Forbidden substrings on weighted alphabets

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Abstract

In an influential 1981 paper, Guibas and Odlyzko constructed a generating function for the number of length n strings over a finite alphabet that avoid all members of a given set of forbidden substrings. Here we extend this result to the case in which the strings are weighted. This investigation was inspired by the problem of counting compositions of an integer n that avoid all compositions of a smaller integer m, a notion which arose from the consideration of *one-sided* random walks.

1 Introduction

In [3] Guibas and Odlyzko construct a generating function for the number of length n strings over a finite alphabet that avoid all members of a given set of forbidden substrings. Here we assign a weight to each letter of the alphabet, define the weight of a string to be the sum of the weights of its letters, and determine a generating function for the number of *weight* n strings that avoid a particular set of forbidden substrings. This investigation was inspired by the problem of counting compositions of an integer n (which can be viewed as weight n strings over the alphabet $\{1, 2, 3, \ldots\}$) that do not contain a composition of a smaller integer m occurring in consecutive positions (*i.e.*, avoid a substring of weight m). This latter problem arose from the consideration of *one-sided* random walks, which are introduced here, and further investigated by Bender, Lawler, Pemantle, and Wilf in [1].

Heubach and Kitaev have also extended Guibas and Odlyzko's results from words to compositions. In [4] they consider length (number of parts) and weights in compositions over alphabets of the form $\{1, 2, \ldots, n\}$. In this paper we consider arbitrary weighted alphabets. For more on the combinatorics of compositions and words, see [5].

2 Forbidden Substrings

In this section we recall Guibas and Odlyzko's theorem concerning forbidden substrings.

A set $S = \{A, B, \ldots, T\}$ of strings over an alphabet Ω is *reduced* if no string contains another as a substring. (In particular, no string in S is empty.) Let $f(n)$ denote the number of length n strings that avoid each member of S. For each string H in S let $f_H(n)$ denote the number of length n strings that end with H and avoid all members of S except for the single appearance of H at the end.

Define generating functions $F(z) = \sum_{n \geq 0} f(n)/z^n$ and $F_H(z) = \sum_{n \geq 0} f_H(n)/z^n$.

The *correlation* of two strings G and H, denoted GH, is a string over $\{0, 1\}$ with the same length as G. The ith character from the left in GH is determined by placing H under G so that the leftmost character of H is under the ith character (from the left) in G. If all the pairs of characters in the overlapping segment are identical, then the ith character of GH is 1. If not, it is 0. For example if $\Omega = \{a, b\}$, $G = ababba$, and $H = abbab$, then $GH = 001001$ as illustrated below.

$$
\begin{array}{llllll}
 & a & b & a & b & b & a \\
0 & a & b & b & a & b \\
0 & & a & b & b & a & b \\
1 & & & a & b & b & a & b \\
0 & & & & a & b & b & a & b \\
0 & & & & & a & b & b & a & b \\
1 & & & & & & a & b & b & a & b \\
\end{array}
$$

Let GH_z denote the correlation of G and H interpreted as a polynomial in the variable z. In the above example, $GH_z = z^3 + 1$.

Theorem 1 *(Guibas, Odlyzko) Given a reduced set $S = \{A, B, \ldots, T\}$ of strings over an alphabet of $q \geq 2$ characters, the generating functions $F(z)$, $F_A(z)$, $F_B(z)$, $\ldots$, $F_T(z)$ satisfy the following system of linear equations:*

$$
\begin{array}{rcrcccrcl}
(z-q)F(z) & + & zF_A(z) & + & \ldots & + & zF_T(z) & = & z \\
F(z) & - & zAA_zF_A(z) & - & \ldots & - & zTA_zF_T(z) & = & 0 \\
F(z) & - & zAB_zF_A(z) & - & \ldots & - & zTB_zF_T(z) & = & 0 \\
& \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
F(z) & - & zAT_zF_A(z) & - & \ldots & - & zTT_zF_T(z) & = & 0 \\
\end{array}
$$

The fact that S is reduced guarantees this system is nonsingular, and we can solve for the generating functions as rational functions of z.

