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An algebraic approach for finding balance index sets*

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Abstract

Any vertex labeling $f : V \to \{0, 1\}$ of the graph $G = (V, E)$ induces a partial edge labeling $f^* : E \to \{0, 1\}$ defined by $f^*(uv) = f(u)$ if and only if $f(u) = f(v)$. The balance index set of G is defined as $\{|f^{*-1}(0) - f^{*-1}(1)| \mid |f^{-1}(0) - f^{-1}(1)| \le 1\}$. In this paper, we propose a new and easier approach to find the balance index set of a graph. This new method makes it possible to determine the balance index sets of a large number of families of graphs in an unified and uniform manner.

1 Introduction

Lee, Liu and Tan considered the following graph labeling problem in [11]. Given any vertex labeling $f : V \to \{0, 1\}$ of a simple graph $G = (V, E)$, define a partial edge labeling f^* of G according to

$$f^*(uv) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that the edge uv is unlabeled if $f(u) \neq f(v)$.

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Denote by $v_f(0)$ and $v_f(1)$ the number of vertices of G that are labeled 0 and 1, respectively, under the mapping f. In a similar manner, let $e_f(0)$ and $e_f(1)$ denote, respectively, the number of edges of G that are labeled 0 and 1 by the induced partial function f^*.

We call a vertex labeling f of a graph G *friendly* if $|v_f(0) - v_f(1)| \leq 1$, and *balanced* if, in addition, $|e_f(0) - e_f(1)| \leq 1$. A graph is said to be *balanced* if it admits a balanced labeling. See [4, 5, 16] for further results on balanced graphs. To extend their study of balanced graphs, Lee, Lee and Ng [10] introduced the notion of a *balance index set* of a graph G. It is defined as

$$BI(G) = \{|e_f(0) - e_f(1)| \bigm| \text{the vertex labeling } f \text{ is friendly}\}.$$

In general, it is a difficult task to determine the balance index set of a given graph. Most of existing research on this problem focus on special families of graphs with simple structures [3, 4, 10, 12]. Examples include stars and cycles with a single chord connecting two nonadjacent vertices. Graphs with more complicated structure such as those formed by the amalgamation of complete graphs, stars, and generalized theta graphs, and L-products with cycles and complete graphs were studied in [7–9].

In [17], Zhang, Ho, Lee and Wen investigated some trees of diameter at most four. Their algorithmic approach limits its extension. With a thorough analysis of the vertex labels, Kwong [6] found a complete solution. Although the technique can be extended to any rooted tree, the computational effectiveness makes it impractical as the height of the tree increases.

In this paper, we propose an algebraic approach for finding the balance index set of a graph. The underlying idea is to use an equivalent vertex labeling method which is easier to analyze. This new approach allows us to develop several general results, with which we are able to find the balance index sets of many familiar classes of graphs.

All graphs considered in this paper are simple. Any undefined notation or concept may be found in [1].

2 The New Approach and Its General Properties

Given a vertex labeling $f : V \to \{0, 1\}$ of a graph $G = (V, E)$, define an associated vertex labeling $g : V \to \{-0.5, 0.5\}$ by $g = f - 0.5$. It induces an edge labeling $g^* : E \to \{-1, 0, 1\}$ defined by

$$g^*(uv) = g(u) + g(v)$$

for every $uv \in E$. We would like to remark that, alternatively but equivalently, we may set $g = 2f - 1 : V \to \{-1, 1\}$, and define the induced edge labeling $g^* : E \to \{-2, 0, 2\}$ in a similar manner. Either way, notice that, unlike f, every edge is labeled under g.

The notations $v_g(i)$ and $e_g(i)$ are defined in a similar fashion. Then $g^*(e) = -1$ means $f^*(e) = 0$, $g^*(e) = 1$ means $f^*(e) = 1$, and $g^*(e) = 0$ means the edge e is unlabeled. It is clear that f is friendly if and only if $|v_g(-0.5) - v_g(0.5)| \leq 1$. In such an event, we also say g a *friendly labeling* of G.

Given any vertex labeling g, which needs not be friendly, we call $B_g(G) = e_g(1) - e_g(-1)$ the *balance index* of G under g. If there is no ambiguity, we would simply write $B_g(G)$ as B_g. Thus G is *balanced* if and only if there is a friendly labeling g such that $|B_g(G)| \leq 1$. The *full balance index set* of G is defined as

$$FBI(G) = \{B_g(G) \mid g \text{ is friendly}\}.$$

Then G is *balanced* if and only if $FBI(G) \cap \{-1, 0, 1\} \neq \varnothing$. A graph is called *strongly balanced* if $|B_g(G)| \leq 1$ for any friendly labeling g. In contrast, a graph is called *non-balanced* $|B_g(G)| \geq 2$ for any friendly labeling g.

It was remarked in [6] that the balance index set of a rooted tree depends on its degree sequence. Our first result demonstrates why this is true in general.

Lemma 2.1. *Let g be any labeling of a graph $G = (V, E)$. Then*

$$B_g(G) = \sum_{e \in E} g^*(e) = \sum_{u \in V} \deg(u) g(u).$$

Proof: By definition $B_g(G) = \sum_{e \in E} g^*(e)$. Since

$$2 \sum_{e \in E} g^*(e) = \sum_{u \in V} \sum_{uv \in E} (g(u) + g(v)) = 2 \sum_{w \in V} \deg(w) g(w),$$

the proof is complete. □

The balance index set defined in [7–10, 12, 17, 18] can be stated as

$$BI(G) = \{|B_g(G)| \mid g \text{ is a friendly labeling}\}.$$

From Lemma 2.1, we find $B_{-g} = -B_g$ for any labeling g. Hence $FBI(G) = -BI(G) \cup BI(G)$. Consequently, we will only consider balance index sets from now on.

Theorem 2.2. *For any graphs G and H having the same degree sequence, $BI(G) = BI(H)$.*

Proof: Let ϕ be a degree-preserving bijection from $V(H)$ to $V(G)$. For any friendly labeling g of G, the mapping $g \circ \phi$ is a friendly labeling of H. According to Lemma 2.1, $B_g(G) = B_{g \circ \phi}(H)$. Similarly, for any friendly labeling h of H, we find $B_h(H) = B_{h \circ \phi^{-1}}(G)$. The theorem follows immediately. □

We are now ready to apply these results to some well-known families of graphs.

