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Complete equipartite 3p-cycle systems

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Abstract

Necessary conditions for decomposing a complete equipartite graph $K_n * \overline{K}_m$ (having n parts of size m) into cycles of a fixed length k are that $nm \geqslant k$, the degree of all its vertices are even and its total number of edges is a multiple of k. Determining whether these necessary conditions are sufficient for general cycle length k is an open problem. Recent results by Manikandan and Paulraja (*Discrete Math.* 306, 429–451) and B. R. Smith (*J. Combin. Designs* 16, 244–252) proved the sufficiency of these conditions in cases where the cycle length is respectively an odd prime or twice an odd prime. Here we further extend these results by showing the sufficiency of these conditions for decomposing complete equipartite graphs into cycles of length $3p$ (where p is an odd prime), thus providing the first general family of results for non-prime, *odd* length cycle decompositions of this type.

1 Introduction

The complete graph on n vertices will be denoted by K_n and unless otherwise specified will be assumed to have vertex set $\mathbb{Z}_n$. The k-cycle C_k on the vertices $v_1, v_2, \ldots, v_k$, with edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_k, v_1\}$ will be denoted by $(v_1, v_2, \ldots, v_k)$ or $(v_k, v_{k-1}, \ldots, v_1)$ or by any cyclic shift of these. The k-path P_k on vertices $v_1, v_2, \ldots, v_k$, with edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ will be denoted by $[v_1, v_2, \ldots, v_k]$ or $[v_k, v_{k-1}, \ldots, v_1]$.

A *closed k-trail* T_k is an alternating sequence of $k + 1$ (not necessarily distinct) vertices and k distinct edges which begins and ends at the same vertex. We frequently define a closed trail by listing just the sequence of adjacent vertices (inside square parenthesis as in the case of paths) repeating the first listed vertex at the end. Furthermore we often identify a closed trail with the graph spanned by its edges (necessarily a simple connected graph in which each vertex has even degree). In particular, we identify the degree of a vertex in a closed trail with its degree in this associated graph.

For any integer x we define $[x]_n$ to be the unique $v \in \mathbb{Z}_n$ satisfying $x \equiv v \pmod{n}$. Furthermore, we say integers u and v *differ modulo n by d* where $d = \min\{[u-v]_n, [v-u]_n\}$; hence $0 \leqslant d \leqslant \lfloor n/2 \rfloor$. The circulant graph $C(n, D)$, where $D \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, is the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{v, [v+d]_n\} \mid v \in \mathbb{Z}_n \text{ and } d \in D\}$; or equivalently, the graph with vertex set $\mathbb{Z}_n$ and an edge between every pair of vertices u and v which differ modulo n by d for some $d \in D$.

The lexicographic product $G * H$ of graphs G and H is the graph having vertex set $V(G) \times V(H)$, and with an edge joining (g_1, h_1) to (g_2, h_2) if and only if there is an edge joining g_1 to g_2 in G; or $g_1 = g_2$ and there is an edge joining h_1 to h_2 in H.

A *complete equipartite graph* has nm vertices, partitioned into n disjoint parts of size m, so that any two vertices in different parts have one edge joining them while any two vertices in the same part have no edge joining them. Using the lexicographic product we denote such a graph by $K_n * \overline{K}_m$. It is easy to see that the degree of any vertex in $K_n * \overline{K}_m$ is $m(n-1)$, and the total number of edges in $K_n * \overline{K}_m$ is $n(n-1)m^2/2$. We note also that the lexicographic product has the following useful property: for any graph G and any positive integers m and ℓ, $(G * \overline{K}_m) * \overline{K}_\ell = G * \overline{K}_{m\ell}$.

If we can partition the edge set $E(G)$ of the graph G into cycles of length k, we refer to this as a *k-cycle decomposition* of G, and write $G \cong C_k \oplus C_k \oplus \cdots \oplus C_k$, or simply $C_k \mid G$. Similarly we can write $G \cong H_1 \oplus H_2 \oplus \cdots \oplus H_s$ if $H_1, H_2, \ldots, H_s$ are *any* edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$ (and subsequently $H \mid G$ in the case that each $H_i \cong H$). Note also that for any graphs G and H and any positive integer m, if $H \mid G$ then $(H * \overline{K}_m) \mid (G * \overline{K}_m)$.

A *latin square* of order n is an $n \times n$ array containing each of the elements in some n-set exactly once in each row and exactly once in each column. For any latin square S, we let $S_{i,j}$ denote the entry in row i, column j, of S. Then two latin squares S and T, each of order n, are said to be *orthogonal* if the ordered pairs $(S_{i,j}, T_{i,j})$ are distinct for all $1 \leqslant i, j \leqslant n$. We say a set of latin squares of order n are *mutually orthogonal* if they are pairwise orthogonal. The notation ℓ MOLS(n) denotes a set of ℓ mutually orthogonal latin squares of order n; in the case $\ell = 1$ this will mean any one latin square of order n.