3   Weighted Strings

Theorem 1 shows us how to construct a generating function for the number of length n strings that avoid each member of a given set of forbidden substrings. In this section we extend this result to the case in which the strings are weighted, and count *weight* n strings.

A *weighted* alphabet $w(\Omega)$ has each letter h assigned a weight w_h. The *weight* of a string $H = h_1 h_2 \ldots h_s$ over $w(\Omega)$ is the sum $w_H = \sum_{i=1}^{t} w_{h_i}$ of the weights of the

individual letters. A set S of weighed strings is *reduced* if no string contains any other as a substring.

Given a set S of reduced strings over a weighted alphabet $w(\Omega)$, let $f(n)$ denote the number of weight n strings that do not contain any substring in S. Similarly for each H in S let $f_H(n)$ denote the number of weight n strings that end with H and do not contain any substring in S except for the single appearance of H at the end. Define $F(z) = \sum\limits_{n \geq 0} f(n)/z^n$ and $F_H(z) = \sum\limits_{n \geq 0} f_H(n)/z^n$. Note $f(0) = 1$ counts the empty composition while $f_H(n) = 0$ for n less than the number of letters in H.

Next we extend the notion of correlation for two strings to a weighted version. For the ordinary correlation GH of two strings G and H, the ith character from the left is 1 if and only if G and H overlap on the string $g_i g_{i+1} \ldots g_r$ for some r. The weighted correlation $w(GH)$ is a multiset, and the weight $w_{g_i} + w_{g_{i+1}} + \ldots + w_{g_r}$ of the string on which G and H overlap is in $w(GH)$. More specifically, for any two strings $G = g_1 g_2 \ldots g_r$ and $H = h_1 h_2 \ldots h_t$ over a weighted alphabet $w(\Omega)$, the *weighted correlation* $w(GH)$ is a (possibly empty) mulitset. This multiset contains k if and only if there is an i such that $h_1 = g_i, h_2 = g_{i+1}, \ldots, h_{r-i+1} = g_r$, and $k = w_{h_1} + w_{h_2} + \ldots + w_{h_{r-i+1}}$ is the weight of the overlap.

For example, let $w(\Omega) = \{1, 2, \ldots\}$ with $w_i = i$. Set $A = 3, B = 21, C = 12$, and $D = 111$. Then $w(AA) = w(BB) = w(CC) = \{3\}$, $w(CB) = \{2\}$, $w(BC) = w(DC) = w(BD) = \{1\}$, and $w(DD) = \{1, 2, 3\}$. The remaining weighted correlations are empty. Note neither correlation nor weighted correlation is commutative in general.

Finally we define $w(GH)_z$ to be the polynomial $\sum\limits_{k \in w(GH)} z^k$. When $w(GH) = \emptyset$, the polynomial $w(GH)_z$ is 0. Thus $w(DD)_z = z^3 + z^2 + z$, for example.

We now prove an extension of Theorem 1.

Theorem 2 *Given a reduced set $S = \{A, B, \ldots, T\}$ of strings over a weighted alphabet $w(\Omega)$, the generating functions $F(z)$, $F_A(z)$, $F_B(z)$, $\ldots$, $F_T(z)$ satisfy the following system of linear equations:*

$$
\begin{array}{llllll}
\frac{z-2}{z-1}F(z) & +F_A(z) & +F_B(z) & \cdots & +F_T(z) & = 1 \\
F(z) & -w\,(AA)_z\, F_A(z) & -w\,(BA)_z\, F_A(z) & \cdots & -w\,(TA)_z\, F_T(z) & = 0 \\
F(z) & -w\,(AB)_z\, F_A(z) & -w\,(BB)_z\, F_A(z) & \cdots & -w\,(TB)_z\, F_T(z) & = 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
F(z) & -w\,(AT)_z\, F_A(z) & -w\,(BT)_z\, F_A(z) & \cdots & -w\,(TT)_z\, F_T(z) & = 0
\end{array}
$$

