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Maximum graphs with unique minimum dominating set of size two

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Abstract

We prove that the maximum number of edges of a graph of order n which has a unique minimum dominating set of size two is bounded above by $\binom{n-2}{2}$. As a corollary to this result, we prove a conjecture by Fischermann, Rautenbach and Volkmann that the maximum number of edges of a graph which has a unique minimum dominating set of size two is $\binom{n-2}{2}$.

1 Introduction

Consider a finite, simple graph $G = (V, E)$. For any $v \in V$ let $N(v, G) = N(v) = \{v' \in V : (v, v') \in E\}$ and for any set $D \subseteq V$ let $N(D, G) = N(D) = \cup_{v \in D} N(v, G)$. Thus $N(v, G)$ is the collection of vertices which are adjacent or neighbors to v and $N(D, G)$ is the collection of all vertices which are adjacent or neighbors to at least one vertex in D. Similarly for any $v \in V$ let $N[v, G] = N[v] = N(v, G) \cup \{v\}$ and for any subset $D \subseteq V$ let $N[D, G] = N[D] = \cup_{v \in D} N[v, G]$. Intuitively $N(v)$ is the open neighborhood of v and $N[v]$ is the closed neighborhood of v.

A set of vertices $D \subseteq V$ is a *dominating set* of G if $N[D, G] = V$. A subset of vertices which is a dominating set of minimum cardinality is called a *minimum dominating set* or a gamma-set (γ-set). The *dominating number* of G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G. More precisely, for any finite set X let $|X|$ denote the cardinality of X, thus $\gamma(G) = \min\{|D| \big| D$ is a dominating set of $G\}$. If the minimum dominating set of G is unique, then it is called a *unique minimum dominating set* or a unique γ-set. For an overview of results on γ-sets and unique γ-sets see [2], [5], [4], [1], and [3].

We are interested in constructing finite, simple graphs without isolated vertices which have a unique γ-set of cardinality γ using the maximum number of edges. In this paper we prove the following:

Theorem 1. *Assume $n \geq 6$. Let $m(n,2) = m(n)$ be the maximum number of edges of a finite, simple graph G of order n without isolated vertices which has a unique γ-set of cardinality 2. Then*

$$m(n,2) = m(n) \leq \binom{n-2}{2}.$$

Fischermann, Rautenbach and Volkmann [2] generalized the definition of $m(n,2)$ to cover dominating sets of cardinality γ. Let $m(n,\gamma)$ be the maximum number of edges of a finite, simple graph G of order n without isolated vertices which has a unique γ-set of cardinality γ. In [2], the authors proved that $m(n,1) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$, for $n \geq 3$, and made the following conjecture for $\gamma \geq 2$.

Conjecture 2 ([2])**.** *If $\gamma \geq 2$ and $n \geq 3\gamma$, then $m(n,\gamma) = \binom{n-\gamma}{2} - \gamma(\gamma-2)$.*

In [2], the authors were able to prove the conjecture for the case $n = 3\gamma$. Notice there are no graphs of order $n < 6$ which have a unique γ-set of cardinality 2. Gunther, Hartnell, Markus, and Rall [5] observed that if G has a unique γ-set D, then every vertex in D which is not an isolated point, has at least two *private neighbors*, which are only dominated by that vertex. So if $v \in D$ is not an isolated point, then there exist at least two points a and b such that $a, b \notin \cup_{v' \in D-v} N[v']$. Thus a graph with two unique guards and no isolated vertices must have $n \geq 6$.

One can see that the case where G has isolated vertices can be handled also. If G has k isolated vertices, then remove those k vertices and then apply the theorem to the subgraph which would consist of $n-k$ vertices with no isolated points.

Additionally in [2], the authors constructed graphs $G(n,\gamma)$ which have a unique γ-set of cardinality γ with $\binom{n-\gamma}{2} - \gamma(\gamma-2)$ edges. This proves a lower bound for $m(n,\gamma)$ as follows:

Theorem 3 (Fischermann, Rautenbach, Volkmann [2])**.** *If $\gamma \geq 2$ and $n \geq 3\gamma$, then $m(n,\gamma) \geq \binom{n-\gamma}{2} - \gamma(\gamma-2)$.*

Combining Theorem 3 and Theorem 1 we are able to prove the conjecture is true for $\gamma = 2$.