3 Regular Graphs

For brevity, we write $v_g^+ = v_g(0.5)$ and $v_g^- = v_g(-0.5)$. It is easy to see that

$$\sum_{u \in V} g(u) = \frac{1}{2}(v_g^+ - v_g^-).$$

This observation leads to the following result on regular graphs.

Theorem 3.1. *For any labeling g of an r-regular graph G, we have*

$$B_g(G) = \frac{r}{2}(v_g^+ - v_g^-).$$

Hence $BI(G) = \{0\}$ if G is of even order, and $BI(G) = \left\{\frac{r}{2}\right\}$ if G is of odd order.

Proof: The result is a direct consequence of Lemma 2.1, because

$$B_g(G) = r \sum_{u \in V(G)} g(u) = \frac{r}{2}(v_g^+ - v_g^-).$$

In particular, when g is friendly, $|v_g^+ - v_g^-| = 0$ if G is of even order, and $|v_g^+ - v_g^-| = 1$ if G is of odd order, which leads to the second half of the theorem. □

Corollary 3.2. *The n-cycle C_n is strongly balanced for all $n \geq 3$.*

Proof: The result follows from the fact that $B_g(C_n) = v_g^+ - v_g^-$ for any labeling g of C_n. □

Using the same idea, we deduce three more corollaries.

Corollary 3.3. ([9]). *For $n \geq 2$, $BI(K_n) = \{0\}$ if n is even, and $BI(K_n) = \left\{\frac{n-1}{2}\right\}$ if n is odd.*

Corollary 3.4. *An r-regular graph, where $r \geq 1$, is strongly balanced if it has an even order, and non-balanced if its order is odd and $r \geq 4$.*

Corollary 3.5. *For $n, m \geq 3$, $BI(C_m \times C_n) = \{0\}$ if mn is even, and $BI(C_m \times C_n) = \{2\}$ if mn is odd.*

We close this section with another important lemma, which is easy to deduce, hence we omit its proof.

Lemma 3.6. *Consider $G = (V, E)$ and a labeling $g : V' \to \{-0.5, 0.5\}$, where $V' \subset V$. If $|V'| \leq |V \setminus V'| + 1$, then g can be extended to a friendly labeling of G. Moreover, if $|V'| \leq |V \setminus V'|$, then g can be extended to be a friendly labeling of G such that $v_g^+ \geq v_g^-$.*

4 Biregular Graphs

A graph is (r, s)-*regular* ($r \neq s$) if the degree of each vertex is either r or s. A graph is *biregular* if it is (r, s)-regular for some distinct integers r and s. The simplest biregular graphs are paths and stars. Each of them is $(r, 1)$-regular for some $r \geq 2$.

Given any labeling g of a graph G, denote by $v_{g,k}^+$ and $v_{g,k}^-$ the number of vertices of degree k in G that are assigned positive and negative label, respectively.

Let $G = (V, E)$ be an (r, s)-regular graph and g a labeling of G. From Lemma 2.1, we find

$$\begin{aligned} B_g(G) &= r \sum_{\substack{u\in V \\ \deg(u)=r}} g(u) + s \sum_{\substack{v\in V \\ \deg(v)=s}} g(v) \\ &= (r-s) \sum_{\substack{u\in V \\ \deg(u)=r}} g(u) + s \sum_{v\in V} g(v). \end{aligned}$$

Hence

$$B_g(G) = \frac{r-s}{2}(v_{g,r}^+ - v_{g,r}^-) + \frac{s}{2}(v_g^+ - v_g^-). \tag{4.1}$$

This formula enables us to determine the balance index sets of many biregular graphs.

4.1 Regular Caterpillar

The *derived graph* $D(T)$ of a graph G is the graph obtained from G by deleting all pendant vertices of G. A *caterpillar* is tree T such that $D(T) = P$ is a path. More specifically, let $P = x_1 \cdots x_t$, and assume x_i is adjacent to n_i pendant vertices in T, where $n_1, n_t \geq 1$, and $n_i \geq 0$ for $2 \leq i \leq t-1$. Then we denote T by $cat(n_1, \dots, n_t)$. The path P is called the *central path*. If $n_1 = n_t = r+1$, and $n_i = r$ for $2 \leq i \leq t-1$ for some $r \geq 1$, then $cat(n_1, \dots, n_t)$ is $(r+2, 1)$-regular. If $r = 0$, then $cat(1, 0, \dots, 0, 1)$ reduces to P_{t+2}. If $t = 1$, then $cat(n)$ is the star graphs $S(n)$. The following result is easy to derive.

Theorem 4.1. ([9, 17, 18]). *For $k \geq 1$, $BI(S(2k+1)) = \{k\}$, and $BI(S(2k)) = \{k-1, k\}$.*

Next, we consider $cat(r+1, r, \dots, r, r+1)$ with central path P_t, where $t \geq 2$ and $r \geq 0$. It contains t vertices of degree $r+2$, and $tr+2$ vertices of degree 1. Let g be any labeling.

For $r = 0$, the caterpillar becomes the path P_{t+2}. It contains t vertices of degree 2 and two vertices of degree 1. Using (4.1), we find

$$B_g(P_{t+2}) = -\frac{1}{2}(v_{g,1}^+ - v_{g,1}^-) + (v_g^+ - v_g^-) = (1 - v_{g,1}^+) + (v_g^+ - v_g^-).$$

Because of Lemma 3.6, we know $B_g(P_{t+2})$ attains the values of $1+(v_g^+ - v_g^-)$, $(v_g^+ - v_g^-)$, and $-1 + (v_g^+ - v_g^-)$. We have proved the following.

Theorem 4.2. ([9]). *For $n \geq 4$, $BI(P_n) = \{0, 1, 2\}$ if n is odd, and $BI(P_n) = \{0, 1\}$ if n is even. Moreover, $BI(P_2) = \{0\}$ and $BI(P_3) = \{0, 1\}$.*

For $r \geq 1$, let $B_g = B_g(cat(r+1, r, \dots, r, r+1))$. Then

$$\begin{aligned} B_g &= \frac{r+1}{2}(v_{g,r+2}^+ - v_{g,r+2}^-) + \frac{1}{2}(v_g^+ - v_g^-) \\ &= \frac{r+1}{2}(2v_{g,r+2}^+ - t) + \frac{1}{2}(v_g^+ - v_g^-). \end{aligned}$$

Assume g is friendly with $B_g \geq 0$. Then $2v^+_{g,r+2} - t \geq 0$. When t is even, the caterpillar is of even order. In this case $B_g \in \left\{(r+1)i \;\middle|\; 0 \leq i \leq \frac{t}{2}\right\}$. When t is odd and r is odd, the caterpillar is of even order. In this case, $B_g \in \left\{(r+1)i + \frac{r+1}{2} \;\middle|\; 0 \leq i \leq \frac{t-1}{2}\right\}$. When t is odd and r is even, the caterpillar is of odd order. In this case, $B_g \in \left\{(r+1)i + \frac{r}{2},\ (r+1)i + \frac{r}{2} + 1 \;\middle|\; 0 \leq i \leq \frac{t-1}{2}\right\}$. Lemma 3.6 asserts that all these values are attainable, which leads to our next result.

