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Enumeration and dichromatic number of tame tournaments

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Abstract

The concept of molds, introduced by the authors in a recent preprint, break regular tournaments naturally into big classes: cyclic tournaments, tame tournaments and wild tournaments. We enumerate completely the tame molds, and prove that the dichromatic number of a tame tournament is 3.

1 Introduction

A *mold* is a regular tournament M such that all the paths of the domination digraph $\mathfrak{D}(M)$ are of order at most 2. The molds were defined and studied in [12]. Two vertices u, v form a *dominant pair* of a regular tournament T if

$$N^{-}(u, T \setminus \{u, v\}) = N^{+}(v, T \setminus \{u, v\})\,[12].$$

The *domination graph* of a tournament T, denoted by $dom(T)$, is the graph on the vertex set $V(T)$ with edges between dominant pairs of T. The domination graph was defined in [7] and the domination graph of regular tournaments were characterized by Cho et al. in [4, 5]. Recently the authors have given a simpler proof of this characterization by the use of molds [12]. The *domination digraph* of a tournament T, denoted by $\mathfrak{D}(T)$, is the domination graph on the vertex set $V(T)$ with the orientation induced by T. The domination digraph was defined in [6]. Since the paths of a domination graph $dom(T)$ are all directed in the tournament T, the characterization of domination graphs induces a characterization of domination digraphs.

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The cyclic tournament $\overrightarrow{C}_{2m+1}\langle\emptyset\rangle$ is the tournament on the vertex set $\mathbb{Z}_{2m+1}$ and $(u,v)\in A\left(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle\right)$ if $v-u\in\mathbb{Z}_m$. We assign a mold M^T and associate a weight function φ_T, to every regular non-cyclic tournament T, and every regular non-cyclic tournament T can be reconstructed from its mold M^T and the weight function φ_T [12]. We say that a mold is *tame* if it has a dominant pair such that the residue is isomorphic to a cyclic tournament (the residue of a subdigraph H of T, is $T\setminus V(H)$). A regular tournament is tame if its mold is tame. Equivalently a regular tournament is tame if $\mathfrak{D}(T)$ has a maximal path P such that $T\setminus P$ is a cyclic tournament. We enumerate the tame molds and prove that all tame tournaments are 3-dichromatic, that is, 3 is the smallest number of colors needed to color the vertices such that no directed monochromatic cycles are formed. It has recently been proved that tame tournaments are tight [8]; a tournament is tight if every vertex coloring with exactly 3 colors induces a cyclic triangle with the 3 colors.

The standard notation follows [1, 2, 3, 9].

2 Preliminaries

In this section, we review some basic facts about domination digraphs of regular tournaments and molds. Clearly the automorphism group of a regular tournament T acts on the domination digraph $\mathfrak{D}(T)$ of T, so it fixes the set of $\mathfrak{D}$-arcs. We will need the following results.

Lemma 1 *[7] If T' is a subtournament of T, then $\mathfrak{D}(T)|_{T'}$ is a subdigraph of $\mathfrak{D}(T')$.*

Let T be a tournament, $u,v\in V(T)$ and $S\subset V(T)$. We say that u and v are *concordant* with respect to S if

$$N^-(u,S\setminus\{u,v\})=N^-(v,S\setminus\{u,v\}).$$

We say that u and v are *discordant* with respect to S if

$$N^+(u,S\setminus\{u,v\})=N^-(v,S\setminus\{u,v\}).$$

Note that are u and v are discordant with respect to T if and only if $(u,v)\in$ *dom* (T).

The components of $\mathfrak{D}(T)$ are directed paths or cycles (Lemma 2.2, [5]). If P is a path of $\mathfrak{D}(T)$, then $T[V(P)]$ is completely determined by the following lemma.

Lemma 2 *[12] Let $P=(u_0,u_1,\ldots,u_k)$ be a path of $\mathfrak{D}(T)$, where T is a regular tournament, $P_{i,j}=(u_i,u_{i+1},\ldots,u_j)$ an induced subpath of P with $0\le i<j\le k$, and int $(P_{i,j})=P_{i,j}\setminus\{u_i,u_j\}$.*

(i) If $j-i$ is even, then u_i and u_j are concordant with respect to $T\setminus P_{i,j}$, $(u_j,u_i)\in A(T)$ and u_i and u_j are discordant with respect to $P_{i,j}$.