Corollary 4. *If $\gamma = 2$ and $n \geq 6$, then $m(n,\gamma) = \binom{n-2}{2}$.*

2 Preliminaries

Throughout this paper we will assume $G = (V,E)$ is a finite, simple graph with n vertices, no isolated vertices, and a unique γ-set $\{x,y\}$. We will refer to x and y as the *unique guards* for G. Let $\mathcal{A} = N(x) \setminus N[y]$, $\mathcal{B} = N(y) \setminus N[x]$, and $\mathcal{C} = N(x) \cap N(y)$. For any vertex set $A \subset V$, let $G \setminus A$ denote the subgraph of G induced by the vertex set $V \setminus A$. Let $G_{xy} = G \setminus \{x,y\}$.

We show that for any G as above the number of edges in G is at most $\binom{n-2}{2}$, or, equivalently, that the total degree of the graph is less than $(n-2)(n-3)+2$.

Denote the *degree* of a vertex $v \in G$ as $d(v,G) = d(v)$. Define the *interior degree* of a vertex $v \in G_{xy}$ to be $Id(v) := d(v, G_{xy})$. Similarly for any set $D \subseteq V$, let $d(D) = \sum_{v \in D} d(v)$ and for $D \subseteq V(G_{xy})$, let $Id(D) = \sum_{v \in D} Id(v)$.

To prove the result we will make use of the following lemmas:

Lemma 5. *Given G as above, for all $c \in \mathcal{C}$, there is a vertex $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a, b \notin N(c)$.*

Proof. Assume there is a vertex $c \in \mathcal{C}$ such that $\mathcal{A} \subset N(c)$. Then $\{c, y\}$ is a γ-set of cardinality two which contradicts our assumption that G has a unique γ-set of cardinality two. Similarly, $\mathcal{B} \not\subset N(c)$. □

Note that Lemma 5 implies that the interior degree of any element of $\mathcal{C}$ is at most $n-5$.

Lemma 6. *Given G as above, let $C = \{c \in \mathcal{C} | Id(c) = n-5\}$. Then there exists $a \in \mathcal{A}$ such that $a \notin N[C]$ or there exists $b \in \mathcal{B}$ such that $b \notin N[C]$.*

Proof. If $|C| = 1$, then the proof is complete by Lemma 5. Assume $|C| \geq 2$. Choose two vertices $c, c' \in C$ such that $a, b \notin N[c]$ where $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and $a', b' \notin N[c']$ where $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$, and $b \neq b'$. If no such pair c and c' exist, then there exists a vertex $b \in \mathcal{B}$ such that $b \notin N[C]$ which completes the proof. If $a \neq a'$, then $\{c, c'\}$ would be a γ-set which is a contradiction. Therefore we can assume $a = a'$. (This completes the case when $|C| = 2$.)

If $|C| > 2$, it suffices to show that for every other vertex $v \in C \setminus \{c, c'\}$, $a \notin N[v]$. Let c'' be any other vertex in $C \setminus \{c, c'\}$. Assume $a'', b'' \notin N[c'']$ where $a'' \in \mathcal{A}$ and $b'' \in \mathcal{B}$. Since $b' \neq b$, we can assume without loss of generality that $b'' \neq b$. If $a'' \neq a$, then $\{c, c''\}$ would be a γ-set which is a contradiction. Therefore $a = a''$ which completes the proof. □

Lemma 7. *Given G as above, $|\mathcal{A}| \geq 2$ and $|\mathcal{B}| \geq 2$.*

Proof. If $\mathcal{A} = \emptyset$, then either $\mathcal{C} \neq \emptyset$ or $(x, y) \in E$ (since x cannot be isolated). If $\mathcal{A} = \emptyset$ and there is a $c \in \mathcal{C}$, then $\{c, y\}$ is a γ-set which is a contradiction. If $\mathcal{A} = \emptyset$ and $(x, y) \in E$, then $\{y\}$ is a γ-set which is a contradiction. Thus $\mathcal{A} \neq \emptyset$.

If $|\mathcal{A}| = 1$ and $a \in \mathcal{A}$, then $\{a, y\}$ is a γ-set which is a contradiction. Thus $|\mathcal{A}| \geq 2$. Similarly, $|\mathcal{B}| \geq 2$. □

Lemma 8. *Given G as above, for all $a \in \mathcal{A}$, there is an $a' \in \mathcal{A}$ such that $a' \notin N[a]$. For all $b \in \mathcal{B}$, there is a $b' \in \mathcal{B}$ such that $b' \notin N[b]$.*

Proof. Given $a \in \mathcal{A}$, if $\mathcal{A} \subset N[a]$, then $\{a, y\}$ is a γ-set for G. Similarly, for all $b \in \mathcal{B}$, $\mathcal{B} \not\subset N[b]$. □

Lemma 8 implies that the interior degree of any element of $\mathcal{A}$ or $\mathcal{B}$ is at most $n-4$.