Theorem 4.3. *For $t \geq 1$ and $r \geq 1$, let $G = cat(r+1, r, \ldots, r, r+1)$ with central path P_t. Then*

$$BI(G) = \begin{cases} \left\{(r+1)i \;\middle|\; 0 \leq i \leq \frac{t}{2}\right\} & \text{if } t \text{ is even,} \\ \left\{(r+1)i + \frac{r+1}{2} \;\middle|\; 0 \leq i \leq \frac{t-1}{2}\right\} & \text{if } t \text{ and } r \text{ are odd,} \\ \left\{(r+1)i + \frac{r}{2},\ (r+1)i + \frac{r}{2} + 1 \;\middle|\; 0 \leq i \leq \frac{t-1}{2}\right\} & \text{if } t \text{ is odd and} r \text{ is even.} \end{cases}$$

4.2 Complete Bipartite Graphs

Another famous biregular graph is the complete bipartite graph $K_{m,n}$, in which we may assume $2 \leq m \leq n$. From (4.1) we have $B_g(K_{m,n}) = \frac{n-m}{2}(2v^+_{g,n} - m) + \frac{m}{2}(v^+_g - v^-_g)$ for any labeling g. The balance index set of $K_{m,n}$ can now be easily obtained from Lemma 3.6.

Theorem 4.4. *For $2 \leq m \leq n$,*

$$BI(K_{m,n}) = \begin{cases} \left\{\frac{n-m}{2}(2i-m) \;\middle|\; \lceil \frac{m}{2} \rceil \leq i \leq m\right\} & \text{if } m \equiv n \pmod 2, \\ \left\{\frac{n-m}{2}(2i-m) + \frac{m}{2} \;\middle|\; \lceil \frac{m}{2} - \frac{m}{2(n-m)} \rceil \leq i \leq m\right\} & \text{if } m \not\equiv n \pmod 2. \end{cases}$$

4.3 Join of Two Regular Graphs

Let G and H be r-regular and s-regular graphs, of order p_G and p_H respectively. Then their join graph $G \vee H$ is $(r+p_H, s+p_G)$-regular. Furthermore, it contains p_G vertices of degree $r + p_H$, and p_H vertices of $s + p_G$. Without loss of generality, we may assume $p_G \leq p_H$. Let g be any labeling of $G \vee H$. It follows from (4.1) that

$$B_g(G \vee H) = \frac{r + p_H - s - p_G}{2}(v^+_{g,r+p_H} - v^-_{g,r+p_H}) + \frac{s + p_G}{2}(v^+_g - v^-_g). \tag{4.2}$$

For example, the wheel graph W_n is $K_1 \vee C_{n-1}$, where $n \geq 4$. From (4.2) we find

$$B_g(W_n) = \frac{n-4}{2}(v^+_{g,n-1} - v^-_{g,n-1}) + \frac{3}{2}(v^+_g - v^-_g).$$

Assume g is friendly. When n is even, $B_g(W_n) = \frac{n-4}{2}(v^+_{g,n-1} - v^-_{g,n-1})$, hence $B_g(W_n) = \frac{n-4}{2}$. When n is odd, without loss of generality, we may assume $v^+_{g,n-1} - v^-_{g,n-1} = 1$. Then $B_g(W_n) = \frac{n-4}{2} \pm \frac{3}{2}$; both values are attainable, according to Lemma 3.6. Hence we have

Theorem 4.5. *For $n \geq 4$,*

$$BI(W_n) = \begin{cases} \left\{\frac{n-4}{2}\right\} & \text{if } n \text{ is even,} \\ \left\{\frac{n-1}{2},\ \frac{n-7}{2}\right\} & \text{if } n \text{ is odd.} \end{cases}$$

4.4 One Point Union of Regular Graphs

Given $m \geq 2$ simple graphs $G_1, \ldots, G_m$, pick a specified vertex x_i from each graph. The graph $\mathrm{Amal}(G_1, \ldots, G_m)$ is the amalgamation of $G_1, \ldots, G_m$ formed by identifying the vertices x_i. We shall denote this vertex as c and call it the *core* of $\mathrm{Amal}(G_1, \ldots, G_m)$. If $G_1, \ldots, G_m$ are the same graph, say H, then $\mathrm{Amal}(G_1, \ldots, G_m)$ is simply denoted as $\mathrm{Amal}(H, m)$.

For example, if G_i are paths, and x_i is one of its two pendants, then $\mathrm{Amal}(G_1, \ldots, G_m)$ is a spider with m legs. If G_i are n-cycles, then $\mathrm{Amal}(C_n, m)$ is called a *flower graph* [9] which is denoted $F(n, m)$. If G_i are the complete graph K_n, then $\mathrm{Amal}(K_n, m)$ is called a *regular windmill graph* [9] which is denoted $WM(n, m)$.

Theorem 4.6. *For $m, r \geq 2$, let $G_1, \ldots G_m$ be r-regular graphs. Let $G = Amal(G_1, \ldots, G_m)$. Then*

$$BI(G) = \begin{cases} \left\{\frac{(m-1)r}{2}\right\} & \text{if } G \text{ is of even order,} \\ \left\{\frac{mr}{2}, \frac{(m-2)r}{2}\right\} & \text{if } G \text{ is of odd order.} \end{cases}$$

Proof: Clearly $G = (V, E)$ is (mr, r)-regular, and only the core c is of degree mr. For any friendly labeling g, apply (4.1) to obtain

$$B_g = \frac{(m-1)r}{2}(v^+_{g,mr} - v^-_{g,mr}) + \frac{r}{2}(v^+_g - v^-_g).$$