(*ii*) *If* $j-i$ *is odd, then* u_i *and* u_j *are discordant with respect to* $T\setminus P_{i,j}$*,* $(u_i,u_j)\in A(T)$ *and* u_i *and* u_j *are concordant with respect to* $P_{i,j}$*.*

The following proposition is a summary of results from [5, 7, 12]

Proposition 1 *A regular tournament* T*, of odd order, is a cyclic tournament if and only if* $dom(T)\cong C_n$*.*

Remark 1 *Let* $m\geq 2$*; then* $\vec{C}_{2m+1}\langle\emptyset\rangle\setminus\{i,i+m\}\cong\vec{C}_{2m-1}\langle\emptyset\rangle$*, and moreover*

$$\mathfrak{D}\left(\vec{C}_{2m+1}\langle\emptyset\rangle\right)=(0,m,2m,m-1,\ldots,1,m+1,0).$$

We will use the following construction:

Remark 2 *[12] Let* (u,v) *be a* $\mathfrak{D}$*-arc of a regular tournament* T *of order* $2m+3$ $(m\geq 1)$ *and denote by* $T'=T\setminus\{u,v\}$ *the residual tournament of* (u,v)*. By the definition of* $\mathfrak{D}(T)$*, there exists a natural partition of* $V(T')$ *into the sets* $V^-=N^+(u;T')$ *and* $V^+=N^+(v;T')$*. Moreover,* $V^-=N^-(v,T')$ *and* $V^+=N^-(u,T')$*. Since* T *is regular,* $|V^-|=m$ *and* $|V^+|=m+1$*.*

Note that in Lemma 1, $\mathfrak{D}(T)|_{T'}$ is not necessarily induced in $\mathfrak{D}(T)$.

The molds were defined in [12]. A *mold* is a regular tournament M such that all the paths of the domination digraph $\mathfrak{D}(M)$ are of order at most 2. Let T be a regular non-cyclic tournament; then we assign the mold M^T and associate a weight function φ_T to T.

Proposition 2 *[12] Let* T *be a non-cyclic regular tournament* T*, and let* M^T *be the mold of* T *and* φ_T *the weight function of* T*. Then*

(*i*) M^T *is a regular subtournament of* T*, and* $\mathfrak{D}(T)|_M$ *is induced in* $\mathfrak{D}(T)$*, and*

(*ii*) T *can be reconstructed from its mold* M^T *and the weight function* φ_T*.*

Note that if $\phi:V(T)\to V(T')$ is an isomorphism, then $(u,v)\in A(\mathfrak{D}(T))$ if and only if $(\phi(u),\phi(v))\in A(\mathfrak{D}(T'))$.

Corollary 1 *Let* T,T' *be regular non-cyclic tournaments and* $(M,\varphi),(M',\varphi')$ *their corresponding molds and weight functions. Then* $T\cong T'$ *if and only if there is an isomorphism* $\phi:V(M)\to V(M')$ *such that* $\varphi'\circ\phi(v)=\varphi(v)$ *for every trivial component* v *of* $\mathfrak{D}(M)$*, and* $\varphi'(\phi(u),\phi(v))=\varphi(u,v)$ *for every non-trivial component* (u,v) *of* $\mathfrak{D}(M)$*.*

We say that a $\mathfrak{D}$-arc is a $\mathfrak{C}$-arc if the residue of the $\mathfrak{D}$-arc is a cyclic tournament.

We study some properties of the regular tournaments with $\mathfrak{C}$-arcs (for more details, see [12]).

Proposition 3 *[12] Let T be a regular non-cyclic tournament with a $\mathfrak{C}$-arc (ξ^-, ξ^+). Then M^T has an odd number of $\mathfrak{D}$-arcs.*

Recall that there are three non-isomorphic regular tournaments of order 7: a cyclic tournament, a Payley tournament and a third tournament W_0 defined in [11]. The regular tournament $W_0 = (V, A)$ is defined as follows:

$$\begin{aligned} V(W_0) &= \left\{w_1^-, w_2^-, w_3^-, w_1^+, w_2^+, w_3^+, w_0\right\}, \\ A(W_0) &= \left\{\left(w_i^-, w_0\right)\right\} \cup \left\{\left(w_0, w_i^+\right)\right\} \cup \left\{\left(w_i^-, w_i^+\right)\right\} \cup \left\{\left(w_i^+, w_{i-1}^-\right)\right\} \cup \\ &\quad \left\{\left(w_i^+, w_{i+1}^-\right)\right\} \cup \left\{\left(w_i^-, w_{i+1}^-\right)\right\} \cup \left\{\left(w_i^+, w_{i+1}^+\right)\right\}, \text{with } i-1, i, i+1 \in \mathbb{Z}_3. \end{aligned}$$

Note that $\mathfrak{D}(W_0) = \left\{\left(w_2^+, w_1^-\right), \left(w_3^+, w_2^-\right), \left(w_1^+, w_3^-\right), \{w_0\}\right\}$, the $\mathfrak{D}$-arcs of W_0 are all $\mathfrak{C}$-arcs and W_0 is transitive in $\mathfrak{D}$-arcs.