Obvious necessary conditions for the existence of a k-cycle decomposition of a simple connected graph G are that G has at least k vertices (or trivially, just one vertex), the degree of every vertex in G is even and the total number of edges in G is a multiple of the cycle length k. These conditions are sufficient in the case that G is the complete graph K_n, or the complete graph minus a 1-factor $K_n - I$ ([3], [10], [7]). It is still however an open problem to determine whether a k-cycle decomposition of the complete equipartite graph $K_n * \overline{K}_m$ exists whenever the obvious necessary conditions are satisfied. Recently, Manikandan and Paulraja [9] (together with results by Hanani [6] and Billington, Hoffman, Maenhaut [4]) have shown that when the cycle length is an odd prime p the necessary conditions are indeed sufficient. This result was extended by Smith [11] to include cycles of length $2p$ — twice an odd prime. Here we prove the sufficiency of the necessary conditions for the decomposition of complete equipartite graphs into cycles of length $3p$, with p an odd prime. This is the first general family of results for non-prime, *odd* length cycle decompositions of this type.

2 "Blowing up" lemmas

We begin with several useful results.

THEOREM 2.1 *([11]) Let T_k be a closed trail of length k having maximum degree $\Delta(T_k) = \Delta$ and (vertex) chromatic number $\chi(T_k) = \chi$. Then for all $\ell \geqslant \Delta/2$, the graph $T_k * \overline{K}_\ell$ can be decomposed into cycles of length k whenever there exist at least $\chi - 2$ MOLS(ℓ).*

In particular, as a special case of the above theorem in which the closed trail itself is a cycle, we get the following.

COROLLARY 2.2 *For each $k \geqslant 3$ and each $a \geqslant 1$, the graph $C_k * \overline{K}_\ell$ has a decomposition into k-cycles.*

Proof: The maximum degree Δ of C_k is 2. Then, when k is even $\chi(C_k) = 2$ and when k is odd $\chi(C_k) = 3$. Since latin squares of order ℓ exist for all $\ell \geq 1 = \Delta/2$, the result follows from Theorem 2.1. □

Using this fact we obtain the following well known result, often referred to as "blowing up" points ℓ-fold.

LEMMA 2.3 *If there exists a k-cycle decomposition of $K_n * \overline{K}_m$, then there exists a k-cycle decomposition of $K_n * \overline{K}_{m\ell}$.*

Proof: Let $K_n * \overline{K}_m$ admit a k-cycle decomposition; that is

$$K_n * \overline{K}_m \cong C_k \oplus C_k \oplus \cdots \oplus C_k.$$

Then, from the properties of $*$ and $\oplus$,

$$\begin{aligned} K_n * \overline{K}_{m\ell} &\cong (K_n * \overline{K}_m) * \overline{K}_\ell \\ &\cong (C_k * \overline{K}_\ell) \oplus (C_k * \overline{K}_\ell) \oplus \cdots \oplus (C_k * \overline{K}_\ell). \end{aligned}$$

The result follows from Corollary 2.2. □

The following result is due to Laskar [8].

LEMMA 2.4 *For any positive integers k and ℓ, the graph $C_k * \overline{K_\ell}$ has a decomposition into $k\ell$-cycles.*

Throughout this paper we also make use of the following result (commonly known as "Brooks' Theorem") relating the colourability of a connected graph to its maximum degree.

THEOREM 2.5 *(Brooks' Theorem, [5]) Let G be a connected graph with maximum degree $\Delta = \Delta(G)$. Then $\chi(G) \leqslant \Delta$ if and only if G is neither a complete graph or an odd cycle.*

3 Preliminary results

The following results will be used in conjunction with Theorem 2.1 in the proof of this section's main result.

LEMMA 3.1 *For any odd $n \geqslant 13$ and any i in the range $3 \leqslant i \leqslant (n-1)/2$, the circulant graph $C(n, \{i-2, i-1, i\})$ is a 6-regular, 5-colourable closed $3n$-trail.*

Proof: The graph is clearly 6-regular and connected and hence we need only show that we can partition its vertices into sets $X_0, X_1, \ldots, X_4$ in such a way that no two vertices in the same X_j are connected by an edge.

We deal first with the case $i = 3$. The graph $C(n, \{1, 2, 3\})$ thus contains edges between all vertices u and v which differ modulo n by 1, 2 or 3.

If $n \equiv 1 \pmod 4$ we define

$$\begin{aligned} X_0 &= \{0, 4, \ldots, n-5\}; \\ X_1 &= \{1, 5, \ldots, n-4\}; \\ X_2 &= \{2, 6, \ldots, n-3\}; \\ X_3 &= \{3, 7, \ldots, n-2\}; \\ X_4 &= \{n-1\}. \end{aligned}$$

It is easy to see that vertices in the same X_j differ modulo n by at least 4. The result follows.

If $n \equiv 3 \pmod 4$ we define

$$\begin{aligned} X_0 &= \{0, 4, \ldots, n-15, n-10, n-5\}; \\ X_1 &= \{1, 5, \ldots, n-14, n-9, n-4\}; \\ X_2 &= \{2, 6, \ldots, n-13, n-8, n-3\}; \\ X_3 &= \{3, 7, \ldots, n-12, n-7, n-2\}; \\ X_4 &= \{n-11, n-6, n-1\}. \end{aligned}$$

Again it is easy to see that vertices in the same X_j differ modulo n by at least 4 and the result follows.

