AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **46** (2010), Pages 37–49

Irredundance saturation number of a graph

S. Arumugam

Core Group Research Facility (CGRF)
National Center for Advanced Research in Discrete Mathematics
(n-CARDMATH)
Kalasalingam University, Krishnankoil-626190
India
s_arumugam_akce@yahoo.com

Odile Favaron

LRI, Bât 490, Université Paris-Sud
91405 Orsay Cedex
France
of@lri.lri.fr

S. Sudha

Core Group Research Facility (CGRF)
National Center for Advanced Research in Discrete Mathematics
(n-CARDMATH)
Kalasalingam University, Krishnankoil-626190
India

Abstract

Let $G = (V, E)$ be a graph and let $v \in V$. Let $IRS(v, G)$ denote the maximum cardinality of an irredundant set in G which contains v. Then $IRS(G) = \min\{IRS(v, G) : v \in V\}$ is called the irredundance saturation number of G. In this paper we initiate a study of this parameter.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching.

An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al. [10]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al. [11].

Definition 1.1. *Let $G = (V, E)$ be a graph. A subset S of V is said to be a dominating set in G if every vertex in $V - S$ is adjacent to some vertex in S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets in G. The upper domination number $\Gamma(G)$ is the maximum cardinality taken over all minimal dominating sets in G.*

Definition 1.2. *Let S be a subset of vertices of a graph G and let $u \in S$. A vertex v is called a private neighbor of u with respect to S if $N[v] \cap S = \{u\}$. The private neighbor set of u with respect to S is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. The set S is called an irredundant set if for every $u \in S$, $pn[u, S] \neq \emptyset$. An irredundant set S is called a maximal irredundant set if no proper superset of S is irredundant.The minimum cardinality of a maximal irredundant set in G is called the irredundance number of G and is denoted by $ir(G)$. The maximum cardinality of an irredundant set in G is called the upper irredundance number of G and is denoted by $IR(G)$.*

Definition 1.3. *A subset S of V in a graph G is said to be independent if no two vertices in S are adjacent. An independent set S is called a maximal independent set if no proper superset of S is independent. The minimum cardinality of a maximal independent set is called the independent domination number of G and is denoted by $i(G)$. The maximum cardinality of an independent set in G is called the independence number of G and is denoted by $\beta_0(G)$.*

The six parameters concerning domination, independence and irredundance are related by the chain of inequalities, $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$. This inequality has been the focus of more than 100 research papers. These parameters can be considered to be the basic building blocks of domination, independence and irredundance.

Acharya [1] introduced the concept of domsaturation number $ds(G)$ of a graph, which is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. Arumugam and Kala [2] observed that for any graph G, $ds(G) = \gamma(G)$ or $\gamma(G) + 1$, and obtained several results on $ds(G)$. Motivated by this concept Arumugam and Subramanian [3] introduced the concept of independence saturation number of a graph.

Definition 1.4. *Let $G = (V, E)$ be a graph and let $v \in V$. Let $IS(v, G)$ denote the maximum cardinality of an independent set in G which contains v. Then $IS(G) = \min\{IS(v, G) : v \in V\}$ is called the independence saturation number of G.*

Thus $IS(G)$ is the largest positive integer k such that every vertex of G lies in an independent set of cardinality k.

The independence saturation number $IS(G)$ extends the domination chain, as shown in the following theorem.

Theorem 1.5. [3] *For any graph G we have $ir(G) \leq \gamma(G) \leq i(G) \leq IS(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$.*

In this paper we introduce the concept of irredundance saturation in graphs and initiate a study of the corresponding parameter. We need the following theorems and definitions.

Definition 1.6. *The middle graph of a graph $G = (V, E)$ is the graph $M(G) = (V \cup E, E')$, where $uv \in E'$ if and only if either u is a vertex of G and v is an edge of G containing u, or u and v are edges in G having a vertex in common.*

Definition 1.7. *The independence graph of a graph $G = (V, E)$ is the graph $I(G) = (I, E'')$ where the vertices correspond one-to-one with independent sets of vertices in G and $uv \in E''$ if and only if the independent sets corresponding to u and v have a vertex in common.*

Definition 1.8. *Let $G = (V, E)$ be an arbitrary graph and let k be any positive integer. The trestled graph of index $k, T_k(G)$, is the graph obtained from G by adding k copies of K_2 to each edge uv of G and joining u and v to the end vertices of each K_2, respectively.*

Theorem 1.9. [5] *For any graph G,*

(a) $\beta_0(M(G)) = IR(M(G)) = n$, *and*

(b) $\beta_0(I(G)) = IR(I(G)) = n$.