Proposition 4 *The number of $\mathfrak{C}$-arcs in a mold M is at most one, except when $M \cong W_0$, and in that case M has three $\mathfrak{C}$-arcs.*

Proof Suppose that the mold M has two $\mathfrak{C}$-arcs $\left(\xi_0^-, \xi_0^+\right)$ and $\left(\xi_1^-, \xi_1^+\right)$. Let $M_0 = M \setminus \left\{\xi_1^-, \xi_1^+\right\}$, $M_1 = M \setminus \left\{\xi_0^-, \xi_0^+\right\}$ and $M' = M \setminus \left\{\xi_0^-, \xi_0^+, \xi_1^-, \xi_1^+\right\}$. By Remark 1, M' is a cyclic tournament. Since M_0 is vertex transitive, we can assume that $\mathfrak{D}(M_0) = \left(0, \xi_0^-, \xi_0^+, m, 2m, \ldots, 0\right)$. Let $\mathfrak{D}(M_1) = \left(0, m, 2m, \ldots, k, \xi_1^-, \xi_1^+, k+m, \ldots, 0\right)$, $0 \leq k \leq 2m$. Then $(j, j+m) \in \mathfrak{D}(M)$ if $j \neq 0, k$ and $\left(\xi_0^-, \xi_0^+\right), \left(\xi_1^-, \xi_1^+\right) \in \mathfrak{D}(M)$. We know that $\left(0, \xi_0^-\right), \left(\xi_0^+, m\right) \in \mathfrak{D}(M_0)$ and $\left(k, \xi_1^-\right), \left(\xi_1^+, k+m\right) \in \mathfrak{D}(M_1)$.

Since $d^-\left(\xi_i^-; T_j\right) = m+1$, $d^-\left(\xi_i^-; T\right) = m+2$, with $i \in \{1, 2\}$, and M_0, M_1, M are regular tournaments, then $M\left[\left\{\xi_0^-, \xi_0^+, \xi_1^-, \xi_1^+\right\}\right] \setminus \left\{\left(\xi_0^-, \xi_0^+\right), \left(\xi_1^-, \xi_1^+\right)\right\} \cong \vec{C}_4$. We assume that $\left(\xi_0^+, \xi_1^+\right) \in M$.

Let $k \in N^+\left(\xi_0^+\right)$. Since $\mathfrak{D}(M) = \left(0, \xi_0^-, \xi_0^+, m, 2m, \ldots, k, \xi_1^-, \xi_1^+, k+m, \ldots, 0\right)$, then T would be a cyclic tournament.

So $k \notin N^+\left(\xi_0^+\right)$ and $k+m \in N^+\left(\xi_0^+\right)$. Then $\mathfrak{D}(M)$ is the disjoint union of the two arcs $\left(\xi_0^-, \xi_0^+\right), \left(\xi_1^-, \xi_1^+\right)$ and two paths $((m, 2m, \ldots, k)$ and $(k+m, k+2m, \ldots, 0))$, one of even order and the other of odd order. Since M is a mold, then by Proposition 2, $\mathfrak{D}(M)$ is the disjoint union of three arcs and a vertex, and then $M \cong W_0$. □

As a consequence of Propositions 3 and 4, we have the following.

Corollary 2 *If M is a mold of order at least 9, with a $\mathfrak{C}$-arc (ξ^-, ξ^+), then (ξ^-, ξ^+) is the only $\mathfrak{C}$-arc of M, and M has an odd number (at least three) of $\mathfrak{D}$-arcs.*

A digraph is *rigid* if its only automorphism is the identity.

Proposition 5 *If T is a tournament with $M^T \cong W_0$, then T is rigid if and only if there are two $\mathfrak{D}$-paths of even order with different length.*

Proof Let T be a tame mold. Let φ_T be the associated weight function and $M^T \cong W_0$. Then $\mathfrak{D}(T)$ has four $\mathfrak{D}$-paths, one of odd order P_1 and three of even orders P_2, P_3, P_4. The cyclic rotation of the $\mathfrak{D}$-arcs of M^T is an automorphism of M^T.

For necessity: Let $l(P_2) = l(P_3) = l(P_4) = 2k$; then the weight function $\varphi_T(a) = 2k$, for all $\mathfrak{D}$-arcs a. Then by Corollary 1 the cyclic rotation of the $\mathfrak{D}$-arcs of M^T induces an automorphism of T, and T is not rigid.

