n]$ there exist blocks $A, B \in \mathcal{C}$ such that $a \in A$, $b \notin A$, $b \in B$ and $a \notin B$. Even if one of A or B is removed, a will still be separated from b or vice versa. ☐

3.1 Some values of $K(n,k)$

In this section some values of $K(n,k)$ are determined. In each case a construction for a class of (n,k)KSSs, is given.

The first values considered are the "diagonal" values $n = 2k$. These are diagonal in the sense that they are the median value of k for each even n.

Lemma 5. *For* $k \geq 1$, $K(2k,k) = 2 + \lfloor \log_2 k \rfloor$.

Proof. The case $k = 1$ is easily checked. Assume that $k > 1$. Let $\mathcal{K}$ be the $(2^r - 2, 2^{r-1} - 1)$KSS of size r obtained from $\mathcal{F}_r$ by removing the r-point. Define a pairing of the points of $[2^r - 2]$ by the isomorphism $\phi : [2^r - 2] \to [2^r - 2]$ with $\phi(x) \mapsto y$ where y is the unique point in all blocks of $\mathcal{K}$ not containing x. Removing the pair $x, \phi(x)$ from $\mathcal{K}$ gives a $(2^r - 4, 2^{r-1} - 2)$KSS. Removing i such pairs for $0 \leq i \leq 2^{r-1} - 2$ gives a $(2^r - 2 - 2i, 2^{r-1} - 1 - i)$KSS of size r.

When $k < 2^{r-1}$, that is $\log_2 k < r-1$, there is a $(2k, k)$KSS of size r. In particular, when $\lfloor \log_2 k \rfloor = r-2$ there is a $(2k, k)$KSS of size r. Hence $K(2k, k) \leq 2 + \lfloor \log_2 k \rfloor$.

For $2^{r-1} + 2 \leq 2k \leq 2^r - 2$ this KSS is minimal by part (ii) of Lemma 3. For $n = 2k = 2^{r-1}$ the KSS described is also minimal because an (n)KSS of size r does not exist for $n > 2^{r-1} - 1$. □

The lower bound for $K(n, k)$ given in Lemma 3(i), is achieved for $k = 1, 2$.

Lemma 6. *For all* $n > 1$

$$K(n, 1) = n,$$

and for all $n \geq 3$

$$K(n, 2) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. The first statement is clear. Lemma 3 with $k = 2$ implies that $K(n, 2) \geq \lceil \frac{2n}{3} \rceil$. To show equality consider the $(3m, 2)$KSS

$$\begin{array}{cc} 1 & 2 \\ 1 & 3 \\ 4 & 5 \\ 4 & 6 \\ \vdots & \vdots \\ 3m-2 & 3m-1 \\ 3m-2 & 3m \end{array}.$$

The bound above is achieved when $n \equiv 0 \mod 3$. When $n = 3m + 1$ and $3m + 2$ add blocks $\{3m + 1, 3m\}$ and $\{3m + 1, 3m\}, \{3m + 1, 3m + 2\}$ respectively. □

A simple construction is now given in which integers are placed consecutively into pairs of rows of an array M in such a way that no two integers appear in the same two rows. This construction will be used in a variety of ways to construct minimal KSSs.

Construction C (From Ramsay and Robert [3].) Let $r > s \geq 2$, $n = \lceil \frac{rs}{2} \rceil$ and $M = (m_{ij})$ be the $r \times s$ array with every entry 0.

For each $t \in [\lfloor \frac{rs}{2} \rfloor]$, in numeric order, include t in the two places of M defined by

$$\min_j \min_i \{m_{ij} : m_{ij} = 0\},$$
$$\min_i \min_j \{m_{ij} : m_{ij} = 0\}.$$

Each of the points $1, \ldots, \lfloor \frac{rs}{2} \rfloor$ now appears exactly twice.

Each 0 will have been replaced if rs is even, otherwise there will be one 0 remaining which is to be replaced by n.

Lemma 7. *Construction C yields an (n, s)KSS consisting of n 2-points when rs is even and $(n-1)$ 2-points and one 1-point, m_{rs}, when rs is odd.*

Proof. Section 3.2 of Ramsey and Roberts ([3]) shows that the 2-points are separated from each other.