Suppose then that $i > 3$ and hence $i-2 > 1$. We write $n-i = x(i-2)+r$ where $1 \leqslant r \leqslant i-2$, and we partition the vertices $1, 2, \ldots, n-i$ into sets $V_0, V_1, \ldots, V_x$ by defining

$$V_j = \{j(i-2) + \ell \mid 1 \leqslant \ell \leqslant i-2\}$$

for each j in the range $0 \leqslant j \leqslant x-1$, and

$$V_x = \{x(i-2)+1, x(i-2)+2, \ldots, x(i-2)+r\}.$$

Clearly no two vertices in the same V_j are connected since they differ modulo n by at most $i-3$ and no vertex in V_0 is connected to any vertex in V_x since they differ modulo n by at least $i+1$. Then for $t = 0, 1$ and 2 we define $X_t = \bigcup_{j \equiv t \pmod 3} V_j$. Now, no two vertices in the same X_t are connected since they either belong to the same V_j or they belong to different V_j (say V_{j_1} and V_{j_2} where $j_1 \equiv j_2 \pmod 3$) and hence differ by at least $2(i-2)+1 > i$ if $x \not\equiv 0 \pmod 3$, or at least $i+1$ if $x \equiv 0 \pmod 3$.

Finally, we partition the remaining i vertices into the sets

$$X_3 = \{x(i-2)+r+1, x(i-2)+r+2, \ldots, n-2\},$$
$$X_4 = \{n-1, 0\},$$

and the result follows. □

Example 3.2 *A proper* 5*-colouring of the graph* $C(19, \{3, 4, 5\})$.
We apply Lemma 3.1 with $n = 19$ and $i = 5$. Then $n - i = 14 = 4(3) + 2$ and hence $x = 4$ and $r = 2$. Thus we have

$$V_0 = \{1, 2, 3\}; \quad V_1 = \{4, 5, 6\}; \quad V_2 = \{7, 8, 9\};$$
$$V_3 = \{10, 11, 12\}; \quad V_4 = \{13, 14\}.$$

Clearly vertices in each V_j differ modulo 19 by at most $2 < 3 = i-2$ and no vertex in V_0 is connected to any vertex in V_4 since they differ modulo 19 by at least $6 > 5 = i$. We then define our sets X_t by

$$X_0 = V_0 \cup V_3 = \{1, 2, 3, 10, 11, 12\};$$
$$X_1 = V_1 \cup V_4 = \{4, 5, 6, 13, 14\};$$
$$X_2 = V_2 = \{7, 8, 9\};$$

and finally

$$X_3 = \{15, 16, 17\};$$
$$X_4 = \{18, 0\}.$$

Now no two vertices in the same X_t but *different* V_j are connected since they differ by at least $7 = 2(3) + 1 > 5$. Hence this is a proper colouring; see Figure 1. □

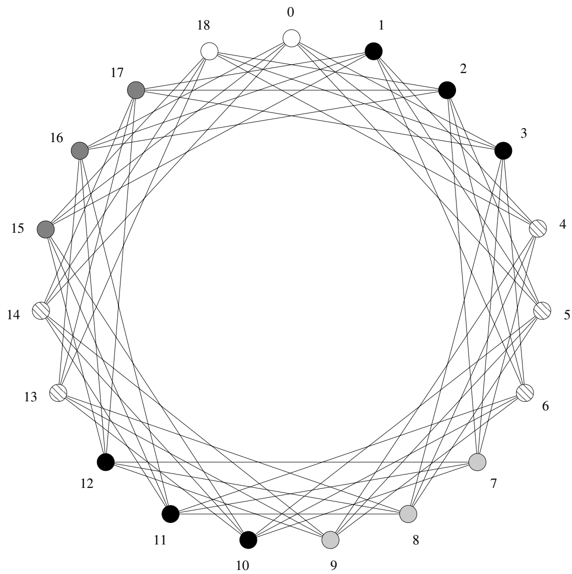

Figure 1: A proper 5-colouring of $C(19, \{3, 4, 5\})$

COROLLARY 3.3 *For each odd $n \geqslant 13$ satisfying $n \equiv 1 \pmod 6$, the graph K_n has a decomposition into 6-regular, 5-colourable, closed $3n$-trails.*

Proof: Letting $n = 6t + 1$ we have

$$K_n = C(n, \{1, 2, 3\}) \oplus C(n, \{4, 5, 6\}) \oplus \cdots \oplus C(n, \{3t-2, 3t-1, 3t\}).$$

The result then follows by Lemma 3.1. □

LEMMA 3.4 *For each odd $n \geqslant 7$ satisfying $n \equiv 1 \pmod 6$, the graph K_{2n+1} has a decomposition into 6-colourable, closed $3n$-trails with maximum degree $\Delta = 6$.*

Proof: Let $n \geqslant 7$ satisfy $n \equiv 1 \pmod 6$ and ρ be the permutation $(0\ 1\ \cdots\ 2n)$ of order $2n + 1$ on the vertex set of K_{2n+1}.

By [3], there is some n-cycle C which generates a decomposition of the graph K_{2n+1} into $2n + 1$ cycles of length n under the action of the group $\langle \rho \rangle$ *and* some vertex v such that v and $[v + 1]_{2n+1}$ both belong to the vertex set of C. We label the cycles in this decomposition $C^0, C^1, \dots, C^{2n}$ in the natural way; that is, C^i is the cycle formed under the action of the permutation ρ^i on C. Note then that C^i and C^{i+1} share at least one common vertex.