3 Proving the result

We divide our argument into two main cases based on whether or not x is adjacent to y. In Section 3.1, Proposition 9 establishes our result for the case where $(x, y) \notin E$. Its proof requires cases based on the maximum interior degree of the vertices in G_{xy}. In Section 3.2, Proposition 15 deals with the case where $(x, y) \in E$.

3.1 The unique guards are not adjacent

We begin with the following proposition for $(x, y) \notin E$.

Proposition 9. *Given a graph $G = (V, E)$ as above, if $(x, y) \notin E$, then $|E| \leq \binom{n-2}{2}$.*

The proof of Proposition 9 has four cases which are addressed in Lemmas 11 through 14. First we provide sufficient bounds for the sum of the interior degrees of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, then we will use the bounds on the total interior degree to prove the upper bound for $|E|$.

Lemma 10. *Given a graph $G = (V, E)$ as above with $(x, y) \notin E$, if $Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) < (n-2)(n-5) + 2 - 2|\mathcal{C}|$, then $|E| \leq \binom{n-2}{2}$.*

Proof. Note that the total degree of G, $d(V)$, is

$$\begin{aligned} d(V) &= d(x) + d(y) + d(\mathcal{A}) + d(\mathcal{B}) + d(\mathcal{C}) \\ &= (|\mathcal{A}| + |\mathcal{C}|) + (|\mathcal{B}| + |\mathcal{C}|) + (Id(\mathcal{A}) + |\mathcal{A}|) + (Id(\mathcal{B}) + |\mathcal{B}|) + (Id(\mathcal{C}) + 2|\mathcal{C}|) \\ &= 2(n-2) + 2|\mathcal{C}| + Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}). \end{aligned}$$

Now the hypothesis leads to the desired bound. □

We now begin to look for the largest interior degree of a vertex in G_{xy}. As noted above, Lemma 8 implies that the largest interior degree of a vertex in $\mathcal{A}$ or $\mathcal{B}$ is $n-4$. However, note that there cannot be a vertex $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $Id(a) = Id(b) = n-4$ since a and b would form a γ-set for G. Also, as a result of Lemma 5, any $c \in \mathcal{C}$ has $Id(c) \leq n-5$. So without loss of generality, we may assume any vertices with interior degree $n-4$ are elements of $\mathcal{A}$.

Lemma 11. *Given G as above with $(x, y) \notin E$, if there is at least one vertex in G_{xy} with interior degree $n-4$, then $|E| \leq \binom{n-2}{2}$.*

Proof. As mentioned above, without loss of generality we may assume that all vertices of interior degree $n-4$ are in $\mathcal{A}$. Let $A = \{a \in \mathcal{A} | Id(a) = n-4\}$. By Lemma 8, for every $a \in A$, there is exactly one $a' \in \mathcal{A}$ such that $a' \notin N[a]$. Let $A' =$

$\{a' \in \mathcal{A} | (a, a') \notin E$ for some $a \in A\}$. For any $a' \in A'$, $N(a') \cap (\mathcal{B} \cup \mathcal{C}) = \emptyset$ since if $z \in N(a') \cap (\mathcal{B} \cup \mathcal{C})$, then $\{a, z\}$ would be a γ-set. So for all $a' \in A'$, $Id(a') \leq |\mathcal{A}| - 1$. However for each $a' \in A'$, $(a', a) \notin E$ for at least one $a \in A$. Thus $Id(A') \leq |A'|(|\mathcal{A}| - 1) - |A|$. Since all other vertices in $\mathcal{A}$ have degree $n - 5$ or less, we have the bound

$$Id(\mathcal{A}) \leq |A|(n-4) + [|A'|(|\mathcal{A}|-1) - |A|] + (|\mathcal{A}| - |A| - |A'|)(n-5). \quad (1)$$