Theorem 1.10. [6] *If G is a bipartite graph, then $\beta_0(G) = IR(G)$.*

Theorem 1.11. [12] *If G is a chordal graph, then $\beta_0(G) = IR(G)$.*

Theorem 1.12. [8] *Let $G = (V, E)$ be any graph and let $T_2(G) = (V', E')$ be the corresponding trestled graph of index 2. Then $\beta_0(T_2(G)) = IR(T_2(G))$.*

Theorem 1.13. [3] *The problem of determining whether $IS(G) \geq k$ for any graph G is NP-complete.*

Theorem 1.14. [7] *For any n-vertex graph G with minimum degree δ, $IR(G) \leq n - \delta$, where equality holds if and only if G is one of the following graphs:*

(i) $V(G) = X \cup W$, *where* $|X| = n - \delta$, X *is independent in* G *and each vertex in* X *is joined to each vertex in* W. *The vertices in* W *are joined to one another arbitrarily, subject to deg* $w \geq \delta$ *for each* $w \in W$.

(ii) $V(G) = X \cup Y \cup Z$, *where* $|X| = |Y| = n - \delta (i.e., \delta \geq \frac{1}{2}n), \langle X \rangle \cong \langle Y \rangle \cong K_{n-\delta}$ *and the vertices in* X *are joined to the vertices in* Y *by a matching. Further,* $|Z| = 2\delta - n$, *each vertex in* Z *is joined to each vertex in* $X \cup Y$ *and the vertices in* Z *are joined to one another arbitrarily, subject to deg* $z \geq \delta$ *for each* $z \in Z$.

2 First Results

Definition 2.1. *Let $G = (V, E)$ be a graph and let $v \in V$. Let $IRS(v, G)$ denote the maximum cardinality of an irredundant set in G which contains v. Then $IRS(G) = \min \{IRS(v, G) : v \in V\}$ is called the irredundance saturation number of G.*

Thus $IRS(G)$ is the largest positive integer k such that every vertex of G lies in an irredundant set of cardinality k. We observe that $IRS(G) = 1$ if and only if $\Delta = n - 1$.

Theorem 2.2. *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq IS(G) \leq IRS(G) \leq IR(G).$$

Proof. From Theorem 1.5 we have $ir(G) \leq \gamma(G) \leq i(G) \leq IS(G)$. From definition we have $IRS(G) \leq IR(G)$ and hence it is enough to prove that $IS(G) \leq IRS(G)$. Since every independent set is an irredundant set $IS(v, G) \leq IRS(v, G)$ for all $v \in V$, so that $IS(G) \leq IRS(G)$. □

We cannot include $\Gamma(G)$ and $\beta_0(G)$ in the chain given in Theorem 2.2, since there is no relation between $IRS(G)$ and $\beta_0(G)$ as well as $IRS(G)$ and $\Gamma(G)$ as shown in the following theorems.

Theorem 2.3. *Given any positive integer k, there exist graphs G_1 and G_2 such that $\beta_0(G_1) - IRS(G_1) = k$ and $IRS(G_2) - \beta_0(G_2) = k$.*

Proof. For the bistar $G_1 = B(k+1, k+1)$, we have $\beta_0(G_1) = 2k+2$ and $IRS(G_1) = k + 2$. For the direct product $G_2 = K_{k+2} \square K_2$, we have $\beta_0(G_2) = 2$ and $IRS(G_2) = k + 2$. □

The above theorem leads to the following problem.

Problem 2.4. *Characterize graphs G for which $IRS(G) = \beta_0(G)$.*

Theorem 2.5. *Given any positive integer k, there exist graphs G_1 and G_2 such that $\Gamma(G_1) - IRS(G_1) = k$ and $IRS(G_2) - \Gamma(G_2) = k$.*

Proof. For the bistar $G_1 = B(k+1, k+1)$, we have $\Gamma(G_1) = 2k+2$ and $IRS(G_1) = k+2$. Now, for the direct product $G_2 = K_p \square K_q$ with $p \geq q \geq 4$, it is proved in [9] that $\Gamma(G_2) = p$ and $IR(G_2) = p+q-4$. Since G_2 is vertex-transitive, $IRS(G_2) = IR(G_2)$. Hence if we take $q = k + 4$ we have $IRS(G_2) - \Gamma(G_2) = k$. □

Problem 2.6. *Characterize graphs G for which $IRS(G) = \Gamma(G)$.*

Theorem 2.7. *Given any three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a graph G with $ir(G) = a, IRS(G) = b$ and $IR(G) = c$.*

Proof. Let a, b and c be three positive integers with $2 \leq a \leq b \leq c$.