When rs is odd there will be one 0 remaining in M after $1, \ldots, \lfloor \frac{rs}{2} \rfloor$ have been entered. The order in which the 0s are replaced ensures that if entry m_{ij} is 0 then $m_{kl} = 0$ for all $k \geq i$ and $l \geq j$. Therefore this final 0 is m_{rs}. This is replaced by the 1-point n. □

Example 1. *The array below is the result of using Construction C with $r = 5$ and $s = 3$. It is an $(8,3)$KSS (which is not minimal).*

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \\ 4 & 7 & 8 \end{array}$$

The next theorem shows that the lower bound on $K(n,k)$ in Lemma 3(i) is attained for n sufficiently large compared to k. It also shows that for fixed k and $n \geq \frac{k^2}{2}$, $K(n,k)$ is monotonic increasing as n increases. This is not the case for $R(n,k)$ (see Theorem 2 of [3]). It is not known if $K(n,k)$ increases monotonically in n for fixed k for $n \geq 2k$.

Theorem 8. *Let $k \geq 3$ and $n \geq \frac{k^2}{2}$. Then*

$$K(n,k) = \left\lceil \frac{2n}{k+1} \right\rceil .$$

For $\frac{k^2}{2} \leq n \leq \binom{k+1}{2}$ we have $\lceil \frac{2n}{k+1} \rceil = k$ and for $n > \binom{k+1}{2}$ we have $\lceil \frac{2n}{k+1} \rceil > k$.

Proof. Let $r = \lceil \frac{2n}{k+1} \rceil$. By Lemma 3 it is sufficient to construct an (n,k)KSS with size r. This will be done by constructing two arrays M_1 and M_2 whose juxtaposition $M = M_1|M_2$ will give the required KSS.

As $n \geq \frac{k^2}{2}$, $\frac{2n}{k+1} \geq \frac{k^2}{k+1} = k - 1 + \frac{1}{k+1}$. Therefore $\lceil \frac{2n}{k+1} \rceil > k-1$. So Construction C can be used create an $r \times (k-1)$ array M_1 representing an $(n', k-1)$KSS with r blocks and $n' = \lceil \frac{r(k-1)}{2} \rceil$. Note that n' is the $(r, k-1)$ entry of M_1. It is the only entry that may be common to M_1 and M_2.

Construct an $r \times 1$ array M_2 which contains each of $n'+1, \ldots, n$ at least once. For this to be possible it needs to be shown that $n - n' \leq r$. That is $n \leq n' + r$, which is $n \leq \left\lceil \lceil \frac{2n}{k+1} \rceil \frac{(k-1)}{2} \right\rceil + \lceil \frac{2n}{k+1} \rceil$. It is sufficient to show that $n \leq \frac{2n}{k+1}\frac{(k-1)}{2} + \frac{2n}{k+1}$, which is trivial.

When $r(k-1)$ is even set ${M_2}^T$, the transpose of M_2, to

$${M_2}^T = \begin{cases} n, \ldots, n, n-1, \ldots, n'+2, n'+1 & n - n' \neq r-1, \\ n', n, n-1, \ldots, n'+2, n'+1 & n - n' = r-1 \end{cases}$$

and when $r(k-1)$ is odd (so $r \geq k+1$) set

$${M_2}^T = \begin{cases} n', n, \ldots, n, n-1, \ldots, n'+2, n'+1 & n - n' \neq r-2, \\ n, n-1, \ldots, n'+2, n'+1, n'+1, n'+1 & n - n' = r-2. \end{cases}$$

Each point of $[n]$ appears at least once in $M = M_1|M_2$. To show that M represents an (n,k)KSS it remains to show that each pair of points is separated.

With the possible exception of n', M_1 and M_2 have no points in common. Also, again with the possible exception of n', every point of M_1 is a 2-point. The restrictions on the values of $n - n'$ ensure that M_2 contains no 2-points. Hence $\{1, 2, \ldots, n' - 1, n' + 1, \ldots, n - 1, n\}$ are separated.

Note that when n' appears in M_2 it does so in the first row; in M_1 it occurs in the last row and may occur in the last two rows. As no other 2-point can occur in the first and last row of M, and if there is another 3-point, it does not occur in the first row, n' is separated from all other points. □

By Lemma 6 and Theorem 8, the value of $K(n, k)$, k fixed, is now known for all but a finite number of values of n.