Since $2n + 1 \equiv 0 \pmod 3$ we have that

$$K_{2n+1} = \bigcup_{0 \leqslant i \leqslant 2} C^i \oplus \bigcup_{3 \leqslant i \leqslant 5} C^i \oplus \cdots \oplus \bigcup_{2n-2 \leqslant i \leqslant 2n} C^i$$

and each $\bigcup_{j-2\leqslant i\leqslant j} C^i$ is even and connected with $3n$ edges and maximum degree $\Delta = 6$. The colourability condition then follows easily by Theorem 2.5. □

LEMMA 3.5 *For each odd* $n \geqslant 5$ *satisfying* $n \equiv 5 \pmod 6$, *the graph* $K_{n+1} * \overline{K}_2$ *has a decomposition into* 3*-colourable, closed* $3n$*-trails with maximum degree* $\Delta = 4$.

Proof: Let $n+1 = 6t$ so that $3n = 15+18(t-1)$, and let the vertex set of $K_{n+1} * \overline{K}_2$ be $\{\{v_i, (v+6)_i\} \mid 0 \leqslant v \leqslant 5 \text{ and } i \in \mathbb{Z}_t\}$. For each $i \in \mathbb{Z}_t$ let $G_i \cong K_6 * \overline{K}_2$ be the graph spanned by the vertices $\{\{v_i, (v+6)_i\} \mid 0 \leqslant v \leqslant 5\}$, and for each $i, j \in \mathbb{Z}_t$ with $i \neq j$, let $H_{i,j} \cong K_2 * \overline{K}_{12}$ be the complete bipartite graph with vertex set

$$V(G_i) \cup V(G_j) = \{0_i, 1_i, \ldots, 11_i\} \cup \{0_j, 1_j, \ldots, 11_j\}.$$

Furthermore, when $t > 2$, let $R_t \subset \mathbb{Z}_t \times \mathbb{Z}_t$ be the set of $t(t-1)/2$ ordered pairs induced by any orientation of the edges of K_t which contains the directed t-cycle $\vec{C} = (0, 1, 2, \ldots, t-1)$. Hence $(i, [i+1]_t) \in R_t$ for each $i \in \mathbb{Z}_t$. Then for all $i, j \in \mathbb{Z}_t$ exactly one of the following holds: $(i,j) \in R_t$, $(j,i) \in R_t$ or $i = j$. Thus

$$K_{n+1} * \overline{K}_2 \cong \bigcup_{i\in\mathbb{Z}_t} G_i \oplus \bigcup_{(i,j)\in R_t} H_{i,j}. \tag{1}$$

For each $i \in \mathbb{Z}_t$, the four closed 15-trails

$$\begin{aligned}
T_i^1 &= [0_i, 3_i, 6_i, 9_i, 2_i, 5_i, 8_i, 11_i, 4_i, 1_i, 10_i, 7_i, 0_i, 1_i, 2_i, 0_i];\\
T_i^2 &= [3_i, 8_i, 1_i, 6_i, 11_i, 2_i, 7_i, 4_i, 9_i, 0_i, 5_i, 10_i, 3_i, 4_i, 5_i, 3_i];\\
T_i^3 &= [6_i, 4_i, 0_i, 11_i, 3_i, 1_i, 5_i, 7_i, 9_i, 8_i, 10_i, 2_i, 6_i, 7_i, 8_i, 6_i];\\
T_i^4 &= [9_i, 5_i, 6_i, 10_i, 0_i, 8_i, 4_i, 2_i, 3_i, 7_i, 11_i, 1_i, 9_i, 10_i, 11_i, 9_i];
\end{aligned}$$

decompose G_i and are each 3-colourable (see Figure 2) with maximum degree $\Delta = 4$.

Furthermore, using the construction outlined in Sotteau [12], each $H_{i,j}$ decomposes into the eight 18-cycles

$$\begin{aligned}
C_{i,j}^1 &= (3_i, 1_j, 6_i, 5_j, 9_i, 0_j, 4_i, 3_j, 7_i, 4_j, 10_i, 9_j, 5_i, 10_j, 8_i, 11_j, 11_i, 2_j);\\
C_{i,j}^2 &= (0_i, 7_j, 9_i, 6_j, 6_i, 4_j, 2_i, 0_j, 11_i, 3_j, 8_i, 5_j, 1_i, 2_j, 10_i, 1_j, 7_i, 8_j);\\
C_{i,j}^3 &= (0_i, 9_j, 3_i, 8_j, 10_i, 10_j, 1_i, 11_j, 4_i, 5_j, 11_i, 6_j, 2_i, 7_j, 5_i, 4_j, 9_i, 3_j);\\
C_{i,j}^4 &= (0_i, 2_j, 8_i, 8_j, 5_i, 1_j, 2_i, 9_j, 7_i, 7_j, 4_i, 6_j, 1_i, 0_j, 6_i, 11_j, 3_i, 10_j);\\
D_{i,j}^1 &= (3_i, 4_j, 11_i, 7_j, 8_i, 6_j, 5_i, 5_j, 10_i, 11_j, 7_i, 10_j, 4_i, 8_j, 9_i, 9_j, 6_i, 3_j);\\
D_{i,j}^2 &= (0_i, 0_j, 7_i, 6_j, 10_i, 7_j, 1_i, 9_j, 8_i, 1_j, 11_i, 8_j, 2_i, 2_j, 6_i, 10_j, 9_i, 11_j);\\
D_{i,j}^3 &= (0_i, 1_j, 9_i, 2_j, 5_i, 11_j, 2_i, 10_j, 11_i, 9_j, 4_i, 4_j, 1_i, 3_j, 10_i, 0_j, 3_i, 5_j);\\
D_{i,j}^4 &= (0_i, 6_j, 3_i, 7_j, 6_i, 8_j, 1_i, 1_j, 4_i, 2_j, 7_i, 5_j, 2_i, 3_j, 5_i, 0_j, 8_i, 4_j).
\end{aligned}$$

Note that for each $1 \leqslant \ell \leqslant 4$, the three vertices that have degree 4 in the trail T_i^ℓ are exactly those vertices in $V(G_i)$ that are *not used* in the cycles $C_{i,j}^\ell$ and $D_{i,j}^\ell$.