Case i. $a = 2$.

$$\text{Let } k = \begin{cases} 0 & \text{if } c \leq 2b-1 \\ c-2b+1 & \text{if } c > 2b-1 \end{cases}$$

$$\text{and let } \alpha = \begin{cases} 2b-1-c & \text{if } c \leq 2b-1 \\ 0 & \text{if } c > 2b-1. \end{cases}$$

Let $P_3 = (v_1, v_2, v_3)$ be a path on three vertices. Attach $b-1$ pendant vertices $u_1, u_2, \ldots, u_{b-1}$ to v_1 and $b-1+k$ pendant vertices $w_1, w_2, \ldots, w_{b-1+k}$ to v_3. Add the edges $u_1w_1, u_2w_2, \ldots, u_\alpha w_\alpha$. For the resulting graph G, we have $ir(G) = 2 = a$, $IRS(G) = b$ and $IR(G) = c$.

Case ii. $a > 2$.

$$\text{Let } k = \begin{cases} 0 & \text{if } c \leq 2b-a \\ c-2b+a & \text{if } c > 2b-a \end{cases}$$

$$\text{and let } \alpha = \begin{cases} 2b-a-c & \text{if } c \leq 2b-a \\ 0 & \text{if } c > 2b-a. \end{cases}$$

Let $P = (v_1, v_2, \ldots, v_a)$ be a path on a vertices. Attach $b-(a-1)$ pendant vertices $u_1, u_2, \ldots, u_{b-(a-1)}$ to v_1, attach $b-(a-1)+k$ pendant vertices $w_1, w_2, \ldots, w_{b-(a-1)+k}$ to v_a and attach a pendant vertex x_i to each v_i, $2 \leq i \leq a-1$. If $c \leq 2b-a$, we add the edges $u_1w_1, u_2w_2, \ldots, u_\alpha w_\alpha$. For the resulting graph G, we have $ir(G) = a, IRS(G) = b$ and $IR(G) = c$. □

Theorem 2.8. *Let G be a connected graph. Then $IRS(G) = 2$ if and only if $\Delta < n-1$ and there exists a vertex $v \in V$ such that the following conditions hold.*

(i) The induced subgraph $\langle V(G) - N[v] \rangle$ is complete.

(ii) For any subset $S \subseteq N(v)$ with $|S| \geq 2$, one of the following holds.

- *(a) Every vertex of $N(v) - S$ is adjacent to at least one vertex of S.*
- *(b) There exists a vertex $w \in S$ such that either $N(w) \cap (V - N[v]) = \emptyset$ or for every $x \in N(w) \cap (V - N[v])$, we have $|N(x) \cap S| \geq 2$.*

Proof. Suppose $IRS(G) = 2$. Clearly $\Delta < n-1$. Now let $v \in V$ be such that $IRS(v, G) = 2$.

(i) If $\langle V(G) - N[v] \rangle$ is not complete, then there exist at least two vertices u and w in $V(G) - N[v]$ which are non-adjacent and hence $\{v, w, u\}$ is an irredundant set of cardinality 3, which is a contradiction. Hence $\langle V(G) - N[v] \rangle$ is complete.

(ii) For any subset $S \subseteq N(v)$ with $|S| \geq 2$, $S_1 = S \cup \{v\}$ is not an irredundant set. Hence there exists $w \in S_1$ such that $pn[w, S_1] = \emptyset$. If $w = v$, then every vertex of $N(v) - S$ is adjacent to at least one vertex of S. Suppose $w \in S$. If $N(w) \cap [V - N[v]] \neq \emptyset$, let $x \in N(w) \cap (V - N[v])$. Since x is not a private neighbor of w, it follows that $|N(x) \cap S| \geq 2$.

To prove the converse, let $\Delta < n-1$ and there exists a vertex v satisfying the conditions of the theorem. Then $\{v, w\}$, where $w \in V - N[v]$, is an irredundant set so that $IRS(v,G) \geq 2$. Now let R be any irredundant set of maximum cardinality containing the vertex v. The set R contains at most one element of $V-N[v]$, by (i). If $|R\cap(V-N[v])| = 1$, then $|R| = 2$ for if $x \in R\cap N(v)$, we would have $pn(x,R) = \emptyset$. Suppose now $R \subseteq N[v]$. Since $IRS(v,G) \geq 2$, $R \cap N(v) \neq \emptyset$ and $\langle N(v)\rangle$ is not complete. If $|R \cap N(v)| > 1$, then either $pn(v,R) = \emptyset$ if (ii) (a) holds for $S = R$, or $pn(w,R) = \emptyset$ for some $w \in R\cap N(v)$ if (ii) (b) holds. Therefore $|R\cap N(v)| = 1$ and $|R| = 2$. In both the cases $IRS(v,G) = |R| = 2$. Hence $IRS(G) \leq 2$ and since $\Delta < n-1$, $IRS(G) = 2$. □

Observation 2.9. *It follows from Theorem 2.8 that if $IRS(G) = 2$ and $\delta = 0$, then G has a unique vertex x of degree 0, $V - N[v] = \{x\}$ for the vertex v of Theorem 2.8, and $\Delta = n-2$.*