Theorem 9. *For* $\binom{k}{2} \leq n \leq \binom{k}{2} + \lfloor \frac{k-1}{3} \rfloor$ *and* $k \geq 5$

$$K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil = k - 1.$$

Proof. Use Construction C to fill a $(k-1) \times (k-2)$ array M_1 on $[\binom{k-1}{2}]$ in which each point occurs twice. The aim is to create an (n, k)KSS with $(k-1)$ blocks represented by a $(k-1) \times k$ array M.

It remains to create a $(k-1) \times 2$ array M_2 with entries from the set $\{\binom{k-1}{2} + 1, \ldots, \binom{k-1}{2} + q\}$ which has no 2-points, and which is a $(q, 2)$KSS.

The value of q is greatest when there are $(k-1)$ 1-points in one column and $\frac{k-1}{3}$ 3-points in the other. Therefore $q \leq k - 1 + \lfloor \frac{k-1}{3} \rfloor$. The smallest q considered here is $q = k - 1$ as this will yield a KSS with smallest n for which $\lceil \frac{2n}{k+1} \rceil = k - 1$.

For $k - 1 \leq q \leq k - 1 + \lfloor \frac{k-1}{3} \rfloor$ fill M_2 with

$$\begin{cases} (\frac{3q+2}{2} - k) \text{ 1-points and } (k - 1 - \frac{q}{2}) \text{ 3-points} & q \text{ even} \\ (\frac{3q+3}{2} - k) \text{ 1-points, } (k - \frac{5}{2} - \frac{q}{2}) \text{ 3-points and 1 4-point} & q \text{ odd.} \end{cases}$$

Place a 3-point in rows 1, 2 and 3 of the first column, another in rows 4, 5 and 6, and so on until there are no more 3-points or there are fewer than 3 spaces remaining in the first column. Do the same in the second column, but work from the bottom up, putting in a 1-point or a 4-point first if a 3-point occurs in the last three rows of the previous column. When there are no more 3-points or 4-points, fill the remaining spaces with 1-points.

The number of 1-points is at least $\frac{3q+2}{2} - k \geq \frac{3k-1}{2} - k = \frac{k-1}{2}$. This is enough to ensure that the last one or two spaces in the first column can be filled, and that the second column can have a 1-point at the bottom when the first column is filled entirely with 3-points. Therefore all 3-points are separated from each other.

The smallest possible number of 3-points and 4-points is $k - \frac{3}{2} - \frac{q}{2} = k - \frac{3}{2} - \frac{1}{2}(k - 1) - \frac{1}{2}(\lfloor \frac{k-1}{3} \rfloor) \geq \frac{2k-3}{6}$. These take up at least $k - 1$ places in M_2. A consequence of this is that each row of M_2 will contain at most one 1-point.

Let $M = M_1 | M_2$ be the $(k-1) \times k$ array obtained by placing M_1 beside M_2. Then M represents an (n, k)KSS with $n = \binom{k-1}{2} + q$. The restriction $k - 1 \leq q \leq k - 1 + \lfloor \frac{k-1}{3} \rfloor$ implies that $\binom{k}{2} \leq n \leq \binom{k}{2} + \lfloor \frac{k-1}{3} \rfloor$. □

One might ask if it is also true that $K(n, k) = \lceil \frac{2n}{k+1} \rceil$ for $\binom{k}{2} + \lfloor \frac{k-1}{3} \rfloor < n < \frac{k^2}{2}$. Theorems 8 and 9 suggest that this may be true. Surprisingly this is not the case, as the next theorem shows.

Theorem 10. *Let $k \geq 3$ and $\binom{k}{2} + \lfloor \frac{k-1}{3} \rfloor < n < \frac{k^2}{2}$. Then*

$$K(n,k) = \left\lceil \frac{2n}{k+1} \right\rceil + 1 = k.$$

Proof. The condition $\binom{k}{2} + \lfloor \frac{k-1}{3} \rfloor < n < \frac{k^2}{2}$ implies that $k(k-1) + 2\lfloor \frac{k-1}{3} \rfloor < 2n < k^2$. As both $2n$ and k^2 are integers, this forces $2n \leq k^2 - 1$. Consequently $k(k-1) + 2\lfloor \frac{k-1}{3} \rfloor < 2n \leq k^2 - 1$, which implies that $k - 1 - \frac{1}{k+1}(k - 1 - 2\lfloor \frac{k-1}{3} \rfloor) < \frac{2n}{k+1} \leq k - 1$. Hence $\lceil \frac{2n}{k+1} \rceil = k - 1$.