Proof. The first equation in the above system follows from the observation that $f(n+1) + f_A(n+1) + \ldots + f_T(n+1) = f(n) + f(n-1) + \ldots + f(0)$. This recurrence holds because any string $h_1 h_2 \ldots h_t$ counted by one of $f(n+1)$, $f_A(n+1)$, $f_B(n+1)$, $\ldots$, $f_T(n+1)$ arises by appending the character h_t to the string $h_1 h_2 \ldots h_{t-1}$ counted by $f(n + 1 - h_t)$. The right hand side of the recurrence equation is the coefficient of $1/z^n$ in $z/(z-1)F(z)$, and the left hand side of the equation is the coefficient of $1/z^n$ in $z[F(z) - 1] + zF_A(z) + zF_B(z) + \ldots + zF_T(z)$. (Recall $f(0) = 1$, but $f_A(0) = f_B(0) = \ldots f_T(0) = 0$.)

The remaining equations result from the fact for any H in S we have $f(n) = \sum_{k \in w(AH)} f_A(n+k) + \sum_{k \in w(BH)} f_B(n+k) + \ldots + \sum_{k \in w(TH)} f_T(n+k)$. To see this let $H = h_1 h_2 \ldots h_t$ and suppose $Y = y_1 y_2 \ldots y_s$ is any string counted by $f(n)$. Let $Z = z_1 z_2 \ldots z_{s+t} = y_1 y_2 \ldots y_s h_1 h_2 \ldots h_t$ denote the concatenation of strings Y and H. Now Z contains at least one string in S as a substring. Let $G = g_1 g_2 \ldots g_r$ denote the leftmost such substring. The for some $u > s$ we have $g_1 g_2 \ldots g_r = z_{u-r+1} \ldots z_{u-1} z_u$, and $z_1 z_2 \ldots z_u$ is counted by $f_G(n+k)$ for some $k \in w(GH)$.

Conversely if $k \in w(GH)$, then any string counted by $f_G(n)$ arises from the concatenation of a string Y counted by $f(n)$ and H. Thus the equality holds. Since $\sum_{n \geq 0} \sum_{k \in w(GH)} f_G(n+k)/z^n = \sum_{k \in w(GH)} z^k \sum_{n \geq 0} f_G(n+k)/z^{n+k} = w(GH)_z F_G(z)$, we obtain the remaining equations in the system. ∎

As was the case with Theorem 1, the fact that S is reduced guarantees the system is nonsingular. To see this, consider the determinant

$$\phi(z) = \det \begin{bmatrix} \frac{z-2}{z-1} & 1 & 1 & \cdots & 1 \\ 1 & -w(AA)_z & -w(BA)_z & \cdots & -w(TA)_z \\ 1 & -w(AB)_z & -w(BB)_z & \cdots & -w(TB)_z \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -w(AT)_z & -w(BT)_z & \cdots & -w(TT)_z \end{bmatrix}.$$

Since S is reduced the highest degree polynomial in each column occurs on the diagonal. When we expand $\phi(z)$, we have $z - 1$ in the denominator and a unique highest degree monomial produced by the product of the diagonal terms in the numerator. The degree of this monomial is the sum $1 + w_A + w_B + \ldots + w_T$. We can therefore solve for $F(z), F_A(z), F_B(z), \ldots, F_T(z)$ and find that each is a rational function of z.

4 Compositions

The inspiration for extending Theorem 1 to Theorem 2 came from the problem of counting compositions of an integer n that avoid compositions of a smaller integer m occurring in consecutive positions. For example, the composition $2+4+1+1+4$ of $n = 12$ contains the compositions $2+4$, $4+1+1$, and $1+1+4$ of 6 in consecutive positions, while avoiding all compositions of $m = 3$ in consecutive positions. (Note it does contain the composition $2 + 1$ of 3 in nonconsecutive positions.)