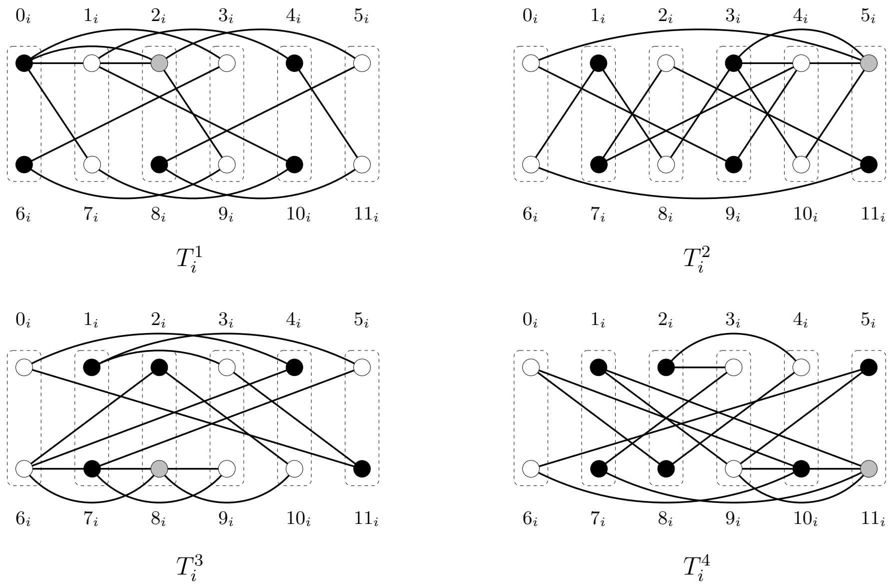

Figure 2: Proper 3-colourings of the trails T_i^ℓ

Importantly this means that the graph spanned by the edges of T_i^ℓ and one of $C_{i,j}^\ell$ or $D_{i,j}^\ell$ has maximum degree 4. This graph is also clearly connected and even. To see that it is also 3-colourable we consider, without loss of generality, the graph $T_i^\ell \cup C_{i,j}^\ell$. We first 3-colour the vertices of T_i^ℓ (as in Figure 2). The vertices remaining, that is those in $V(G_j) \cap V(C_{i,j}^\ell)$, are each adjacent to only two vertices in $V(G_i)$ and thus can easily be coloured as required. In addition, note if we were to form a new graph by attaching *any* other cycle $C_{j,k}^{\ell'}$ or $D_{j,k}^{\ell'}$, satisfying $k \neq i$, to $T_i^\ell \cup C_{i,j}^\ell$, then by the same arguments, this graph would also be even, connected and 3-colourable with maximum degree 4.

If $t = 1$, the trails T_0^1, T_0^2, T_0^3 and T_0^4 have length 15 and are 3-colourable with maximum degree $\Delta = 4$. If $t = 2$, the trails

$$\begin{array}{llll} T_0^1 \cup C_{0,1}^1; & T_0^3 \cup C_{0,1}^3; & T_1^1 \cup D_{0,1}^1; & T_1^3 \cup D_{0,1}^3; \\ T_0^2 \cup C_{0,1}^2; & T_0^4 \cup C_{0,1}^4; & T_1^2 \cup D_{0,1}^2; & T_1^4 \cup D_{0,1}^4; \end{array}$$

have length 33 and are 3-colourable with maximum degree $\Delta = 4$. Suppose then that $t > 2$.

Let $2K_t$ be the complete multigraph with vertex set $\mathbb{Z}_t$ and exactly two edges between each pair of distinct vertices. For each $i \in \mathbb{Z}_t$ we define a hamilton path P^i on $2K_t$ by

$$P^i = \begin{cases} [i, i+1, i-1, i+2, i-2, \ldots, i+(t+1)/2], & \text{for } t \text{ odd}; \\ [i, i+1, i-1, i+2, i-2, \ldots, i+(t/2)], & \text{for } t \text{ even}; \end{cases}$$

where entries are taken modulo t from the residues $\mathbb{Z}_t$. Then when t is odd, for each $d \in \{1, 2, \ldots, (t-1)/2\}$ there is exactly two distinct pairs of adjacent vertices in each P^i which differ modulo n by d. Similarly when t is even, for each $d \in \{1, 2, \ldots, t/2-1\}$ there is exactly two distinct pairs of adjacent vertices in each P^i which differ modulo n by d and exactly one pair which differ modulo n by $t/2$. Thus the paths P^i together decompose $2K_t$.

We 2-colour the edges of our paths P^i with the colours α and β in such a way that the graph $2K_t$ formed by taking the union of the *coloured* paths P^i has exactly one edge of each of the colours α and β between each pair of distinct vertices.

Then for each $1 \leqslant \ell \leqslant 4$ and each $i \in \mathbb{Z}_t$, we form a closed $3n$-trail with the required properties as follows.