Now assume that $K(n,k) \leq \lceil \frac{2n}{k+1} \rceil = k - 1$. Then there is an (n,k)KSS $\mathcal{K}$ of size $k-1$, and so of volume $k(k-1)$.

The volume required for $\mathcal{K}$ is at least the volume taken by $(k-1)$ 1-points, $\binom{k-1}{2}$ 2-points and $n - ((k-1) + \binom{k-1}{2})$ 3-points. That is, $1(k-1) + 2\binom{k-1}{2} + 3(n - ((k-1) + \binom{k-1}{2}))) = 3n - (k-1)(1 + \frac{k}{2})$. For $\mathcal{K}$ to exist its volume must be at least the volume calculated, that is,

$$k(k-1) \geq 3n - (k-1)(1 + \frac{k}{2}).$$

This is equivalent to $\frac{k-1}{3} + \binom{k}{2} \geq n$. As n is an integer this implies $\lfloor \frac{k-1}{3} \rfloor + \binom{k}{2} \geq n$. This contradicts our assumption on n. Therefore $K(n,k) \geq \lceil \frac{2n}{k+1} \rceil + 1$.

To construct an (n,k)KSS of size k use Construction C to construct $k \times (k-1)$ array M_1 with each of $1, \ldots, \binom{k}{2}$ appearing twice. Take the $k \times 1$ array M_2 to be given by $M_2^T = \binom{k}{2}+1, \binom{k}{2}+2, \ldots, n-2, n-1, n, n, n, \ldots, n$. As $\lfloor \frac{k-1}{3} \rfloor < n - \binom{k}{2} < \frac{k}{2}$ this is possible and n will be a p-point for some $p \geq 3$.

The array $M = M_1|M_2$ represents an (n,k)KSS of size k. □

The value of $K(n,k)$ has now been determined for all $n \geq \binom{k}{2}$.

4 Near Symmetry

There is a symmetry between the values of $R(n,k)$ and $R(n, n-k)$, namely $R(n,k) = R(n, n-k)$ (Lemma 2 of [3]). However, there is only "near-symmetry" between the values of $K(n,k)$ and $K(n, n-k)$. In this section it is shown that they differ by at most 1.

The **complement** of an (n)KSS $\mathcal{K} = \{A_1, \ldots, A_r\}$ is the collection of subsets of $[n]$ defined by $\mathcal{K}' = \{[n] - A_1, \ldots, [n] - A_r\}$.

Theorem 11. *Let $\mathcal{K}$ be a minimal (n,k)KSS and let $\mathcal{K}'$ be its complement.*

(i) If no point of $[n]$ occurs in every block of $\mathcal{K}$ then $\mathcal{K}'$ is an $(n, n-k)$KSS.

(ii) If there is a point of $[n]$ which occurs in every block then $\mathcal{K}'$ is isomorphic to an $(n-1, n-k)$KSS.

In either case, if $\mathcal{K}'$ is not minimal then there is a minimal $(n, n-k)$KSS, respectively $(n-1, n-k)$KSS, of size $|\mathcal{K}'| - 1$.

Proof. (i) Since there is no point of $[n]$ occurring in every block of $\mathcal{K}$, each point occurs at least once in $\mathcal{K}'$. Suppose that for distinct $a, b \in [n]$, there exists $A \in \mathcal{K}$ such that $a \in A$ and $b \notin A$. Then $a \notin [n] \setminus A$ and $b \in [n] \setminus A$. Thus $\mathcal{K}'$ is an $(n, n-k)$KSS.

(ii) Assume that it is n that occurs in each block. Removing it from each block of $\mathcal{K}$ gives an $(n-1, k-1)$KSS with the same complement $\mathcal{K}'$. By part (i) $\mathcal{K}'$ is an $(n-1, n-k)$KSS.