For sufficiency: We assume that $l(P_2) \neq l(P_3)$, and $l(P_2) \neq l(P_4)$. Then any automorphism of T fixes the $\mathfrak{D}$-path P_2 and, any automorphism of M^T fixes the $\mathfrak{C}$-arc (v_0, v_1), where $P_2 = (v_0, v_1, v_2, \ldots, v_{2k-1})$. Then M^T with the weight function φ_T is rigid and by Corollary 1, so is T. □

3 Tame molds

A mold with a $\mathfrak{C}$-arc is called a *tame* mold. The $\mathfrak{C}$-arcs induce a natural partition of the molds in three classes: the family of cyclic tournaments $\vec{C}_{2m+1}\langle\varnothing\rangle$, the tame tournaments and otherwise, the *wild* tournaments. A tame tournament is a regular tournament with a tame mold. We associate to each tame mold an even directed cycle and a weight function. We prove that there is a bijection between the tame molds and this set of even directed cycles with their weight function. Using this fact we enumerate the tame molds.

Theorem 1 *There exists a bijection between the tame molds and the set of even directed cycles* $\vec{C}_{2s} = \left(u_1^-, u_1^+, u_2^-, u_2^+, \ldots, u_s^-, u_s^+, u_1^-\right)$, $s \geq 1$, *with the weight function* $\rho_M : V\left(\vec{C}_{2s}\right) \to \mathbb{N}$ *with the following properties*

(i) $\rho_M(v) \geq 2$ *for* $v \in V\left(\vec{C}_{2s}\right)$, *and*

(ii) $\sum_{i=1}^{s} \rho_M\left(u_i^+\right) = \sum_{i=1}^{s} \rho_M\left(u_i^-\right) + 1$.

Proof Given a tame mold M we will prove that we can associate to M an even directed cycle $\vec{C}_{2s}$ and a weight function ρ_M and that this cycle with its weight function is unique.

Let M be a tame mold of order $2m+3$ $(m \geq 2)$, with $2s+1$ $\mathfrak{D}$-arcs, where $0 < s \leq m/2$. Let (ξ^-, ξ^+) be a $\mathfrak{C}$-arc, and $M' = M \setminus \{\xi^-, \xi^+\}$. We denote $\vec{C}_{2m+1}\langle\varnothing\rangle$ by M'. By Remark 1,

$$\mathfrak{D}(M') = (0, m, 2m, m-1, 2m-1, \ldots, m+1, 0).$$

We will associate an even directed cycle $\vec{C}_{2s}$ and a weight function ρ_M as follows:

Let $V^- = N^+(\xi^-, M')$, $V^+ = N^+(\xi^+, M')$. Since M' is vertex-transitive and by Corollary 2, we may assume that $(0, m) \in \mathfrak{D}(M)$ such that $m \in V^-$ and $0 \in V^+$.

Let $\{P_i^\varepsilon\}_{i=1}^s$ be the set of maximal paths of $\mathfrak{D}(M') \cap V^\varepsilon$, with $\varepsilon \in \{-,+\}$. Note that $P_1^- = (m, 2m, \ldots, lm)$ and $P_s^+ = (km, \ldots, m+1, 0)$, for some $1 \le l < k \le 2m$.

Let $\mathbf{C}_M$ be the resulting cycle when we contract each path P_i^- and P_i^+ to the vertex v_i^- and v_i^+ respectively, so $\mathbf{C}_M = \left(u_1^-, u_1^+, u_2^-, u_2^+, \ldots, u_s^-, u_s^+, u_1^-\right)$; see Figure 1.

The weight function ρ_M assigns to each vertex u_i^ε the order of the path P_i^ε, $\rho_M(u_i^\varepsilon) = |V(P_i^\varepsilon)|$. Note that $\rho_M(u_i^\varepsilon) \ge 2$, $l(\mathbf{C}_M) = 2s$ and $\mathbf{C}_M$ contains all the $\mathfrak{D}$-arcs of M except the $\mathfrak{C}$-arc (ξ^-, ξ^+). Also note that $\sum_{i=1}^s \rho_M\left(u_i^-\right) = m$, $\sum_{i=1}^s \rho_M\left(u_i^+\right) = m+1$ and $\sum_{u \in \mathbf{C}_M} \rho_M(u) = 2m+1$ (see Figure 1).