Without lost of generality, we may assume $B_g \geq 0$. Then $v^+_{g,mr} = 1$, and $v^-_{g,mr} = 0$. Thus, $B_g = \frac{(m-1)r}{2} + \frac{r}{2}(v^+_g - v^-_g)$. Hence

$$B_g = \begin{cases} \frac{(m-1)r}{2} & \text{if } |V| \text{ is even,} \\ \frac{(m-1)r}{2} \pm \frac{r}{2} & \text{if } |V| \text{ is odd.} \end{cases}$$

The proof is completed by observing that both values in the odd case are attainable. □

Corollary 4.7. ([17]). *For $m \geq 2$ and $n \geq 3$,*

$$BI(F(n, m)) = \begin{cases} \{m-1\} & \text{if } n \text{ is even and } m \text{ is odd,} \\ \{m,\ m-2\} & \text{otherwise.} \end{cases}$$

Corollary 4.8. ([9]). *For $m \geq 2$ and $n \geq 3$,*

$$BI(WM(n, m)) = \begin{cases} \left\{\frac{(m-1)(n-1)}{2}\right\} & \text{if } WM(n, m) \text{ of even order,} \\ \left\{\frac{(m-2)(n-1)}{2}, \frac{m(n-1)}{2}\right\} & \text{if } WM(n, m) \text{ of odd order.} \end{cases}$$

4.5 Connected Bicyclic Graph without Pendant

A connected $(p, p+1)$-graph G is called a *bicyclic* graph. Here are three families of examples.

- The *one-point union of two cycles* C_m and C_n, where $m, n \geq 3$, is denoted $U(m, n)$. It is sometimes referred to as a *double cycle*.

- A *cycle with a long chord* is the graph obtained from an m-cycle, where $m \geq 4$, by adding a chord of length $l \geq 1$. If the m-cycle is $u_0u_1 \cdots u_{m-1}u_0$, without loss of generality, we may assume the chord joins u_0 to u_i, where $2 \leq i \leq m-2$. We denote this graph $C_m(i; l)$.

- For $m, n \geq 3$, a *long dumbbell graph* is the graph obtained from two cycles C_m and C_n by joining them with a path of length $l \geq 1$. Without loss of generality, we may assume

  $$C_m = u_0u_1 \cdots u_{m-1}u_0, \quad P_l = u_{m-1}u_m \cdots u_{m+l-1}$$

  $$\text{and} \quad C_n = u_{m+l-1}u_{m+l} \cdots u_{m+n+l-2}u_{m+l-1}.$$

  We denote this graph with the notation $D(m, n; l)$.

Because of the next result, any $(p, p+1)$-graph must belong to one of these three groups.

Theorem 4.9. ([14]). *A bicyclic graph without pendant is either a one-point union of two cycles, a long dumbbell graph, or a cycle with a long chord.*

From Theorem 4.6 we have

Corollary 4.10. *For* $m, n \geq 3$,

$$BI(U(m, n)) = \begin{cases} \{1\} & \text{if } m+n-1 \text{ is even,} \\ \{0, 2\} & \text{if } m+n-1 \text{ is odd.} \end{cases}$$

Let G be $C_m(i; l)$ or $D(m, n; l)$. It is a $(3, 2)$-regular graphs with exactly two vertices of degree 3. Let g be any labeling of G, because of (4.1) we find $B_g(G) = (v^+_{g,3} - 1) + (v^+_g - v^-_g)$. Hence we obtain the following result.

Theorem 4.11. *Let* G *be* $C_m(i; l)$ *or* $D(m, n; l)$. *Then*

$$BI(G) = \begin{cases} \{0, 1\} & \text{if } G \text{ is of even order,} \\ \{0, 1, 2\} & \text{if } G \text{ is of odd order.} \end{cases}$$

4.6 Sun Graphs

Let $C_n = v_1v_2\cdots v_nv_1$ be an n-cycle ($n \geq 3$) and $\langle e_1, \ldots, e_n\rangle$ be the ordered set of $E(C_n)$ where $e_i = v_iv_{i+1}$ (with the conventiont that $v_{n+1} = v_1$). Let $t_1, \ldots, t_n$ be positive integers. The *sun graph* with *parameters* $(t_1, t_2, \ldots, t_n)$, denoted $C_n(t_1, \ldots, t_n)$, is obtained from C_n by connecting both ends of each e_i with a path H_i of length t_i. This construction was introduced in [15]. Since we only consider simple graph, we assume $t_1, \ldots, t_n > 1$.

The sun graph $C_n(t_1, \ldots, t_n)$ is a $(4, 2)$-regular graph with n vertices of degree 4. It follows from (4.1) that $B_g(C_n(t_1, \ldots, t_n)) = (2v^+_{g,4} - n) + (v^+_g - v^-_g)$ for any labeling g. Since the order of $C_n(t_1, \ldots, t_n)$ is $\sum_{i=1}^n t_i$, which is at least n, Lemma 3.6 yields

Theorem 4.12. *For $n \geq 3$ and $t_1, \ldots, t_n > 1$,*

$$BI(C_n(t_1, \ldots, t_n)) = \begin{cases} \{2i - n \mid \lceil \frac{n}{2} \rceil \leq i \leq n\} & \text{if } \sum_{i=1}^n t_i \text{ is even,} \\ \{2i + 1 - n \mid \lfloor \frac{n}{2} \rfloor \leq i \leq n\} & \text{if } \sum_{i=1}^n t_i \text{ is odd.} \end{cases}$$

4.7 Generalized Theta Graphs

For $k \geq 2$, let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ be k positive integers. The *generalized theta graph* $\Theta(a_1, a_2, \ldots, a_k)$ is obtained by connecting two vertices v_1 and v_2 with k parallel and non-intersecting paths of length $a_1, a_2, \ldots, a_k$ (see [13]). Obviously, $\deg(v_1) = \deg(v_2) = k$, and all other vertices are of degree 2. Since we only consider simple graphs, we also assume $a_2 \geq 2$. From (4.1), we learn that $B_g(\Theta(a_1, a_2, \ldots, a_k)) = (k - 2)(v^+_{g,k} - 1) + (v^+_g - v^-_g)$ for any labeling g of the graph. We deduce from Lemma 3.6 the following.

Theorem 4.13. **([7]).** *For $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$, where $a_2 \geq 2$,*

$$BI(\Theta(a_1, a_2, \ldots, a_k)) = \begin{cases} \{0, k-2\} & \text{if } \sum_{i=1}^k a_i - k \text{ is even,} \\ \{1, k-3, k-1\} & \text{if } \sum_{i=1}^k a_i - k \text{ is odd.} \end{cases}$$

4.8 Cartesian Product of Regular Graphs with Biregular Graphs

Let G be a k-regular graph and H an (r, s)-regular graph. Then $G \times H$ is $(k+r, k+s)$-regular. In particular, the cylinder graph $C_m \times P_n$ is a $(3, 4)$-regular graph when $n \geq 3$.