If the conditions of part (i) hold and if also $\mathcal{K}'$ is not minimal, let $\mathcal{M}$ be a minimal $(n, n-k)$KSS. Then $\mathcal{M}'$ can't be an (n, k)KSS as this would contradict the minimality of $\mathcal{K}$. So $\mathcal{M}$ must have a point common to each block and hence $\mathcal{M}'$ is isomorphic to an $(n-1, k)$KSS.

If the block $1, 2, \ldots, k-1, n$ is added to $\mathcal{M}'$ one gets an (n, k)KSS. Hence, in part (i), if the $(n, n-k)$KSS $\mathcal{K}'$ is not minimal then $K(n, n-k) = |\mathcal{K}'| - 1$.

If the conditions of part (ii) hold and if $\mathcal{K}'$ is not minimal, let $\mathcal{M}$ be a minimal $(n-1, n-k)$KSS. Then $\mathcal{M}'$ is either an $(n-1, k-1)$KSS or an $(n-2, k-1)$KSS. Adding n to each block of an $(n-1, k-1)$KSS yields an (n, k)KSS, contradicting the minimality of $\mathcal{K}$. So $\mathcal{M}'$ must be an $(n-2, k-1)$KSS.

Adding the block $1, 2, \ldots, k-2, n-1$ to $\mathcal{M}'$ yields an $(n-1, k-1)$KSS. As $\mathcal{K}'$ has no point common to all blocks, n can included in each block to obtain an (n, k)KSS. As above, if the $(n-1, n-k)$KSS $\mathcal{K}'$ is not minimal then $K(n-1, n-k) = |\mathcal{K}'| - 1$. □

Example 2. *A minimal KSS with complement not minimal is* $\mathcal{K} = \{12, 23, 34, 45\}$. *Here* $\mathcal{K}' = \{123, 125, 145, 345\}$ *is not a minimal* $(5, 3)$*KSS as* $\{123, 124, 135\}$ *is also a* $(5, 3)$*KSS.*

We have seen in Theorems 8, 9 and 10 that the lower bound of Lemma 3(i) is in fact $K(n, k)$ or $K(n, k) - 1$ for $n \geq \binom{k}{2}$. It is not a good bound for values of k which are larger with respect to n. However, this bound, together with Theorem 11 implies that either $K(n, n-k) \geq \lceil \frac{2n}{k+1} \rceil - 1$ if there is a minimal (n, k)KSS in which no point of $[n]$ occurs in every block, or $K(n-1, n-k) \geq \lceil \frac{2n}{k+1} \rceil - 1$ if there is a minimal (n, k)KSS in which some point occurs in every block.

The next Lemma is an example of this.

Lemma 12. *For all* $n \geq 3$,

$$K(n, n-1) = n - 1.$$

Proof. Equation 2 of Section 3, with $k = n-1$, represents an $(n, n-1)$KSS with $n-1$ blocks. It is minimal by Lemma 6 and Theorem 11. □

Corollary 13. *Assume that a minimal* (n, k)*KSS* $\mathcal{K}$ *has no* $K(n, k)$*-point and that* $K(n-1, k) = K(n, k)$. *Then* $\mathcal{K}'$ *is minimal and so* $K(n, k) = K(n, n-k)$.

Proof. Let $r = K(n, k)$. By Theorem 11(i) the complement $\mathcal{K}'$ of $\mathcal{K}$ is an $(n, n-k)$KSS with r blocks.

If $\mathcal{K}'$ is not minimal, let $\mathcal{M}$ be a minimal $(n, n-k)$KSS. From Theorem 11 and its proof it has $r-1$ blocks, can not be an (n, k)KSS, and so $\mathcal{M}'$ must be an $(n-1, k)$KSS with $r-1$ blocks. This contradicts the assumption that $K(n-1, k) = r$. Hence $\mathcal{K}'$ must be minimal. □