We can apply Theorem 2 to find, for example, a generating function for the numbers of compositions of n that avoid all compositions of $m = 3$ occurring in consecutive positions. To do so we view compositions as words over $w(\Omega) = \{1, 2, 3, \ldots\}$ with $w_i = i$. For example, we identify the composition $2 + 4 + 1 + 1 + 4$ as the word 24114. Set $S = \{A = 3, B = 21, C = 12, D = 111\}$. The number of compositions of n that avoid all compositions of 3 occurring in consecutive positions is given by the number of weight n strings over $w(\Omega)$ which do not contain any substring in S. Let $f(n)$ denote this number. For each H in S, let $f_H(n)$ denote the number

of weight n strings which end with H and contain no substring in S except for the single occurrence of H at the end.

Set $F(z) = \sum_{n \geq 0} f(n)/z^n$ and $F_H(z) = \sum_{n \geq 0} f_H(n)/z^n$. Earlier we recorded the weighted correlation $w(GH)$ for each pair of strings in S. We use this information to form the table below. The polynomial $w(GH)_z$ appears in row H and column G.

	A	B	C	D
A	z^3	0	0	0
B	0	z^3	z^2	0
C	0	z	z^3	z
D	0	z	0	$z^3 + z^2 + z$

Theorem 2 guarantees the generating functions satisfy the following system of equations:

$$
\begin{array}{llllll}
\frac{z-2}{z-1}F(z) & +F_A(z) & +F_B(z) & +F_C(z) & +F_D(z) & = 1 \\
F(z) & -z^3 F_A(z) & & & & = 0 \\
F(z) & & -z^3 F_B(z) & -z^2 F_C(z) & & = 0 \\
F(z) & & -z F_B(z) & -z^3 F_C(z) & -z F_D(z) & = 0 \\
F(z) & & -z F_B(z) & & -(z^3 + z^2 + z)F_D(z) & = 0
\end{array}
$$

Solving this system yields

$$
\begin{aligned}
F(z) &= (z^8 - 2z^5 + z^3)/(z^8 - z^7 - z^6 + z^5 - z^4 - z^3 - z^2 + z + 1) \\
&= 1 + 1/z + 2/z^2 + 2/z^4 + 3/z^5 + 9/z^6 + 12/z^7 + 20/z^8 \ldots
\end{aligned}
$$

as desired.

5   Motivating Problem

The question of composition avoidance arose from the consideration of board games in which a roll of one or more (fair, 6-sided) dice determines the number of "squares" a player moves forward on a given turn. Some squares are undesirable to land on, and one would like to know the probability of avoiding them, given the number of squares separating a particular "bad" square from one's current square.

To solve this problem, we replace the board with a finite number of squares, through which we cycle repeatedly, with an infinite succession of squares extending in one direction. A sequence of dice rolls determines a *one-sided random walk* which begins on square 0, and continues through squares 1, 2, 3, and so on, landing on some squares while avoiding others. What is the probability that a one-sided random walk avoids square m?

We record a one-sided random walk as an "infinite composition" of positive integer parts. For example, $1 + 2 + 2 + \ldots$ indicates a sequence of rolls beginning with a roll of 1 followed by two rolls of 2. What is the probability that a one-sided random walk avoids an initial composition of m?