For all $b \in \mathcal{B}$, there is some $b' \in \mathcal{B}$ with $(b, b') \notin E$, and for all $a' \in A'$, $(a', b) \notin E$. Finally, by Lemma 5, for all $c \in \mathcal{C}$, there is a $b \in \mathcal{B}$ such that $(c, b) \notin E$ and so we have:

$$Id(\mathcal{B}) \leq |\mathcal{B}|(n - 4 - |A'|) - |\mathcal{C}|. \quad (2)$$

Now, by Lemma 5, simply note that every vertex in $\mathcal{C}$ has interior degree at most $n - 5$. Thus

$$Id(\mathcal{C}) \leq |\mathcal{C}|(n-5). \quad (3)$$

Combining (1), (2), (3) and $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = n - 2$, we have

$$\begin{aligned} &Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) \\ &\quad \leq |A|(n-4) + [|A'|(|\mathcal{A}|-1) - |A|] + (|\mathcal{A}| - |A| - |A'|)(n-5) \\ &\qquad + |\mathcal{B}|(n-4-|A'|) - |\mathcal{C}| + |\mathcal{C}|(n-5) \\ &\quad = (n-2)(n-5) + |\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|). \end{aligned}$$

By Lemma 10, it remains to show that $|\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|) < 2 - 2|\mathcal{C}|$. If we recall $n = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + 2$, then

$$\begin{aligned} |\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|) &= |\mathcal{B}| - |\mathcal{C}| + |A'|(-2|\mathcal{B}| - |\mathcal{C}| + 2) \\ &= |\mathcal{B}|(1 - |A'|) + |A'|(2 - |\mathcal{B}|) - (|A'| + 1)|\mathcal{C}|. \end{aligned}$$

Note by assumption $|A| \geq 1$ which implies $|A'| \geq 1$, so $|\mathcal{B}|(1 - |A'|) \leq 0$ and $-(|A'| + 1)|\mathcal{C}| \leq -2|\mathcal{C}|$. From Lemma 7 we know $|\mathcal{B}| \geq 2$ so $|A'|(2 - |\mathcal{B}|) \leq 0$. Thus by Lemma 10 we have our result. □

Now we consider the case where there are no vertices of interior degree $n-4$. We will consider subcases determined by the highest interior degree present in $\mathcal{C}$. First we will consider the case where $\mathcal{C}$ has at least one vertex of interior degree $n - 5$.

Lemma 12. *Given G as above with $(x, y) \notin E$, if there are no vertices in G_{xy} with interior degree $n - 4$ and there is at least one vertex in $\mathcal{C}$ of interior degree $n - 5$, then $|E| \leq \binom{n-2}{2}$.*

Proof. Let $C = \{c \in \mathcal{C} \mid Id(c) = n - 5\}$. By assumption, $C \neq \emptyset$. Fix some $c \in C$. By Lemma 5, there exists a vertex $a \in \mathcal{A}$ such that $a \notin N(c)$ and a vertex $b \in \mathcal{B}$ such that $b \notin N(c)$. In addition, if $z \in N[a] \cap N[b]$, then $\{z, c\}$ would be a γ-set.

Thus $N[a] \cap N[b] = \emptyset$ which implies that $Id(a) + Id(b) \leq n - 5$. In fact, by Lemma 6, without loss of generality, $a \notin N[C]$. Let $M = \{v \in C | b \notin N(v)\}$. Then

$$Id(a) + Id(b) \leq (n-5) - (|M| - 1).$$

By Lemma 5, for all $v \in C \setminus M$, there is a $b' \in \mathcal{B}$ such that $b' \neq b$ and $b' \notin N(v)$. Thus,

$$\begin{aligned} Id(\mathcal{B} \cup \{a\}) &\leq (n-5) - (|M| - 1) + (|\mathcal{B}| - 1)(n-4) - (|C| - |M|) \\ &= (|\mathcal{B}| - 1)(n-4) + (n-5) - (|C| - 1). \end{aligned}$$

Finally, by assumption there are no vertices in $\mathcal{A}$ of degree $n-4$ and every vertex in $\mathcal{C} \setminus C$ has degree at most $n-6$, so

$$\begin{aligned} Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) &= Id(\mathcal{A} \setminus \{a\}) + Id(\mathcal{B} \cup \{a\}) + Id(C) + Id(\mathcal{C} \setminus C) \\ &\leq (|\mathcal{A}| - 1)(n-5) + (|\mathcal{B}| - 1)(n-4) + (n-5) - (|C| - 1) \\ &\qquad + |C|(n-5) + (|\mathcal{C}| - |C|)(n-6) \\ &= (n-2)(n-5) - |\mathcal{C}| + |\mathcal{B}| - n + 5 \\ &= (n-2)(n-5) - 2|\mathcal{C}| + 2 + |\mathcal{C}| + |\mathcal{B}| - n + 3. \end{aligned}$$

By Lemma 10, it remains to show that $|\mathcal{C}| + |\mathcal{B}| - n + 3 < 0$. But recall that $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = n - 2$, so $|\mathcal{C}| + |\mathcal{B}| - n + 3 = -|\mathcal{A}| + 1$ which is negative by Lemma 7, completing the proof. □

Now we consider the case where every vertex in $\mathcal{C}$ has interior degree less than $n-5$ and there is at least one vertex in $\mathcal{C}$ with interior degree $n-6$.