	k														
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2														
3	3	2													
4	4	3	*3*												
5	5	4	3	4											
6	6	4	3	4	5										
7	7	5	4			6									
8	8	6	4	4	4		7								
9	9	6	5	4	4		6	8							
10	10	7	5	4	4	4	5		9						
11	11	8	6	5	4					10					
12	12	8	6	5	*5*	4		5	6	8	11				
13	13	9	7	6	5			5				12			
14	14	10	7	6	5		4		5	6	7		13		
15	15	10	8	6	5	5				5	6		10	14	
16	16	11	8	7	6	5		5		5			8		15
17	17	12	9	7	6	*6*						6	7		
18	18	12	9	8	6	6			5			6	6		9
19	19	13	10	8	7	6							6		8
20	20	14	10	8	7	6				5				6	7
21	21	14	11	9	7	6	6								6
22	22	15	11	9	8	7	6				5				6
23	23	16	12	10	8	7	6								
24	24	16	12	10	8	7	*7*					5			
25	25	17	13	10	9	8	7								

Table 1: Values of $K(n,k)$ calculated. Diagonal values are underlined.

Constructions for minimal (n,k)KSSs with no point common to every block have been given for the following values of n and k:

$(n,1)$ with $n>1$ and $(n,2)$ with $n>2$ (Lemma 6),

$(n,3)$ with $n>5$, $(n,4)$ with $n>7$, $(n,5)$ with $n>11$ and for $k\geq 6$ with $n>\binom{k}{2}$ (Theorems 8, 9 and 10),

The diagonal values $n=2k$ (Lemma 5).

Corollary 13 has been applied where possible.

Table 1 shows the values of $K(n,k)$ for $1\leq n\leq 25$ and $1\leq k\leq 15$ calculated in this paper. Apart from the values of $K(n,n-1)$ and $K(5,3)$, the values to the right of the diagonal entries (underlined) were given by Corollary 13. The values in italics were determined by Theorem 10.

5 Search Theory

Search Theory had its foundations in the Second World War. Dorfmann proposed a way of minimising the number of tests for syphilis in draftees in a short paper on the use of group testing of blood samples which appeared in the Annals of Mathematical Statistics. See [1] for a brief historical account of this and subsequent developments.

Group testing can be adaptive—the results of previous tests (choices of subsets) are used to inform the choice of subsequent tests; or non-adaptive—which is effectively equivalent to using simultaneous tests. Non-adaptive group testing is considered here.

The basic search problem can be stated as follows (based upon [2]):

An unknown element $d \in [n]$, called the defective element, is to be identified by asking questions of the type "Is $d \in A$?" where A is a subset of $[n]$ with $|A| \leq k$. In this formulation it is assumed that a defective element actually exists.

The original model proposed by Dorfmann is an example of Combinatorial Group Testing (CGT)—subsets of a given set are chosen for testing as a group, rather than conducting tests on individual elements. Given a search space of size n, each test can be interpreted as a block of $[n]$, and each collection of tests can be interpreted as a combinatorial design on $[n]$, with the tests being carried out in parallel or sequentially. The aim is to minimise the number of tests required, subject to keeping block sizes constant.

Unlike the basic search problem stated above, it may not be known if a defective element actually exists. In this case the question can be modified to be

"If there is a defective element, is it in A?"

where A is a subset of $[n], |A| \leq k$. A minimal KSS provides a suitable strategy for non-adaptive group testing. Consider the problem of finding a defective element, if one exists in $[n]$, where it is desired to test exactly k-points in each test. An SS will not suffice for this problem unless it contains all points including the possible defective point. Hence the need for a KSS.

For example, to find a defective point, if one exists, in the set [6] with 3 points in each test, the minimal $(6,3)$KSS $\{\{1,2,3\},\{1,4,5\},\{2,4,6\}\}$ can be used. This shows that three tests suffice.

The next theorem shows that this is optimal. That is, to minimise the number of tests used for CGT a KSS of minimal size is required.

Theorem 14. *Assume that $[n]$ contains at most one defective element. A collection of sets $\mathcal{K}$ can be used to determine the defective element, or show that none exists, if and only if $\mathcal{K}$ is an (n)KSS.*

Proof. Let $\mathcal{K}$ be a collection of sets which can be used to determine a defective element as above. Every element must be in a least one set, for if an element was not in any set it would be impossible to determine if it was defective or if there were in fact no defective elements.

Now assume that there is a defective element, and call it x. For each $a \neq x$, there must be a set containing one, but not the other. That is, a is separated from x or x is separated from a. Hence $\mathcal{K}$ is a KSS. The converse is clearly true. □

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