For any labeling g, from (4.1) we have $B_g(C_m \times P_3) = \frac{1}{2}(2v^+_{g,4} - m) + \frac{3}{2}(v^+_g - v^-_g)$, and

$$B_g(C_m \times P_n) = -(v^+_{g,3} - m) + 2(v^+_g - v^-_g) = (m - v^+_{g,3}) + 2(v^+_g - v^-_g)$$

when $n \geq 4$. For $n = 3$ and odd m, if g is friendly, then

$$|B_g(C_m \times P_3)| \leq \left|v^+_{g,4} - \frac{m}{2}\right| + \frac{3}{2} \leq \frac{m}{2} + \frac{3}{2} = \frac{m+3}{2}.$$

Similarly, if $n \geq 4$ and mn is odd, then $|B_g(C_m \times P_n)| \leq m+2$. For brevity, given two nonnegative integers $k \leq l$, we define

$$[\boldsymbol{k}, \boldsymbol{l}] = \{i \in \mathbb{N} \mid k \leq i \leq l\}.$$

We obtain the following result with the aid of Lemma 3.6.

Theorem 4.14. *For $n \geq 3$ and $m \geq 3$,*

$$BI(C_m \times P_n) = \begin{cases} [\mathbf{0}, \frac{\boldsymbol{m}}{\mathbf{2}}] & \text{if } m \text{ is even and } n = 3, \\ [\mathbf{0}, \frac{\boldsymbol{m}+\mathbf{3}}{\mathbf{2}}] & \text{if } m \text{ is odd and } n = 3, \\ [\mathbf{0}, \boldsymbol{m}] & \text{if } mn \text{ is even and } n \geq 4, \\ [\mathbf{0}, \boldsymbol{m}+\mathbf{2}] & \text{if } mn \text{ is odd and } n \geq 4. \end{cases}$$

4.9 Earth Graphs

The graph obtained from the cylinder graph $P_m \times C_n$, where $m, n \geq 3$, by contracting the each of the top and the bottom cycles into a single vertex is called a *earth graph* $E(m,n)$. The cycles come from C_n are called the *latitudes* and the paths come from P_m are called the *longitudes*. The earth graph $E(m,n)$ is a $(4,n)$-regular graph if $n \neq 4$. There are two vertices of degree n, and $n(m-2)$ vertices of degree 4. Therefore $B_g(E(m,n)) = \frac{n-4}{2}(2-2v_{g,n}^-) + 2(v_g^+ - v_g^-)$, where $v_{g,n}^- = 0,1,2$.

For even n, $B_g(E(m,n)) = (n-4)-(n-4)v_{g,n}^-$ if g is friendly, hence $BI(E(m,n)) = \{0, n-4\}$. For odd n, since $n(m-2) > 2$, Lemma 3.6 states that any pre-labeling of the two vertices of degree n can be extended to a friendly labeling with $v_g^+ - v_g^- = 1$. Thus $B_g(E(m,n)) = (n-4)-(n-4)v_{g,n}^- + 2$ for $v_{g,n}^- = 0,1,2$. Hence $BI(E(m,n)) = \{2, n-2\}$.

Theorem 4.15. *For $n \geq 5$,*

$$BI(E(m,n)) = \begin{cases} \{0, n-4\} & \text{if } n \text{ is even}, \\ \{2, n-2\} & \text{if } n \text{ is odd}. \end{cases}$$

4.10 Regular Halin Graphs

Let T be a tree all of whose interior vertices (also called nodes) are of degree at least three. A *Halin graph* $G = T \cup C$ is a planar graph that consists of a planar embedding of T and a cycle C connecting the leaves of T such that C is the boundary of the exterior face. The tree T and the cycle C are called the *characteristic tree* and the *adjoint cycle* of G, respectively. If all nodes of T are of the same degree r, we call G an *r-regular Halin graph.*

If $r = 3$, then G is a cubic graph, which was studied in Section 3; so we assume $r \geq 4$. Then G is a $(r,3)$-regular graph. Assume G is of order n. It is clear that there are $\frac{n-2}{r-1}$ nodes and $n - \frac{n-2}{r-1}$ leaves on the characteristic tree. From (4.1) we find, for any labeling g,

$$B_g(G) = \frac{r-3}{2}\left(2v_{g,r}^+ - \frac{n-2}{r-1}\right) + \frac{3}{2}(v_g^+ - v_g^-),$$

We leave this problem as an exercise to the reader.

5 Triregular Graphs

Let r, s and t be three distinct positive integers. A graph is (r,s,t)-*regular* if the degree of each vertex is either r, s or t. A graph is *triregular* if it is (r,s,t)-regular. Many famous classes of graphs are triregular. For examples, m-ary trees, spider graphs, grids (that is, $P_m \times P_n$), and Halin graph with an m-ary tree as its characteristic tree.

Let $G=(V,E)$ be an (r,s,t)-regular graph with a labeling g. From Lemma 2.1 we have

$$\begin{aligned} B_g(G) &= r\sum_{\deg(u)=r} g(u) + s\sum_{\deg(v)=s} g(v) + t\sum_{\deg(w)=t} g(w) \\ &= r\sum_{u\in V} g(u) + (s-r)\sum_{\deg(v)=s} g(v) + (t-r)\sum_{\deg(w)=t} g(w). \end{aligned}$$

Hence

$$B_g(G) = \frac{r}{2}(v_g^+ - v_g^-) + \frac{s-r}{2}(v_{g,s}^+ - v_{g,s}^-) + \frac{t-r}{2}(v_{g,t}^+ - v_{g,t}^-). \tag{5.1}$$

We shall use this formula to find the balance index sets of the families of graphs mentioned above.

5.1 m-ary Trees

A rooted tree T is called an m-ary tree if every vertex that is not a leaf has $m \geq 2$ children. Hence T is a $(m+1, m, 1)$-regular graph. Moreover, only the root is of degree m.