Take the 2-coloured path P^i and associate each vertex v in the path with the set of vertices $V(G_v)$ and each edge $\{u, v\}$, where without loss of generality we assume $(u, v) \in R_t$, coloured α (respectively β) with the cycle $C^\ell_{u,v}$ (respectively $D^\ell_{u,v}$). We then take this new graph together with the trail T^ℓ_i, thus forming a closed trail with $18(t-1) + 15 = 3n$ edges. The maximum degree and colourability conditions then follow by the comments above (recall in particular that the edge $\{i, [i+1]_t\}$ is in the path P^i, that $(i, [i+1]_t) \in R_t$ and that the graphs $T^\ell_i \cup C^\ell_{i,j}$ and $T^\ell_i \cup D^\ell_{i,j}$ are 3-colourable with maximum degree $\Delta = 4$). That these trails together decompose $K_{n+1} * \overline{K}_2$ follows from (1). □

LEMMA 3.6 *The graph $C_5 * \overline{K}_3$ has a decomposition into cycles of length* 9.

Proof: Let $C_5 * \overline{K}_3$ be the graph, with vertex set $\{\{v_1, v_2, v_3\} \mid 0 \leqslant v \leqslant 4\}$, induced by blowing up 3-fold the 5-cycle $(0, 1, 2, 3, 4)$, and let ρ be the permutation

$$(0_1\, 1_1\, 2_1\, 3_1\, 4_1)\,(0_2\, 1_2\, 2_2\, 3_2\, 4_2)\,(0_3\, 1_3\, 2_3\, 3_3\, 4_3)$$

of order 5 on the vertices of $C_5 * \overline{K}_3$. The 9-cycle

$$C = (0_1, 1_2, 0_2, 1_1, 2_3, 1_3, 2_2, 3_3, 4_1),$$

then generates a decomposition of $C_5 * \overline{K}_3$ under the action of the group $\langle \rho \rangle$. □

We now give some decompositions of specific "small cases" which will be needed in the proof of our main result.

LEMMA 3.7 *There exist* 21*-cycle decompositions of the graphs (i) $K_7 * \overline{K}_4$ and (ii) $K_7 * \overline{K}_5$.*

Proof: (i) We give a decomposition of $K_7 * \overline{K}_2$ into four 3-colourable, closed 21-trails with maximum degree $\Delta = 4$ and the result then follows by Theorem 2.1. Let the vertex set of $K_7 * \overline{K}_2$ be $\{\{v_1, v_2\} \mid 0 \leqslant v \leqslant 6\}$. Then the four closed 21-trails

$$T^1 = [0_1, 1_1, 2_1, 0_1, 3_1, 0_2, 3_2, 2_1, 5_1, 2_2, 5_2, 6_1, 1_1, 6_2, 0_2, 6_1, 4_1, 5_2, 6_2, 4_2, 2_2, 0_1];$$
$$T^2 = [0_1, 5_1, 4_1, 3_1, 2_2, 1_1, 0_2, 5_2, 2_1, 1_2, 6_2, 3_1, 4_2, 6_1, 3_2, 4_1, 1_2, 6_1, 2_2, 6_2, 3_2, 0_1];$$
$$T^3 = [0_1, 4_1, 2_1, 3_1, 5_1, 4_2, 2_2, 0_2, 1_2, 2_2, 3_2, 1_2, 5_1, 1_1, 4_1, 0_2, 2_1, 4_2, 1_1, 3_1, 5_2, 0_1];$$
$$T^4 = [0_1, 2_2, 4_1, 6_2, 5_1, 3_2, 1_1, 5_2, 4_2, 3_2, 5_2, 1_2, 3_1, 6_1, 2_1, 6_2, 0_1, 6_1, 5_1, 0_2, 4_2, 0_1];$$

decompose $K_7 * \overline{K}_2$ and are each 3-colourable (see Figure 3) with maximum degree $\Delta = 4$ as required.

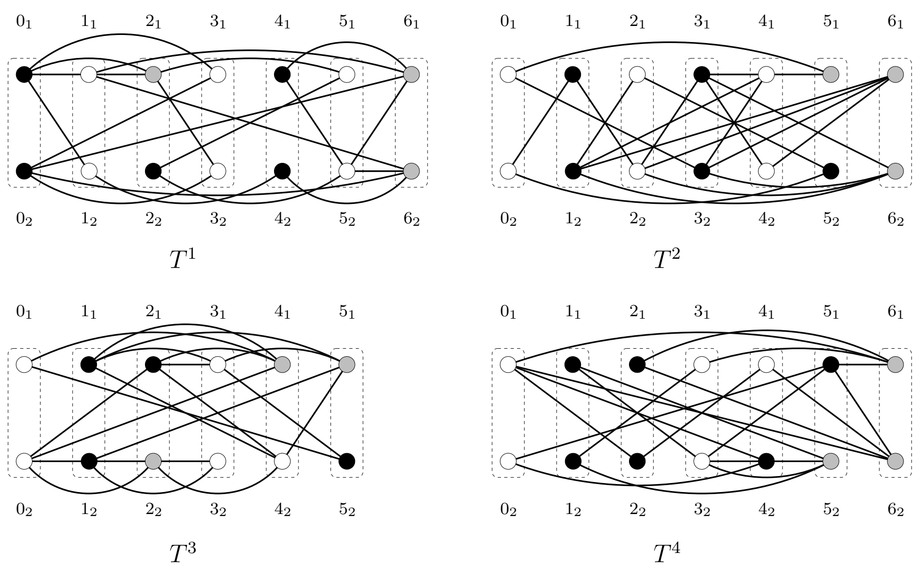

Figure 3: Proper 3-colourings of the trails T^ℓ

(ii) Let the vertex set of $K_7 * \overline{K}_5$ be $\{\{v_1, v_2, v_3, v_4, v_5\} \mid 0 \leqslant v \leqslant 6\}$ and let τ be the permutation