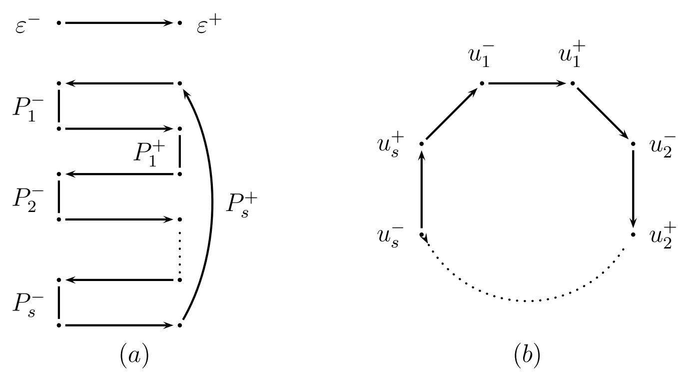

Figure 1: (a) The tame tournament T, (b) the axillary cycle of T.

Observe that if $|M| \ge 9$, then by Corollary 2, the $\mathfrak{C}$-arc is unique, and the cycle $\mathbf{C}_M$ with the associated weight function ρ_M is unique.

Given an even cycle and a weight function with the required properties we can construct a tame mold M.

Let the even cycle $\mathbf{C}_M = \left(u_1^-, u_1^+, u_2^-, u_2^+, \ldots, u_s^-, u_s^+, u_1^-\right)$, $s \ge 1$, with the weight function $\rho_M : V\left(\vec{C}_{2s}\right) \to \mathbb{N}$ with the properties:

(i) $\rho_M(v) \ge 2$ for $v \in V\left(\vec{C}_{2s}\right)$,

(ii) $\sum_{i=1}^s \rho_M\left(u_i^-\right) = m$, and

(iii) $\sum_{i=1}^s \rho_M\left(u_i^+\right) = m+1$.

We will construct a tame mold M of order $\sum_{i=1}^s\left(\rho_M\left(u_i^+\right) + \rho_M\left(u_i^-\right)\right) + 2$, $(m \ge 2)$, with $2s+1$ $\mathfrak{D}$-arcs, where $0 < s \le m/2$.

Let $\{P_i^\varepsilon\}_{i=1}^s = \{v_{i,1}^\varepsilon, v_{i,2}^\varepsilon, \ldots, v_{i,\rho_M\left(u_i^\varepsilon\right)}^\varepsilon\}$ be the set of directed paths such that $|V(P_i^\varepsilon)| = \rho_M(u_i^\varepsilon)$, with $\varepsilon \in \{-,+\}$. Now we construct the cycle C as follows: for each vertex $u_i^\varepsilon \in V\left(\vec{C}_{2s}\right)$, $\varepsilon \in \{-,+\}$, take a copy of the path P_i^ε and connect the terminal vertex of P_i^- with the initial vertex of P_i^+ and the terminal vertex of P_i^+ with the initial vertex of P_{i+1}^+. Note that $|V(C)| = 2m+1$. Let $C = \mathfrak{D}(M')$; then M' is a cyclic tournament. We define the tame mold M as follows:

Let $V(M) = V(M') \cup \{\xi^-, \xi^+\}$, let $N^+(\xi^\varepsilon) = \cup_{i=1}^s \{v_{i,1}^\varepsilon, v_{i,2}^\varepsilon, \ldots, v_{i,\rho_M(u_i^\varepsilon)}^\varepsilon\}$ and let $(\xi^-, \xi^+) \in A(M)$. Note that (ξ^-, ξ^+) is a $\mathfrak{C}$-arc of M and that the tame mold M is completely determined by the cycle $\mathbf{C}_M$ and the weight function ρ_M. □

Proposition 6 *The tame molds of order at least* 9 *are rigid.*

Proof Let M be a tame tournament and $|V(M)| \geq 9$. Let (ξ^-, ξ^+) be the $\mathfrak{C}$-arc of M and let $M' = M \setminus \{\xi^-, \xi^+\}$. Let $\overrightarrow{C}_{2s} = \left(u_1^-, u_1^+, u_2^-, u_2^+, \ldots, u_s^-, u_s^+, u_1^-\right)$ be the even directed cycle associated to the tame mold M, and $\rho_M : V\left(\overrightarrow{C}_{2s}\right) \to \mathbb{N}$ be the weight function associated to the tame mold M. Note that an automorphism of $\overrightarrow{C}_{2s}$ with its weight function ρ_M induces an automorphism of $\mathfrak{D}\left(M \setminus \{\xi^-, \xi^+\}\right)$ that fixes the ex-neighborhoods of ξ^- and ξ^+. Since $\sum_{i=1}^s \rho_M\left(u_i^+\right) = \sum_{i=1}^s \rho_M\left(u_i^-\right) + 1$, an odd rotation of the vertices of $\overrightarrow{C}_{2s}$ does not induce an automorphism of M (nor of T, by Corollary 1). Let φ be an even rotation of the vertices of $\overrightarrow{C}_{2s}$. Since φ fixes the ex-neighborhoods of ξ^- and ξ^+, it follows that φ induces a rotation of the vertices of $N_{M'}^+(\xi^-)$ and a rotation of the vertices of $N_{M'}^+(\xi^+)$. Since $|N_{M'}^+(\xi^-)| = m$, $|N_{M'}^+(\xi^+)| = m+1$ and $\gcd(m, m+1) = 1$, it follows that φ is the identity and then $\overrightarrow{C}_{2s}$ with its associated weight function ρ_M is rigid. Then M is rigid by Corollary 2 and Theorem 1. □