Theorem 5.1. *For $m \geq 2$, let T be an m-ary tree of order n. Then*

$$BI(T) = \begin{cases} \left\{\left|\frac{(m-1)n}{2} + 1 - mi\right| \;:\; 0 \leq i \leq n - \frac{n-1}{m}\right\} & \text{if } n \text{ is even,} \\ \left\{\left|\frac{(m-1)(n+1)}{2} + 1 - mi\right|, \left|\frac{(m-1)(n+1)}{2} + 2 - mi\right| \;:\; 0 \leq i \leq n - \frac{n-1}{m}\right\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Note that $n-1$ is a multiple of m. Let ℓ be the number of leaves, and a the number of non-leaves, so that $a+\ell=n$. It is known that [2]

$$\ell = 2 + (m-2)\times 1 + (m-1)\times(a-1) = ma - a + 1.$$

Since $a+\ell=n$, $a=\frac{n-1}{m}$ and $\ell = n - \frac{n-1}{m}$. Thus $\ell - a = n - \frac{2(n-1)}{m} \geq n-(n-1) = 1$.

Let g be a friendly labeling of T. First consider the case in which n is even. Because of (5.1), we have

$$B_g(T) = -\frac{1}{2}(v_{g,m}^+ - v_{g,m}^-) - \frac{m}{2}(v_{g,1}^+ - v_{g,1}^-).$$

We may assume that the value of the root under g is negative. So the formula above becomes

$$B_g(T) = \frac{1}{2} - \frac{m}{2}(v_{g,1}^+ - v_{g,1}^-) = \frac{m\ell+1}{2} - mv_{g,1}^+ = \frac{(m-1)n}{2} + 1 - mv_{g,1}^+,$$

where $0 \le v^+_{g,1} \le n - \frac{n-1}{m}$. Lemma 3.6 implies that $BI(T) = \big\{\big|\frac{(m-1)n}{2} + 1 - mi\big| : 0 \le i \le n - \frac{n-1}{m}\big\}$.

Next, assume n is odd. We may also assume $v^+_g - v^-_g = 1$. Then, from (5.1), we have

$$B_g(T) = \frac{m+1}{2} - \frac{1}{2}(v^+_{g,m} - v^-_{g,m}) - \frac{m}{2}(v^+_{g,1} - v^-_{g,1}).$$

If $v^+_{g,m} - v^-_{g,m} = 1$, then

$$B_g(T) = \frac{m}{2} - \frac{m}{2}(2v^+_{g,1} - \ell) = \frac{(m-1)(n+1)}{2} + 1 - mv^+_{g,1},$$

where $0 \le v^+_{g,1} \le n - \frac{n-1}{m}$. Similarly, if $v^+_{g,m} - v^-_{g,m} = -1$, then

$$B_g(T) = \frac{(m-1)(n+1)}{2} + 2 - mv^+_{g,1},$$

where $0 \le v^+_{g,1} \le n - \frac{n-1}{m}$. The result follows from Lemma 3.6. □

Example 5.1. Suppose T is a binary tree, that is, $m = 2$. Then T must be an odd tree. It follows from Theorem 5.1 that $BI(T) = \big\{i \,\big|\, 0 \le i \le \frac{n+5}{2}\big\} = \big[\mathbf{0}, \boldsymbol{\frac{n+5}{2}}\big]$.

5.2 Spider Graphs

The spider graphs is a one point union of paths. For each path P_{n_i} with $n_i \ge 2$, we choose one of the two end vertices as the specified vertex. The graph Amal$(P_{n_1}, \dots, P_{n_m})$ is denoted $Sp(n_1, \dots, n_m)$. We call it a *spider graph* with m legs. In particular, $Sp(2, \dots, 2)$ is the star $S(m)$. In this section, we only consider spider graph which is neither a star nor a path. In other words, we assume $m \ge 3$ and $N = \sum_{i=1}^{m} n_i > 2m$. Hence $Sp(n_1, \dots, n_m)$ consists of one vertex of degree m, m vertices of degree 1, and $N - 2m$ vertices of degree 2.

Let $G = Sp(n_1, \dots, n_m)$. The order of G is $N - m + 1$. If $N > 3m$, then there are more vertices of degree 2 than leaves. In this case, we may pre-label the leaves and the core, and, by Lemma 3.6, the labeling can be extended to a friendly labeling of G. Then (5.1) becomes

$$B_g(G) = (v^+_g - v^-_g) + \frac{m-2}{2}(v^+_{g,m} - v^-_{g,m}) - \frac{1}{2}(v^+_{g,1} - v^-_{g,1}).$$

If $N - m + 1$ is even, we may assume $v^+_{g,m} - v^-_{g,m} = 1$; then

$$B_g(G) = (m-1) - v^+_{g,1},$$

where $0 \le v^+_{g,1} \le m$. Thus, $B_g(G)$ may attend the values $m-1, m-2, \dots, 1, 0, -1$. Therefore, $BI(G) = \big\{i \,\big|\, 0 \le i \le m-1\big\}$.

If $N - m + 1$ is odd, we may assume $v^+_g - v^-_g = 1$. Then

$$B_g(G) = 1 + \frac{m-1}{2}(v^+_{g,m} - v^-_{g,m}) - v^+_{g,1}.$$

If $v^+_{g,m} - v^-_{g,m} = 1$, then $B_g(G) = m - v^+_{g,1}$. If $v^+_{g,m} - v^-_{g,m} = -1$, then $B_g(G) = 2 - v^+_{g,1}$. Here $0 \le v^+_{g,1} \le m$. Thus, $BI(G) = \big\{i \,\big|\, 0 \le i \le m\big\}$.

Theorem 5.2. *For $m \geq 3$, let $G = Sp(n_1, \ldots, n_m)$ and $N = \sum_{i=1}^{m} n_i$. If $N > 3m$, then*

$$BI(G) = \begin{cases} [\mathbf{0}, \boldsymbol{m-1}] & \text{if } N-m+1 \text{ is even,} \\ [\mathbf{0}, \boldsymbol{m}] & \text{if } N-m+1 \text{ is odd.} \end{cases}$$

For $2m \leq N < 3m$, (5.1) becomes

$$B_g(G) = \frac{1}{2}(v_g^+ - v_g^-) + \frac{1}{2}(v_{g,2}^+ - v_{g,2}^-) + \frac{m-1}{2}(v_{g,m}^+ - v_{g,m}^-).$$

If $N - m + 1$ is even, we may assume the core is assigned the positive label. Then

$$B_g(G) = \frac{1}{2}(N - m - 1) - v_{g,2}^-,$$

where $0 \leq v_{g,2}^- \leq N - 2m$. Lemma 3.6 asserts that $BI(G) = \big\{ i \mid \frac{1}{2}(3m - N - 1) \leq i \leq \frac{1}{2}(N - m - 1) \big\}$.