Theorem 2.10. *Let G be a connected graph of order n with $IRS(G) = 2$. Then $(n-1) \leq |E(G)| \leq \left\lfloor \frac{n(n-2)}{2} \right\rfloor$. Also $|E(G)| = n-1$ if and only if G is isomorphic to the graph G_1 obtained from the star $K_{1,n-2}$ with exactly one edge subdivided or the graph G_2 obtained from the star $K_{1,n-3}$ with one edge subdivided twice. Further $|E(G)| = \left\lfloor \frac{n(n-2)}{2} \right\rfloor$ if and only if $G \cong K_n - M$ where M is a perfect matching, when n is even and $M = M_1 \cup \{e\}$ where M_1 is a maximum matching and e is any edge incident with the unique M_1-unsaturated vertex, when n is odd.*

Proof. Let G be a connected graph of order n with $IRS(G) = 2$. The lower bound is obvious since G is connected. Since $IRS(G) = 2$, we have $\Delta < n-1$. Hence $deg\ v \leq n-2$ for all $v \in V(G)$, so that $|E(G)| \leq \frac{n(n-2)}{2}$.

Let G be a graph with $IRS(G) = 2$ and $|E(G)| = n-1$. Then G is a tree. Let $v \in V(G)$ such that $IRS(v,G) = 2$. Since $\langle V(G) - N[v]\rangle$ is complete it follows that $deg\ v = n-2$ or $n-3$. If $deg\ v = n-2$, then G is isomorphic to G_1, and if $deg\ v = n-3$, then G is isomorphic to G_2. The converse is obvious.

Now, let G be a connected graph with $IRS(G) = 2$ and $|E(G)| = \left\lfloor \frac{n(n-2)}{2} \right\rfloor$. It follows that if n is even, then $deg\ v = n-2$ for all $v \in V$ and if n is odd, exactly one vertex of G has degree $n-3$ and all the remaining vertices have degree $n-2$. Hence if n is even, G is isomorphic to $K_n - M$, where M is a perfect matching of K_n and if n is odd, then G is isomorphic to $K_n - (M_1 \cup \{e\})$, where M_1 is a maximum matching and e is an edge incident with the unique M_1-unsaturated vertex. The converse is obvious. □

Theorem 2.11. *For any graph $G, IRS(G) \leq n - \Delta$.*

Proof. Let v be a vertex of degree Δ in G. Let S be a maximum irredundant set in G such that $v \in S$. Let $k = |N(v) \cap S|$ so that v is adjacent to $\Delta - k$ vertices in $V - S$. If $k = 0$, then $|V - S| \geq \Delta$ and hence $|S| \leq n - \Delta$. If $k > 0$, then each of the k neighbors of v in S must have a private neighbor in $V - S$ and these private neighbors are distinct. Hence $|V - S| \geq (\Delta - k) + k = \Delta$, so that $|S| \leq n - \Delta$. Thus $IRS(v,G) \leq n - \Delta$ and hence $IRS(G) \leq n - \Delta$. □

Remark 2.12. *The bound given in Theorem 2.11 is sharp. For any graph G with $\Delta = n-1$, we have $IRS(G) = n - \Delta = 1$. Also for the corona $K_k \circ K_1$, $IRS(G) = n - \Delta = k$.*

Theorem 2.13. *A d-regular graph $G = (V, E)$ of order n satisfies $IRS(G) = n-d$ if and only if V is partitioned into $V_1 \cup V_2 \cup \cdots \cup V_p$, $V_1, V_2, \ldots, V_k$ are independent sets $\overline{K_q}$ for some positive integer q and some k with $0 \leq k \leq p$, $V_{k+1}, \ldots, V_p$ induce prisms $K_q \Box K_2$, and for $1 \leq i \neq j \leq p$, every vertex of V_i is adjacent to every vertex of V_j.*

Proof. It is easy to check that a graph with the described structure is d-regular with $d = n - q$ and satisfies $IR(G) = q = n - d$. Moreover $IRS(G) = n - d$ since every vertex x of a part V_i belongs to the independent set $\overline{K_q}$ if $i \leq k$ or to an irredundant set K_q if $i > k$.

To prove the converse, we make an induction on n. For $n \leq 5$, the d-regular graphs such that $IRS(G) = n - d$ are K_n, $\overline{K_n}$, $K_{2,2}$ and thus the property is true. Suppose the property true for regular graphs of order less than n with $n \geq 6$ and let G be a d-regular graph of order n such that $IRS(G) = n-d$. Since $IRS(G) \leq IR(G) \leq n-\delta$ for every graph and $\delta = d$, $IR(G) = n - d$. Hence by Theorem 1.14, we have two cases.

Case i. $V = X \cup W$ with $|X| = n - d$ and $|W| = d \geq |X|$, X is an independent set, and each vertex in W is adjacent to every vertex in X and to $2d - n$ vertices in W.