Proposition 7 *A tame tournament T is rigid, except if $M^T \cong W_0$ and all the $\mathfrak{D}$-paths of even order of T have the same length.*

Proof Let T be a tame tournament. If $\left|V(M^T)\right| \geq 9$, then the result is a consequence of Proposition 6 and Corollary 1. Let $\left|V(M^T)\right| = 7$. It is known that up to isomorphism there are three regular tournaments of order 7: the cyclic, the Paley tournament and W_0. W_0 is the only tame mold of order 7, and the result is a consequence of Proposition 5 and Corollary 1. □

Given the weight function ρ_M of a tame mold M (see Figure 1), we can represent ρ_M by alternating two partitions Π_1 and Π_2, with

$$\begin{aligned} \Pi_1 &= \rho_M\left(u_1^-\right) \mid \rho_M\left(u_2^-\right) \mid \ldots \mid \rho_M\left(u_s^-\right), \\ \Pi_2 &= \rho_M\left(u_1^+\right) \mid \rho_M\left(u_2^+\right) \mid \ldots \mid \rho_M\left(u_s^+\right), \end{aligned}$$

where Π_1 and Π_2 are partitions of m and $m+1$ respectively into s parts. Let π_1 and π_2 be the frequency representations of Π_1 and Π_2 respectively. Then $\pi_i = \left(n_{i,1}^{r_{i,1}}, n_{i,2}^{r_{i,2}}, \ldots, n_{i,k}^{r_{i,k}}\right)$, where $n_{i,j} \geq 2$, for $i = 1,2$ and $1 \leq j \leq k$, such that $\sum_{j=1}^{j=k} n_{1j} r_{1j} = m$, $\sum_{j=1}^{j=k} n_{2j} r_{2j} = m+1$ and $\sum_{j=1}^{j=k} r_{ij} = s$ for each $i = 1,2$. Let $\pi = \left(n_1^{r_1}, n_2^{r_2}, \ldots, n_k^{r_k}\right)$ be the frequency representation of a partition; then define $\varepsilon(\pi)! = r_1! r_2! \ldots r_s!$.

Lemma 3 *The number of non-isomorphic molds of order* $2m+3$ *with* $2s+1$ $\mathfrak{D}$*-arcs, where* $0 < s \leq m/2$*, and frequency representation* π_1, π_2 *is*

$$\frac{(s!)^2}{s \cdot \varepsilon(\pi_1)!\varepsilon(\pi_2)!}.$$

Proof Let π_1 and π_2 be the frequency representation of the partitions Π_1 and Π_2 respectively. Let $\pi_1 = \left(n_{1,1}^{r_{1,1}}, n_{1,2}^{r_{1,2}}, \ldots, n_{1,k}^{r_{1,k}}\right)$ and $\pi_2 = \left(n_{2,1}^{r_{2,1}}, n_{2,2}^{r_{2,2}}, \ldots, n_{2,l}^{r_{2,l}}\right)$, with $n_{i,j} \geq 2$ and $\sum_{j=1}^{j=k} n_{1j}r_{1j} = m$, $\sum_{j=1}^{j=k} n_{2j}r_{2j} = m+1$ and $\sum_{j=1}^{j=k} r_{ij} = s$ for each $i = 1, 2$. Using the counting argument of acyclic permutations with repetition, it follows that the number of tame molds is equal to

$$\frac{1}{s}\binom{s}{r_{1,1}r_{1,2}\ldots r_{1,k}}\binom{s}{r_{2,1}r_{2,2}\ldots r_{2,l}} = \frac{(s!)^2}{s \cdot \varepsilon(\pi_1)!\varepsilon(\pi_2)!}$$

where $n_1 + n_2 + \ldots + n_k = n$ and

$$\binom{n}{n_1 n_2 \ldots n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}.$$