If $N - m + 1$ is odd, then we may assume $v_g^+ - v_g^- = 1$. If $v_{g,m}^+ - v_{g,m}^- = 1$, then $B_g(G) = \frac{N-m}{2} - v_{g,2}^-$. If $v_{g,m}^+ - v_{g,m}^- = -1$, then $B_g(G) = \frac{N-3m}{2} + 1 - v_{g,2}^-$. Here $0 \leq v_{g,2}^- \leq N - 2m$. Combining the two cases we find $BI(G) = \big\{ i \mid \frac{1}{2}(3m - N) - 1 \leq i \leq \frac{1}{2}(N - m) \big\}$.

Theorem 5.3. *For $m \geq 3$, let $G = Sp(n_1, \ldots, n_m)$ and $N = \sum_{i=1}^{m} n_i$. If $2m \leq N < 3m$, then*

$$BI(G) = \begin{cases} \left[\boldsymbol{\frac{3m-N-1}{2}}, \boldsymbol{\frac{N-m-1}{2}}\right] & \text{if } N-m+1 \text{ is even,} \\ \left[\boldsymbol{\frac{3m-N}{2}} - \mathbf{1}, \boldsymbol{\frac{N-m}{2}}\right] & \text{if } N-m+1 \text{ is odd.} \end{cases}$$

5.3 Grids

In this section we consider the grid graph $P_m \times P_n$, where $m \geq n \geq 2$. When $m = n = 2$, the grid graph is a 4-cycle, which has been discussed in Section 2. When $m > n = 2$, the grid graph becomes a $(3, 2)$-regular graph. It is easy to show that $BI(P_m \times P_2) = [\mathbf{0}, \mathbf{2}]$. We leave it as an exercise for the reader.

We now consider $m \geq n \geq 3$, in which case $P_m \times P_n$ is a $(2, 3, 4)$-regular graph. For each $i = 2, 3, 4$, let V_i denote the set of vertices of degree i in $P_m \times P_n$. Then $|V_2| = 4$, $|V_3| = 2m + 2n - 8$, and $|V_4| = (m - 2)(n - 2)$.

Theorem 5.4. *For $m \geq 3$,*

$$BI(P_m \times P_3) = \begin{cases} \left[\mathbf{0}, \boldsymbol{\frac{m}{2}} + \mathbf{1}\right] & \text{if } m \text{ is even,} \\ \left[\mathbf{0}, \boldsymbol{\frac{m+1}{2}} + \mathbf{2}\right] & \text{if } m \text{ is odd.} \end{cases}$$

Proof: Let $V' = V_2 \cup V_4$. Since $n = 3$, we have $|V'| = m + 2$, and $m + 2 \leq (2m - 2) + 1 = |V_3| + 1$ when $m \geq 3$. Hence V' satisfies the condition of Lemma 3.6. Therefore any pre-labeling g of V' can be extended to a friendly labeling of $P_m \times P_3$. From (5.1) we deduce that

$$B_g(P_m \times P_3) = \frac{3}{2}(v_g^+ - v_g^-) + v_{g,4}^+ + v_{g,2}^- - \frac{m}{2} - 1,$$

where $0 \le v^+_{g,4} \le m-2$ and $0 \le v^-_{g,2} \le 4$.

For even m, the formula above becomes $B_g(P_m \times P_3) = v^+_{g,4} + v^-_{g,2} - \frac{m}{2} - 1$ for $0 \le v^+_{g,4} \le m-2$ and $0 \le v^-_{g,2} \le 4$. It will attain all values between $-\frac{m}{2} - 1$ and $\frac{m}{2} + 1$, inclusive. Thus $BI(P_m \times P_3) = \{i \mid 0 \le i \le \frac{m}{2} + 1\}$.

For odd m, we first consider $m \ge 5$. We may assume $v^+_g - v^-_g = 1$. Then the formula above reduces to $B_g(P_m \times P_3) = \frac{3}{2} + v^+_{g,4} + v^-_{g,2} - \frac{m}{2} - 1$ for $0 \le v^+_{g,4} \le m-2$ and $0 \le v^-_{g,2} \le 4$. Hence, $BI(P_m \times P_3) = \{i \mid 0 \le i \le \frac{m+1}{2} + 2\}$.

For $m = 3$, we may first pre-label the vertices of degree 4 with 0.5, and the vertices of degree 2 arbitrarily. This pre-labeling can be extended to a friendly labeling g such that $v^+_g - v^-_g = 1$. Thus $BI(P_3 \times P_3) \supseteq \{0, 1, 2, 3, 4\}$. It is easy to show that $B_g(P_m \times P_3) \le 4$. We conclude that $BI(P_3 \times P_3) = \{0, 1, 2, 3, 4\}$. □

Theorem 5.5. *For $m \ge 4$, $BI(P_m \times P_4) = [\mathbf{0}, \boldsymbol{m}]$.*

Proof: For $n = 4$, we have $|V_2 \cup V_4| = 2m$ and $|V_3| = 2m$. Thus any pre-labeling g of vertices of degree 2 or 4 can be extended to a friendly labeling of $P_m \times P_4$. Since $P_m \times P_4$ is of even order, we have, according to (5.1), $B_g(P_m \times P_4) = v^+_{g,4} + v^-_{g,2} - m$, where $0 \le v^+_{g,4} \le 2m-4$ and $0 \le v^-_{g,2} \le 4$. Since all possible values of $v^+_{g,4}$ and $v^-_{g,2}$ are attainable, $BI(P_m \times P_4) = [\mathbf{0}, \boldsymbol{m}]$. □

Theorem 5.6. *For $m \ge 5$,*

$$BI(P_m \times P_5) = \begin{cases} [\mathbf{0}, \boldsymbol{m} + \mathbf{5}] & \text{if } m \ge 12 \text{ and is even,} \\ [\mathbf{0}, \boldsymbol{m} + \mathbf{7}] & \text{if } m \ge 12 \text{ and is odd,} \\ [\mathbf{0}, \frac{\mathbf{3}\boldsymbol{m}}{\mathbf{2}} - \mathbf{1}] & \text{if } m \le 11 \text{ and is even,} \\ [\mathbf{0}, \frac{\mathbf{3}\boldsymbol{m}+\mathbf{1}}{\mathbf{2}}] & \text{if } m \le 11 \text{ and is odd.} \end{cases}$$

Proof: By (5.1), we have

$$B_g(P_m \times P_5) = 2(v^+_g - v^-_g) + v^-_{g,3} + 2v^-_{g,2} - m - 5, \tag{5.2}$$

where g is friendly, $0 \le v^-_{g,3} \le 2m+2$ and $0 \le v^-_{g,2} \le 4$. Hence $|B_g(P_m \times P_5)| \le m+5$ when m is even, and $|B_g(P_m \times P_5)| \le m+7$ when m is odd.