In this case $\langle W \rangle$ is a d_1-regular graph with $d_1 = 2d-n$ and of order $n_1 = d$. Note that $n_1 - d_1 = n - d$. If $n - d = 2$, then $d_1 = n_1 - 2$ and every vertex w of W is contained in an independent, and thus irredundant, set of order $2 = n - d$ of $\langle W \rangle$. This is also obviously true when $n - d = 1$. Suppose $n - d \geq 3$. Let w be a vertex of W and S an irredundant set of order $n - d$ of G containing w. If S contains two vertices x and y in X, then $pn[x, S] = pn[y, S] = \emptyset$, a contradiction. If $S \cap X = \{x\}$, then $S \cap W$ contains a vertex t different from w and $pn[t, S] = pn[w, S] = \emptyset$, a contradiction. Hence $S \subseteq W$. Moreover the private neighbors of the vertices of S cannot be in X, since every vertex of X is adjacent to every vertex of S, and belong to W. Therefore S is an irredundant set of $\langle W \rangle$.

Case ii. $V = X \cup Y \cup W$ where $\langle X \rangle \cong \langle Y \rangle \cong K_{n-d}$ and the vertices of X and Y are joined by a perfect matching, $|W| = 2d - n \geq |X|$ and each vertex in W is adjacent to all the vertices of $X \cup Y$ and to $3d - 2n$ vertices in W.

In this case $\langle W \rangle$ is a d_1-regular graph with $d_1 = 3d-2n$ and of order $n_1 = 2d-n$. Note that again $n_1 - d_1 = n - d$. If $n - d = 2$, then $d_1 = n_1 - 2$ and every vertex w of W is contained in an independent set of order $2 = n - d$ of $\langle W \rangle$. This is also obviously true when $n - d = 1$. Suppose $n - d \geq 3$. The same argument as in the previous case shows that if S is an irredundant set of order $n - d$ of G containing a vertex w of W, then S cannot contain vertices in X and is an irredundant set of $\langle W \rangle$.

In both cases, every vertex of W is contained in an irredundant set of $\langle W \rangle$ of order $n-d$. Therefore $\langle W\rangle$ is a d_1-regular graph of order n_1 such that $IRS(W) = n-d = n_1-d_1$. By the inductive hypothesis, $\langle W\rangle$ has the form described in the theorem with $q = n-d$. By adding X ($X\cup Y$ respectively) and all the edges between W and X ($X \cup Y$ respectively), we see that G has also the required form. □

Vizing [13] has given an upper bound for the number of edges in a graph of given order and a given domination number. In the following theorem we obtain a similar result for the irredundance saturation number.

Theorem 2.14. *If G is a graph of order n, with irredundance saturation number IRS, then $m \le n(n-IRS)/2$, and equality holds if and only if G is r-regular and V is partitioned into $V_1 \cup V_2 \cup \cdots \cup V_p$ such that $V_1,\ldots,V_k$ are independent sets $\overline{K_q}$ for some positive integer q and some k with $0 \le k \le p, V_{k+1},\ldots,V_p$ induce prisms $K_q\square K_2$, and for $1 \le i \ne j \le p$, every vertex of V_i is adjacent to every vertex of V_j.*

Proof. Since $2m \le n\Delta$ and $IRS \le n-\Delta$, we have $m \le n(n-IRS)/2$, and equality holds if and only if $2m = n(n-IRS)-1$ and $IRS = n-\Delta$. Hence G is regular and the result follows from Theorem 2.13. □

Theorem 2.15. *Let G be any graph with at least three vertices. Then $3 \le IRS + \overline{IRS} \le n+1-(\Delta-\delta)$ and $2 \le IRS\cdot\overline{IRS} \le (n-\Delta)(\delta+1)$.*

Further the following are equivalent.

(a) $IRS + \overline{IRS} = 3$.

(b) $IRS\cdot\overline{IRS} = 2$.

(c) Either G or $\overline{G}$ has the property that it has a unique vertex of degree $n-1$ and has at least one pendant vertex.

Proof. Since $IRS \ge 1$, and $\overline{IRS} \ge 2$ when $IRS = 1$, it follows that $IRS+\overline{IRS} \ge 3$ and $IRS\cdot\overline{IRS} \ge 2$. By Theorem 2.11, $IRS \le n-\Delta$ and hence it follows that $IRS+\overline{IRS} \le (n+1)-(\Delta-\delta)$ and $IRS\cdot\overline{IRS} \le (n-\Delta)(1+\delta)$.

Obviously (a) and (b) are both equivalent to $IRS = 1$ and $\overline{IRS} = 2$ or $IRS = 2$ and $\overline{IRS} = 1$.