□

We conclude that

Theorem 2 *The number of non-isomorphic tame molds of order* $2m+3$*, with* $2s+1$ $\mathfrak{D}$*-arcs, where* $0 < s \leq m/2$ *and* $m \geq 4$*, is*

$$\sum_{\pi_1, \pi_2} \frac{(s!)^2}{s \cdot \varepsilon(\pi_1)!\varepsilon(\pi_2)!}.$$

Proof Let π_1, π_2 be the frequency representation of the partitions Π_1 and Π_2 respectively. Then the result is a consequence of Lemma 3, summing on the frequency representations π_1, π_2. □

Corollary 3 *There is a unique tame mold up to isomorphism, of order* $2m+3$

(i) with 3 $\mathfrak{D}$*-arcs, and*

(ii) $m+1$ $\mathfrak{D}$*-arcs, with* m *an even integer and* $m \geq 4$.

Let M be a tame tournament, and let $\mathfrak{D}(M)$ be the disjoint union of s arcs and $2t+1$ vertices. Then $|V(M)| = 2(s+t)+1$. Let $C(D)$ denote the set of maximal components of the digraph D. Let $T \in \mathfrak{F}(M)$. Let $\varphi : C(\mathfrak{D}(M)) \to \mathbb{N}$ be the weight function that assigns to each component of $\mathfrak{D}(M)$ the length of the corresponding path in $\mathfrak{D}(T)$. Since the components of $\mathfrak{D}(M)$ have order one or two, observe that $|C(\mathfrak{D}(M))| = s+2t+1$. Let $C_1(\mathfrak{D}(M))$ denote the set of trivial components and let $C_2(\mathfrak{D}(M))$ denote the set of components of order 2. Since T has odd order, we have

$$\sum_{c \in C(\mathfrak{D}(M))} \varphi(c) = 2r+1, \text{ where } r \geq s+t.$$

Moreover, if $c \in C_1(\mathfrak{D}(M))$, then $\varphi(c)$ is an odd positive integer and if $c \in C_2(\mathfrak{D}(M))$, then $\varphi(c)$ is an even positive integer.

Let $k = r - s - t$. Note that $|V(T)| = \left|V(M^T)\right| + 2k$. Let n_1 and n_2 be two nonnegative integers such that $n_1 + n_2 = k$,

$$\sum_{c \in C_1(\mathfrak{D}(M))} \varphi(c) = 2(n_1 + t) + 1 \quad \text{and} \quad \sum_{c \in C_2(\mathfrak{D}(M))} \varphi(c) = 2(n_2 + s).$$

Let P^1 and P^2 be the set of maximal paths of odd order and even order respectively of $\mathfrak{D}(T)$; then $2n_1$ is the number of vertices of $V(P^1) \setminus V(C_1(\mathfrak{D}(M)))$ and $2n_2$ is the number of vertices of $V(P^2) \setminus V(C_2(\mathfrak{D}(M)))$.

Lemma 4 *Fix the partition (n_1, n_2) of k, fix the frequency representation $\pi_1 = (n_{11}^{r_{11}}, n_{12}^{r_{12}}, \ldots, n_{1k}^{r_{1k}})$, with $n_{1j}^{r_{1j}}$ an odd integer, of the partition Π_1 of $2(t + n_1) + 1$ into $2t + 1$ parts and fix the frequency representation $\pi_2 = (n_{21}^{r_{21}}, n_{22}^{r_{22}}, \ldots, n_{2l}^{r_{2l}})$ of the partition Π_2 of $s + n_2$ into s parts. Then the number of non-isomorphic tame tournaments $T \in \mathfrak{F}(M)$, with s $\mathfrak{D}$-arcs, order $2(s+t+k)+1$, $\sum |V(P_i^1)| = 2(t+n_1)+1$ and $\sum |V(P_i^2)| = 2(s + n_2)$ is exactly*

$$\frac{s!(2t+1)!}{\varepsilon(\pi_1)!\varepsilon(\pi_2)!}.$$

Proof Fix the frequency representation $\pi_1 = (n_{11}^{r_{11}}, n_{12}^{r_{12}}, \ldots, n_{1k}^{r_{1k}})$, with $n_{1j}^{r_{1j}}$ an odd integer, of the partition Π_1 of $2(t + n_1) + 1$ into $2t + 1$ parts, and fix the frequency representation π_2 of the partition Π_2 of $s + n_2$ into s parts. Note that $r_{11} + r_{12} + \ldots + r_{1k} = 2t+1$ and $r_{21} + r_{22} + \ldots + r_{2l} = s$. Using the counting argument of acyclic permutations with repetition, it follows that the number of tame molds is equal to

$$\binom{2t+1}{r_{11} r_{12} \ldots r_{1k}} \binom{s}{r_{21} r_{22} \ldots r_{2l}} = \frac{s!(2t+1)!}{\varepsilon(\pi_1)!\varepsilon(\pi_2)!}.$$