For $m \ge 12$, we find $|V_2 \cup V_3| = 2m + 6 \le |V_4| = 3m - 6$. Any pre-labeling of $V_2 \cup V_3$ can be extended to a friendly labeling g of $P_m \times P_5$ such that $v^+_g - v^-_g \ge 0$. Since all possible values of $v^-_{g,3}$ and $v^-_{g,2}$ are attainable, $BI(P_m \times P_5) = [\mathbf{0}, \boldsymbol{m} + \mathbf{5}]$ when m is even, and $BI(P_m \times P_5) = [\mathbf{0}, \boldsymbol{m} + \mathbf{7}]$ when m is odd.

It remains to consider $5 \le m \le 11$. Let g be a friendly labeling, we shall study the even and odd cases separately.

Case 1: When m is even, $6 - \frac{m}{2} \le v^-_{g,3} + v^-_{g,2} \le \frac{5m}{2}$. This implies that $|B_g(P_m \times P_5)| \le \frac{3m}{2} - 1$. Since $m < 12$, for each j, where $4 \le j \le \frac{5m}{2}$, we may choose all four vertices from V_2 and any $j - 4$ vertices from V_3, and label them with -0.5. Since $j \le \frac{5m}{2}$, this pre-labeling can be extended to a friendly labeling g of $P_m \times P_5$. In this labeling we have $-m + 3 \le B_g(P_m \times P_5) \le \frac{3m}{2} - 1$. Hence $BI(P_m \times P_5) = [\mathbf{0}, \frac{\mathbf{3}\boldsymbol{m}}{\mathbf{2}} - \mathbf{1}]$.

Case 2: When m is odd, $6-\frac{m+1}{2} \le v^-_{g,3}+v^-_{g,2} \le \frac{5m+1}{2}$. Thus $\frac{-3m-3}{2} \le B_g(P_m \times P_5) \le \frac{3m+3}{2}$. Suppose $B_g(P_m \times P_5) = \frac{3m+3}{2}$. Then from (5.2),

$$4(v^+_g - v^-_g) + 2(v^-_{g,3} + v^-_{g,2}) + 2v^-_{g,2} = 5m + 13;$$

hence $5m+13 \le 4(v^+_g - v^-_g) + 5m + 9$. This implies that $v^+_g - v^-_g = 1$, which forces $v^-_{g,3} + v^-_{g,2} = \frac{5m+1}{2}$, which is impossible. Similarly, $B_g(P_m \times P_5) = -\frac{3m+3}{2}$ is also impossible. Therefore, $|B_g(P_m \times P_5)| \le \frac{3m+1}{2}$.

Since $m < 12$, for each j, where $4 \le j \le \frac{5m-1}{2}$, we may choose all four vertices from V_2 and any $j-4$ vertices from V_3, and label them with -0.5. Since $j \le \frac{5m+1}{2}$, this pre-labeling can be extended to a friendly labeling g of $P_m \times P_5$ such that $v^+_g - v^-_g = 1$. In this labeling we have $-m+5 \le B_g(P_m \times P_5) \le \frac{3m+1}{2}$. Hence $BI(P_m \times P_5) = \left[\mathbf{0}, \mathbf{\frac{3m+1}{2}}\right]$. □

Theorem 5.7. *For $m \ge 6$,*

$$BI(P_m \times P_6) = \begin{cases} [\mathbf{0}, \mathbf{2m-2}] & \text{if } m = 6 \text{ or } 7, \\ [\mathbf{0}, \mathbf{m+6}] & \text{if } m \ge 8. \end{cases}$$

Proof: Since $n = 6$, we find $|V_4| = 4m-8$ and $|V_3| = 2m+4$. If g is a friendly labeling of $P_m \times P_6$, then $B_g(P_m \times P_6) = v^-_{g,3} + 2v^-_{g,2} - m - 6$, where $0 \le v_{g,3} \le 2m+4$ and $0 \le v^-_{g,2} \le 4$.

When $m \ge 8$, we have $|V_2 \cup V_3| \le |V_4|$. Any pre-labeling g of vertices of $V_2 \cup V_3$ can be extended to a friendly labeling of $P_m \times P_6$. Hence $BI(P_m \times P_6) = [\mathbf{0}, \mathbf{m+6}]$.

When $m = 6$ or 7, we have $8-m \le v^-_{g,3}+v^-_{g,2} \le 3m$. Thus $|B_g(P_m \times P_6)| \le 2m-2$. For each j, where $4 \le j \le 3m$, we may choose all four vertices from V_2 and any $j-4$ vertices from V_3, and label them with -0.5. Since $j \le 3m$, this pre-labeling can be extended as a friendly labeling g of $P_m \times P_6$. In this labeling, we have $-m+2 \le B_g(P_m \times P_6) \le 2m-2$. Hence $BI(P_m \times P_6) = [\mathbf{0}, \mathbf{2m-2}]$. □

Theorem 5.8. *For $m \ge n \ge 7$,*

$$BI(P_m \times P_n) = \begin{cases} [\mathbf{0}, \mathbf{m+n}] & \text{if } mn \text{ is even}, \\ [\mathbf{0}, \mathbf{m+n+2}] & \text{if } mn \text{ is odd}. \end{cases}$$

Proof: In any labeling g of $P_m \times P_n$, we find

$$B_g(P_m \times P_n) = 2(v^+_g - v^-_g) + v^-_{g,3} + 2v^-_{g,2} - m - n,$$

where $0 \le v_{g,3} \le 2m+2n-8$ and $0 \le v^-_{g,2} \le 4$. Assume g is friendly. If mn is even, then $|B_g(P_m \times P_n)| \le m+n$. If mn is odd, then $|B_g(P_m \times P_n)| \le m+n+2$. Since $|V_4| - |V_3| - |V_2| = mn+8 > 0$, all possible values of $v^-_{g,3}$ and $v^-_{g,2}$ are attainable. This completes the proof. □

The method outlined above can be applied to many other triregular graphs such as the amalgamation of identical stars [9]. The details are left to the reader as exercises.

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