To prove (a) implies (c), we assume without loss of generality that $IRS = 1$ and $\overline{IRS} = 2$. Since $IRS = 1$, we have $\Delta(G) = n-1$ and thus $\delta(\overline{G}) = 0$. Since $\overline{IRS} = 2$, it follows from Observation 2.9 that G has exactly one vertex of degree Δ and has a pendant vertex. Thus (a) implies (c). Suppose now that G satisfies the property described in (c). Then $IRS = 1$ and every pendant vertex v of G satisfies in $\overline{G}$ the conditions (i) and (ii) (b) of Theorem 2.8. Therefore $\overline{IRS} = 2$, which implies (a). □

Corollary 2.16. *For a tree, $IRS + \overline{IRS} = 3$ if and only if it is a star.*

3 Graphs with $IR(G) = \beta_0(G)$

We now extend for some families of graphs the property $IR(G) = \beta_0(G)$ recalled in Theorems 1.9, 1.10 1.11 and 1.12 to the property $IRS(G) = IS(G)$.

Proposition 3.1. *Let v be a vertex of a bipartite graph $G = (V, E)$ and T an irredundant set of G containing v. Then there exists an independent set S of G containing v and of same order as T. Therefore $IS(v, G) = IRS(v, G)$ for every vertex v of G.*

Proof. Let A and B be two independent sets partitioning V with $v \in A$. Let U be the set of isolated vertices of T and $T_A = T \cap A$, $T_B = T \cap B$, $U_B = U \cap B$. If $T_B - U_B = \emptyset$, then T is independent and we are done. If $T_B - U_B \neq \emptyset$, we consider for each vertex $x \in T_B - U_B$ a T-private neighbor x' of x. Then $x' \in A - T_A$ and $T_A \cup U_B \cup \{x' : x \in T_B - U_B\}$ is an independent set of G containing v and of order $|T|$. □

Corollary 3.2. *For any bipartite graph G, $IS(G) = IRS(G)$.*

Since $IS(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $IS(K_{m,n}) = \min(m, n)$ we have the following corollaries.

Corollary 3.3. *For the path $P_n = (v_1, v_2, \ldots, v_n)$ we have $IRS(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Corollary 3.4. *For the complete bipartite graph $G = K_{m,n}$, $IRS(G) = \min\{m, n\}$.*

Remark 3.5. *For any vertex transitive graph G, we have $IRS(G) = IR(G)$. Hence $IRS(C_n) = IR(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proposition 3.6. *Let v be a vertex of a chordal graph $G = (V, E)$ and T an irredundant set of G containing v. Then there exists an independent set S of G containing v and of same order as T. Therefore $IS(v, G) = IRS(v, G)$ for every vertex v of G.*

Proof. The proof is by induction on the order n of G. The property is easy to check for $n \leq 3$. Suppose that the property is true for chordal graphs of order less than n and let G be chordal of order n. Let $v \in V$ and let T be an irredundant set of G containing v. Let I be an irredundant set of G with same order as T, containing v and under the previous conditions, having the maximum number of isolated vertices. If v is isolated in I let $G' = G - N[v]$. The graph G' is chordal and the set $I' = I - \{v\}$ is irredundant in G'. By the inductive hypothesis, G' contains an independent set S' of order $|I'|$ and $S' \cup \{v\}$ is an independent set of G of order $|T|$. Suppose v is not isolated in I. Let J be the set of non-isolated vertices of I. For each vertex $x \in J$ we choose an I-private neighbor x' and consider the subgraph G' of G induced by the set $V' = J \cup J'$ with $J' = \{x' : x \in J\}$. The graph G' is chordal and the set J is irredundant in G'. If $|V'| < |V|$, then by the inductive hypothesis, v is contained in an independent set S' of G' of order $|J|$. Since $S' \cup (I - J)$ is an independant set of G, we are done.

Assume finally $G' = G$. This means that I has no isolated vertex, each vertex of I has exactly one I-private neighbor, each vertex of $V - I$ has exactly one neighbor in I and the edges of G between I and $I' = V - I$ form a perfect matching. Let $A_1, A_2, \cdots, A_p$ be the connected components of $\langle I \rangle$ with $v \in A_1$, let $B_1, \ldots, B_q$ be the compondents of $\langle I' \rangle$ and for $1 \leq i \leq p$, let $A'_i = \{x'; x \in A_i\}$. The graph H obtained by contracting each component A_i of $\langle I \rangle$ into one vertex a_i, each component B_j of $\langle I' \rangle$ into one vertex b_j, and deleting the loops is chordal and bipartite with classes $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$. Since G is chordal, the I-private neighbors x' of the different vertices x of a component A_i of $\langle I \rangle$ belong to different components B_j of $\langle I' \rangle$. Hence the graph H is simple and $d_H(a_i) = |V(A_i)| \geq 2$ for $1 \leq i \leq p$. Similarly, if two vertices y' and z' of a component B_j of $\langle I' \rangle$ are the respective I-private neighbors of two vertices y and z of I, then y and z belong to different components of $\langle I \rangle$. Hence $d_H(b_j) = |V(B_j)|$ for $1 \leq j \leq q$. Moreover, for each vertex $x \in I - v$, the I-private neighbor x' of x is adjacent to at least one other vertex y' of I', for otherwise $(I - \{x\}) \cup \{x'\}$ is an irredundant set of G containing v and with more isolated vertices than I. Therefore each component B_j of $\langle I' \rangle$ has order at least two, except possibly one equal to $\{v'\}$, and $d_H(b_j) \geq 2$ for $2 \leq j \leq q$. Hence at most one vertex of H has degree one, which contradicts the fact that the simple graph H, which is chordal and bipartite, is acyclic. Therefore the case $G' = G$ is impossible. □