$$(0_1\, 0_2\, 0_3\, 0_4\, 0_5)\, (1_1\, 1_2\, 1_3\, 1_4\, 1_5)\, \cdots\, (6_1\, 6_2\, 6_3\, 6_4\, 6_5)$$

of order 5 on the vertices of $K_7 * \overline{K}_5$. Then the five 21-cycles

$$C^1 = (0_1, 1_1, 0_2, 1_5, 0_3, 1_4, 3_4, 1_3, 3_5, 1_2, 3_1, 2_1, 3_2, 2_5, 3_3, 2_4, 6_4, 2_3, 6_5, 2_2, 6_1);$$
$$C^2 = (0_1, 2_1, 0_2, 2_5, 0_3, 2_4, 4_4, 2_3, 4_5, 2_2, 4_1, 3_1, 4_2, 3_5, 4_3, 3_4, 6_5, 2_3, 6_1, 2_2, 6_2);$$
$$C^3 = (0_1, 3_1, 0_2, 3_5, 0_3, 3_4, 5_4, 3_3, 5_5, 3_2, 5_1, 4_1, 5_2, 4_5, 5_3, 4_4, 6_1, 2_3, 6_2, 2_2, 6_3);$$
$$C^4 = (0_1, 4_1, 0_2, 4_5, 0_3, 4_4, 1_4, 4_3, 1_5, 4_2, 1_1, 5_1, 1_2, 5_5, 1_3, 5_4, 6_2, 2_3, 6_3, 2_2, 6_4);$$
$$C^5 = (0_1, 5_1, 0_2, 5_5, 0_3, 5_4, 2_4, 5_3, 2_5, 5_2, 2_1, 1_1, 2_2, 1_5, 2_3, 1_4, 6_3, 2_3, 6_4, 2_2, 6_5);$$

under the action of the group $\langle\, \tau \,\rangle$ give the required decomposition. □

4 Main theorem

We first deal with $3p$-cycle decompositions of the graph $K_n * \overline{K}_m$ for the specific case $p = 3$.

THEOREM 4.1 *For any* $n \geqslant 3$ *the complete equipartite graph* $K_n * \overline{K}_m$ *has a decomposition into* 9*-cycles if and only if*

(i) $nm \geqslant 9$*;*

(ii) $(n-1)m \equiv 0 \pmod 2$*; and*

(iii) $\dfrac{n(n-1)m^2}{2} \equiv 0 \pmod 9$.

Proof: The necessity of the conditions is obvious, and so we need only prove the sufficiency.

From condition (iii) we have that either $9 \mid n(n-1)/2$ (and hence $n \geqslant 9$) or $3 \mid m$.

Suppose $9 \mid n(n-1)/2$. Then if n is odd the result follows from [3] by blowing up m-fold a 9-cycle decomposition of K_n using Corollary 2.3. Alternatively, if n is even, by condition (ii) we have that m is even and the result follows from [3] by blowing up $(m/2)$-fold a 9-cycle decomposition of $K_n * \overline{K}_2$.

Suppose then that $3 \mid m$. If n is odd then there exists a decomposition of K_n into cycles of length 3 and cycles of length 5 by [2]. Both $C_3 * \overline{K}_3$ and $C_5 * \overline{K}_3$ can be decomposed into 9-cycles by Theorem 2.4 and Lemma 3.6 and the result then follows by blowing up $(m/3)$-fold. Similarly if n is even, then by condition (ii) m is even and the graph $K_n * \overline{K}_2$ can be decomposed into cycles of length 3 and cycles of length 5 (again by [2]). The result then follows by blowing up 9-cycle decompositions of $C_3 * \overline{K}_3$ and $C_5 * \overline{K}_3$ $(m/6)$-fold. □

Finally we present the paper's main result.

THEOREM 4.2 *For any* $n \geqslant 3$ *and prime* $p \geqslant 3$*, the complete equipartite graph* $K_n * \overline{K}_m$ *has a decomposition into cycles of length* $3p$ *if and only if*

(i) $nm \geqslant 3p$*;*

(ii) $(n-1)m \equiv 0 \pmod 2$*; and*

(iii) $\dfrac{n(n-1)m^2}{2} \equiv 0 \pmod{3p}$.

Proof: The necessity of the conditions is obvious, and so we need only prove the sufficiency.

The case $p = 3$ follows from Theorem 4.1. Hence letting n, m and $p \geqslant 5$ satisfy the conditions above, we split the problem into the following three distinct cases:

Case I:	$3 \mid m$;
Case II:	$3 \nmid m$ and $p \mid m$;
Case III:	$3 \nmid m$ and $p \nmid m$.