□

Theorem 3 *Let (n_1, n_2) be a partition of k. Let $\pi_1 = (n_{11}^{r_{11}}, n_{12}^{r_{12}}, \ldots, n_{1k}^{r_{1k}})$ be the frequency representation of a partition Π_1 of $2(t + n_1) + 1$ into $2t + 1$ parts, with $n_{1j}^{r_{1j}}$ an odd integer, and let $\pi_2 = (n_{21}^{r_{21}}, n_{22}^{r_{22}}, \ldots, n_{2l}^{r_{2l}})$ be the frequency representation of a partition Π_2 of $s + n_2$ into s parts. Then the number of non-isomorphic tame tournaments $T \in \mathfrak{F}(M)$, with s $\mathfrak{D}$-arcs and order $2(s + t + k) + 1$ is exactly*

$$\sum_{n_1, n_2} \sum_{\pi_1, \pi_2} \frac{s!(2t+1)!}{\varepsilon(\pi_1)!\varepsilon(\pi_2)!}.$$

Proof Fix the partition (n_1, n_2) of k. Then the number of non-isomorphic tame tournaments $T \in \mathfrak{F}(M)$, such that T has order $2(s + t + k) + 1$ is exactly $\sum_{\pi_1, \pi_2} \frac{s!(2t+1)!}{\varepsilon(\pi_1)!\varepsilon(\pi_2)!}$. Summing on the partitions of k there are

$$\sum_{n_1, n_2} \sum_{\pi_1, \pi_2} \frac{s!(2t+1)!}{\varepsilon(\pi_1)!\varepsilon(\pi_2)!}$$

non-isomorphic tame tournaments of order $2(s + t + k) + 1$ with $T \in \mathfrak{F}(M)$. □

4 Coloring

We prove that all tame tournaments have dichromatic number 3. The *dichromatic number* $dc(D)$ of a digraph D is the smallest number of colors needed to color the vertices of D such that no directed monochromatic cycles are formed [10].

Remark 3 *[12] Let $P = (u_0, u_1, \ldots, u_k)$ be a path of $\mathfrak{D}(T)$, where T is a regular tournament. Let $F_0 = (u_0, u_2, \ldots, u_{2l})$ and $F_1 = (u_1, u_3, \ldots, u_{2l+1})$, where $2l, 2l+1 \leq k$; then $T[F_i]$ is transitive, $i = 0, 1$.*

To put Theorem 5 below in perspective, it is convenient to recall the following:

Theorem 4 *[13] Let T be a regular tournament. Then $dc(T) = 2$ if and only if T is a cyclic tournament.*

Theorem 5 *Let T be a tame tournament. Then $dc\left(M^T\right) = 3 = dc(T)$.*

Proof Let T be a tame tournament; by Theorem 4 $dc(T) > 2$. Let P be a path in $\mathfrak{D}(T)$ such that $T \setminus P$ is a cyclic tournament. Let $P = (u_1, u_2, \ldots, u_{2j})$. We assume that $T \setminus P = \vec{C}_{2m+1}\langle\emptyset\rangle$, $0 \in N^+(u_1)$ and $m \in N^+(u_2)$. Let $\varphi : V(T) \to \{c, c^-, c^+\}$ be the following 3-coloring:

$$\varphi(i) = \begin{cases} c^- & \text{if } i \in (\{0, 1, \ldots, m\} \cap N^+(u_1)) \cup \{u_1, u_3, \ldots, u_{2j-1}\} \\ c^+ & \text{if } i \in (\{0, 1, \ldots, m\} \cap N^+(u_2)) \cup \{u_2, u_4, \ldots, u_{2j}\} \\ c & \text{if } i \in \{m+1, m+2, \ldots, 2m\}. \end{cases}$$

Clearly $\{m+1, m+2, \ldots, 2m\}$ and $\{0, 1, \ldots, m\}$ are acyclic. Since u_{2s+1} and u_{2t+1} are concordant with respect to $T \setminus P$ by Lemma 2 (i), then by Remark 3 the chromatic class $\{c^-\}$ is acyclic. Analogously the chromatic class $\{c^+\}$ is acyclic. It follows that φ is an acyclic coloring with 3 colors, and then $dc(T) = 3$. □

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