Corollary 3.7. *If G is chordal, then $IS(G) = IRS(G)$.*

Theorem 3.8. *For any graph G of order n, $IS(M(G)) = IRS(M(G)) = n - 1$.*

Proof. It follows from the description of the middle graph $M(G)$ by means of n cliques partitioning $E(M(G))$ (see [5]), that the subset $V(G)$ of $V(M(G))$, which is independent in $M(G)$, is the unique $IR(M(G))$-set and that all the independent sets and all the irredundant sets of $M(G)$ containing a vertex $e \in E(G)$ have maximum order $n - 1$. Therefore $IS(v, M(G)) = IRS(v, M(G)) = n$ for every $v \in V(G)$ and $IS(e, M(G)) = IRS(e, M(G)) = n - 1$ for every $e \in E(G)$. Hence $IS(M(G)) = IRS(M(G)) = n - 1$. □

Theorem 3.9. *For any graph G of order n, $IS(I(G)) = IRS(I(G)) = n - \beta_0 + 1$.*

Proof. Let $S_1 \in V(I(G))$ be any independent set of G and let $\{S_1, S_2, \ldots, S_k\}$ with $k = IRS(S_1, I(G))$ be an irredundant set of $I(G)$ containing S_1. For $2 \leq i \leq k$, each vertex S_i of $I(G)$ has $\{S_1, S_2, \ldots, S_k\}$- private neighbor S'_i in $I(G)$, which means that each set S_i intersects an independent set S'_i of G (possibly equal to S_i)which is disjoint from $\bigcup_{j \neq i} S_j$. Therefore $n \geq |\bigcup_{1 \leq i \leq k} S_i| \geq |S_1| + k - 1$. Therefore $IRS(S_1, I(G)) \leq n - |S_1| + 1$. On the other hand, let $V(G) - S_1 = \{x_1, x_2, \ldots, x_{n-|S_1|}\}$. Then $\{S_1, \{x_1\}, \{x_2\}, \ldots, \{x_{n-|s_1|}\}\}$ is an independent set of $I(G)$. Hence $IS(S_1, I(G)) \geq n - |S_1| + 1$. Since $IS(S_1, I(G)) \leq IRS(S_1, I(G))$, both of them are equal to $n - |S_1| + 1$. Hence $IS(I(G)) = IRS(I(G)) = \min\{n - |S_1| + 1 : S_1$ independent in $G\} = n - \beta_0 + 1$. □

Theorem 3.10. *Let $G = (V, E)$ be any graph and let $T_2(G) = (V', E')$ be the corresponding trestled graph of index 2. Let $x \in V'$ and let S' be an irredundant set in $T_2(G)$ of maximum cardinality containing x. Then there exists an independent set S^* in $T_2(G)$ containing x such that $|S'| = |S^*|$. Therefore, $IS(x, T_2(G)) = IRS(x, T_2(G))$ for every vertex x in $T_2(G)$.*

Proof. For any edge uv in G let $u'v'$ and $u''v''$ be the edges in $T_2(G)$ such that u is adjacent to u', u'' and v is adjacent to v', v'' (cf. Figure 1). If other edges ending at v are considered, we denote v' and v'' respectively by $v'_u$and v''_u.