Case I: $3 \mid m$

Consider the graph $K_n * \overline{K}_{m/3}$. We have

$$\begin{aligned} \frac{nm}{3} &\geqslant p && \text{from (i);} \\ \frac{(n-1)m}{3} &\equiv 0 \pmod 2 && \text{from (ii);} \\ \frac{n(n-1)}{2} \cdot \left(\frac{m}{3}\right)^2 &\equiv 0 \pmod p && \text{from (iii).} \end{aligned}$$

Hence $C_p \mid (K_n * \overline{K}_{m/3})$ by [9], and we have

$$K_n * \overline{K}_{m/3} \cong C_p \oplus C_p \oplus \cdots \oplus C_p\,.$$

Now

$$\begin{aligned} K_n * \overline{K}_m &\cong (K_n * \overline{K}_{m/3}) * \overline{K}_3 \\ &\cong (C_p * \overline{K}_3) \oplus (C_p * \overline{K}_3) \oplus \cdots \oplus (C_p * \overline{K}_3)\,, \end{aligned}$$

and each $C_p * \overline{K}_3$ admits a decomposition into cycles of length $3p$ by Lemma 2.4. Thus $C_{3p} \mid (K_m * \overline{K}_n)$ as required.

Case II: $3 \nmid m$ and $p \mid m$

The proof is exactly as in Case I above, interchanging 3 and p throughout.

Case III: $3 \nmid m$ and $p \nmid m$

From (iii) we have

$$\frac{n(n-1)}{2} \equiv 0 \pmod{3p}\,. \tag{2}$$

We further split this case according to the parity of n.

Case IIIa: n odd

If $n \geqslant 3p$ we can easily obtain our decomposition by blowing up a $3p$-cycle decomposition of K_n (which exists by [3]), as detailed in Corollary 2.3. Therefore we need only consider the case $n < 3p$ (which arises only in cases when $n \not\equiv 0 \pmod{3p}$ and $(n-1)/2 \not\equiv 0 \pmod{3p}$). Hence from (2) we have either

- $n \equiv 0 \pmod p$ and $n \equiv 1 \pmod 6$; or
- $n \equiv 1 \pmod{2p}$ and $n \equiv 0 \pmod 3$.

Since n is odd and $n < 3p$, we therefore have either

- $n = p$ and $p \equiv 1 \pmod 6$; or
- $n = 2p + 1$ and $p \equiv 1 \pmod 6$.

Note however that in the case $n = 2p + 1$ and $p \equiv 1 \pmod 6$, we can show for all even m that $C_{3p} \mid (K_{2p+1} * \overline{K}_m)$ by blowing up $(m/2)$-fold a $3p$-cycle decomposition of $K_{2p+1} * \overline{K}_2 \cong K_{4p+2} - I$ which exists by Šajna [10].

This leaves the following two cases to be solved:

- $K_p * \overline{K}_m$, where $p \equiv 1 \pmod 6$, $m \geqslant 4$ and $3 \nmid m$, $p \nmid m$; and
- $K_{2p+1} * \overline{K}_m$, where $p \equiv 1 \pmod 6$, $m \geqslant 5$, m is odd and $3 \nmid m$, $p \nmid m$.

We deal with these as follows. Note that in each of the following two subcases we make use of the well-known fact that there exist $(q-1)$ MOLS(q) whenever q is a prime power (see for example [1]).

Subcase 1: $K_p * \overline{K}_m$, where $p \equiv 1 \pmod 6$, $m \geqslant 4$ and $3 \nmid m$, $p \nmid m$

If $p \geqslant 13$ we have, by Corollary 3.3, that K_p decomposes into 6-regular, 5-colourable, closed $3p$-trails. Since there exist three MOLS(q) for each prime power $q \geqslant 4$, the result follows by applying Theorem 2.1 and blowing up (m/q)-fold for suitable prime power q.

If $p = 7$ we view K_7 as a closed 21-trail having maximum degree $\Delta = 6$ and $\chi(K_7) = 7$. Suitable decompositions of $K_7 * \overline{K}_4$ and $K_7 * \overline{K}_5$ are given in Lemma 3.7 and since there exist five MOLS(q) for each prime power $q \geqslant 7$ the result follows as above.

Subcase 2: $K_{2p+1} * \overline{K}_m$, where $p \equiv 1 \pmod 6$, $m \geqslant 5$, m is odd and $3 \nmid m$, $p \nmid m$

By Lemma 3.4 we have that K_{2p+1} decomposes into 6-regular, 6-colourable, closed $3p$-trails. Since there exist four MOLS(q) for each prime power $q \geqslant 5$, the result follows by applying Theorem 2.1 and blowing up (m/q)-fold for suitable prime power q.

Case IIIb: n even

Note first that since n is even, m is even also. If $n > 3p/2$, we have $C_{3p} \mid (K_n * \overline{K}_m)$ by blowing up $(m/2)$-fold a $3p$-cycle decomposition of $K_n * \overline{K}_2 \cong K_{2n} - I$ which exists by Šajna [10]. Therefore we need only consider the case $n < 3p/2$. Hence from (2) we have either

- $\frac{n}{2} \equiv 0 \pmod p$ and $n \equiv 1 \pmod 3$; or
- $n \equiv 1 \pmod p$ and $\frac{n}{2} \equiv 0 \pmod 3$.

Since n is even and $n < 3p/2$ we therefore need only consider the case $n = p+1$ and $\equiv 5 \pmod 6$. By Lemma 3.5 we have that $K_{p+1} * \overline{K}_2$ decomposes into 3-colourable closed $3p$-trails with maximum degree $\Delta = 4$. Since there exists a latin square of each order $q \geqslant 2 = \Delta/2$, the result follows by applying Theorem 2.1. □

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