Let $P(m)$ denote the probability that a one-sided random walk begins with a composition of m, and define $P(0) = 1$. In the simplest case, we use a single die to determine the size of each step in the walk, and compute $P(m)$ using the observation that $P(m) = \frac{1}{6}P(m-1)+\frac{1}{6}P(m-2)+\frac{1}{6}P(m-3)+\frac{1}{6}P(m-4)+\frac{1}{6}P(m-5)+\frac{1}{6}P(m-6)$. From the recurrence we obtain the generating function

$$g(z) = \sum_{m\geq 0} P(m)z^m = \frac{1}{1 - \frac{1}{6}(z + z^2 + z^3 + z^4 + z^5 + z^6)}$$

which converges for $|z| < 1$. Since $g(z)$ has a simple pole at $z = 1$, and the residue there is $-\frac{2}{7}$, we know $P(m) \approx \frac{2}{7}$ for large m. For large m, the probability that a one-sided random walk avoids square m is therefore $1 - P(m) \approx \frac{5}{7}$. We can arrive at the same result using the fact that the recurrence for $P(m)$ has constant coefficients. Specifically $P(m) = \frac{2}{7}+c_1 r_1^m+c_2 r_2^m+c_3 r_3^m+c_4 r_4^m+c_5 r_5^m$, where $|r_i| < 1$ for $1 \leq i \leq 5$.

More generally, we can consider one-sided random walks in which p_i is the probability of moving i squares on a given turn. When p_i is determined by the roll of *two* dice, we obtain $P(m) \approx \frac{1}{7}$. The notion of one-sided random walks is considered further by Bender, Lawler, Pemantle, and Wilf in [1]. They compute, for instance, the probability of a "collision" when two players take simultaneous one-sided random walks. If $C(m)$ is the probability of a collision for the first time on square m, then

$$\sum_{m\geq 0} C(m)x^{2m} = 1 - \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{1-|p(xe^{i\theta})|^2}}$$

where $p(z) = \sum_{i\geq 1} p_i z^i$.

It is easy to count *finite* compositions of an integer n that avoid an initial composition of $m < n$. Compositions that begin with an initial composition of m have the form $\tau + \sigma$, where τ is a composition of m and σ is a composition of $n - m$. There are $2^{m-1} \cdot 2^{n-m-1} = 2^{n-2}$ compositions of n that begin with a composition of m, and therefore also 2^{n-2} that avoid an initial composition of m, independent of our choice for m. In other words, *the probability that a randomly selected composition of n avoids an initial composition of $m < n$ is the same as the probability that it doesn't, for all such m.*

The mathematical literature contains numerous results concerning permutations and multiset permutations (which can be viewed as compositions) that avoid particular *patterns*, *i.e.*, permutations on fewer letters (see [2], for example, for an introduction to the field). The above investigation can be framed in this context as follows. We know we can easily count compositions of n that avoid an initial composition of $m < n$. This results suggests the more general goal of counting compositions of n that avoid a composition of m anywhere. We can interpret this statement in several ways:

1. Count compositions of n that avoid *all* compositions of m occurring in consecutive positions.

2. Count compositions of n that avoid a *particular* composition τ of m occurring in consecutive positions.

3. Count compositions of n that avoid a *particular* composition τ of m in (possibly) nonconsecutive positions.

4. Count compositions of n that avoid *all* compositions of m in (possibly) nonconsecutive positions.

Here we have solved 1 and 2 with Theorem 2. Problem 3 is straightforward, and 4 is open.

The problems above use the word "avoid" in a narrow sense compared to that for patterns. We can define a notion of composition avoidance analogous to that for pattern avoidance. To do so we view the compositions of n as multiset permutations. For example, the compositions 1+1+2, 1+2+1, and 2+1+1 correspond to the permutations 112, 121, and 211 of the multiset $\{1^2, 2\}$. We identify the compositions of $n = 4$ with permutations of the multisets $\{4\}$, $\{1, 3\}$, $\{2^2\}$, $\{1^2, 2\}$, and $\{1^4\}$. It is well-known that the number of permutations of a set of n letters that avoid a pattern π of 3 letters is independent of π. Since the same result holds for multisets (see [6] or [7]), we see that the number of compositions of n that avoid a "composition pattern" (I suggest the term *motif*) π with 3 distinct parts is independent of the parts. It would be interesting to investigate *motif avoidance* for other motifs. The $(1 + 2)$-avoiding compositions are the partitions. How about the $(1 + 2 + 1)$-avoiding compositions?

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