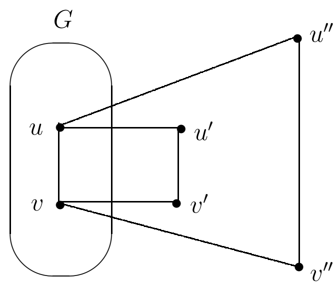

Figure 1

Let $V(T_2(G)) = V'$. Suppose $S' \cap V$ contains a pair of adjacent vertices u, v and let $v \neq x$. Since S' is irredundant, it follows that $S' \cap \{u', v', u'', v''\} = \emptyset$ and u', u'' are two S'- private neighbors of u. Hence $(S' - \{v\}) \cup \{v'\}$ is an irredundant set in $T_2(G)$. By repeating this process for every pair of adjacent vertices in $S' \cap V$, we obtain an irredundant set S_1 such that $x \in S_1$, $S_1 \cap V$ is independent and $|S_1| = |S'|$. Now, suppose $S_1 \cap (V' - V)$ contains a pair of adjacent vertices say u', v' and let $v' \neq x$. Then $S_1 \cap \{u, v, u'', v''\} = \emptyset$ and $(S_1 - \{v'\}) \cup \{v''\}$ is an irredundant set in $T_2(G)$. Again, by repeating this substitution process, we obtain an irredundant set S_2 in $T_2(G)$ such that $x \in S_2$, both $S_2 \cap V$, $S_2 \cap (V' - V)$ are independent and $|S_2| = |S'|$. Finally, if S_2 is not independent, then S_2 contains a vertex u in V and a corresponding adjacent vertex u' in $V - V'$. Then $v' \notin S_2$.

Now, if $u = x$, then we replace S_2 by $(S_2 - \{u'\}) \cup \{v'\}$. Let $u' = x$. If u has S_2-private neighbor $p(u)$ belonging to $V' - V$, say $p(u) = u''_w$ where uw is an edge of G, then, since $S_2 \cap V$ is independent, $w \notin S_2$ and $(S_2 - \{u\}) \cup \{u'_w\}$ is irredundant. If u has no S_2-private neighbor in $V' - V$, let w be an S_2-private neighbor of u in V. Then $N_{T_2(G)}(w) \cap S_2 = \{u\}$ and $\{u'_w, u''_w\} \subseteq S_2$. Suppose some vertex t of $N_{T_2(G)}(w)$ is the unique S_2-private neighbor of a non-isolated vertex y of $S_2 - \{u'_w, u''_w\}$. Since $N_{T_2(G)}(w) \cap S_2 = \{u\}$, $t \in V$ and $S_2 \cap \{w'_t, w''_t, t'_w, t''_w\} = \emptyset$. Hence $(S_2 - \{u\}) \cup \{w'_t, w''_t\}$ is an irredundant set of $T_2(G)$ containing x and larger than S_2, a contradiction. Therefore w is not adjacent to the unique S_2- private neighbor of any vertex of $S_2 - \{u'_w, u''_w\}$ and $(S_2 - \{u\}) \cup \{w\}$ is irredundant. By repeating this process, we ultimately produce an independent set S^* such that $x \in S^*$ and $|S^*| = |S'|$. Hence $IRS(x, T_2(G)) = IS(x, T_2(G))$. □

Corollary 3.11. *For any graph G, $IS(T_2(G)) = IRS(T_2(G))$.*

Remark 3.12. *The results of this section lead to the following question: Does every graph G such that $IR(G) = \beta_0(G)$ satisfy $IRS(G) = IS(G)$? The following example shows that the answer is negative.*

Let G consist of two paths $P = v_1v_2v_3v_4v_5$ and $P' = v_1'v_2'v_3'v_4'v_5'$ forming together a prism by addition of the edges v_iv_i', $1 \leq i \leq 5$, plus the edges $v_1v_3, v_1v_4, v_1v_5, v_5v_2, v_5v_3, v_2'v_4'$. The independent set $\{v_1', v_2, v_3', v_4, v_5'\}$ is maximum and the two irredundant sets $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1', v_2', v_3', v_4', v_5'\}$ are maximum. Therefore $\beta_0(G) = IR(G) = IRS(G) = 5$ while $IS(v_2', G) = 3$. Hence $IS(G) < IRS(G)$.

4 Complexity results

Fellows et al. [8], using the trestled graph $T_2(G)$, have proved that given any positive integer k, the problem of determining whether $IR(G) \geq k$ is NP-complete. We now proceed to establish the NP-completeness for the decision problem corresponding to IRS.

Lemma 4.1. *For any graph G of size $m, IS(T_2(G)) = IS(G) + 2m$.*

Proof. For any edge uv in G let $u'v'$ and $u''v''$ be the edges in $T_2(G)$ such that u is adjacent to u', u'' and v is adjacent to v', v'' (cf. Figure 1). Any maximal independent set of $T_2(G)$ exactly contains two vertices on each pair of edges associated to an edge of G. Therefore the maximum independent sets of $T_2(G)$ containing a given vertex x of G have order $|A| + 2m$ where A is a maximum independent set of G containing x. Hence $IS(x, T_2(G)) = IS(x, G) + 2m$ for all $x \in V$ □

Theorem 4.2. *Given any graph G and a positive integer k, the problem of determining whether $IRS(G) \geq k$ is NP-complete.*

Proof. It follows from Corollary 3.11 and Lemma 4.1 that $IRS(T_2(G)) = IS(G) + 2m$. Hence $IS(G) \geq k$ if and only if $IRS(T_2(G)) \geq k + 2m$ and the result follows from Theorem 1.13 □

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(Received 4 Oct 2008)