Lemma 13. *Given G as above with $(x, y) \notin E$, if there are no vertices in G_{xy} with interior degree $n-4$, no vertices in $\mathcal{C}$ of interior degree $n-5$, and there is at least one vertex in $\mathcal{C}$ of interior degree $n-6$, then $|E| \leq \binom{n-2}{2}$.*

Proof. Let $C = \{v \in \mathcal{C} \mid Id(v) = n - 6\}$. By assumption, $C \neq \emptyset$. Fix some $c \in C$; then by Lemma 5 there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $z \in G_{xy}$ such that $a, b, z \notin N[c]$.

Clearly $N(a) \cap N(b) \cap N(z) = \emptyset$. If not, then $\{w, c\}$ would be a γ-set for any $w \in N(a) \cap N(b) \cap N(z)$. Note that among a, b, and z, we may connect to every vertex in G_{xy} at most twice except c, a, b, and z (i.e. for all $v' \in G_{xy} \setminus \{a, b, c, z\}$, $|N[v'] \cap \{a, b, z\}| \leq 2$). But we may also have up to one edge among a, b, and z. Thus

$$Id(a) + Id(b) + Id(z) \leq 2(n-6) + 2 = 2n - 10.$$

We consider several cases based on which set contains z. First, if $z \in \mathcal{C} \setminus C$, then there are $|\mathcal{C}| - |C| - 1$ vertices in $\mathcal{C}$ with degree at most $n-7$ and there are $|C|$

vertices of degree at most $n-6$. Now by assumption, every vertex in $\mathcal{A}\cup\mathcal{B}$ has degree at most $n-5$. This give us the following:

$$\begin{aligned} Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C}) &= Id(\mathcal{A}\setminus\{a\})+Id(\mathcal{B}\setminus\{b\})+Id(a)+Id(b)+Id(z)\\ &\quad +Id(C)+Id(\mathcal{C}\setminus(C\cup\{z\}))\\ &\le (|\mathcal{A}|-1)(n-5)+(|\mathcal{B}|-1)(n-5)+2n-10\\ &\quad +|C|(n-6)+(|\mathcal{C}|-|C|-1)(n-7)\\ &=(n-2)(n-5)-2|\mathcal{C}|+|C|-(n-7). \end{aligned}$$

By Lemma 10, it remains to show that $|C|-(n-7)<2$. But by Lemma 7, $|\mathcal{C}|\le n-6$ so clearly $|C|<n-5$ and our result is proved for the case where $z\in\mathcal{C}\setminus C$.

The cases where z is an element of C, $\mathcal{A}$, or $\mathcal{B}$ are proved with a similar computation. □

The next lemma considers the case where there are no vertices with interior degree $n-4$ and either all vertices in $\mathcal{C}$ have interior degree less than $n-6$ or $\mathcal{C}=\emptyset$. This will complete the case where $(x,y)\notin E$.

Lemma 14. *Given G as above with $(x,y)\notin E$, if there are no vertices in G_{xy} with interior degree $n-4$, and either $\mathcal{C}=\emptyset$ or all vertices in $\mathcal{C}$ have interior degree less than $n-6$, then $|E|\le\binom{n-2}{2}$.*

Proof. With the given assumptions we have the following bound:

$$\begin{aligned} Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C}) &\le |\mathcal{A}|(n-5)+|\mathcal{B}|(n-5)+|\mathcal{C}|(n-7)\\ &=(n-2)(n-5)-2|\mathcal{C}|. \end{aligned}$$

So by Lemma 10 we have our result. □

We see that Lemmas 11, 12, 13, and 14 prove all possible cases when $(x,y)\notin E$. Thus we have proved Proposition 9.

3.2 The unique guards are adjacent

Here we prove the following proposition for $(x,y)\in E$.

Proposition 15. *Given a graph $G=(V,E)$ as above, if $(x,y)\in E$, then $|E|\le\binom{n-2}{2}$.*

Before beginning this proof we provide sufficient bounds for the sum of the interior degrees of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. The proof of the following lemma is similar to the proof of Lemma 10.

Lemma 16. *Given a graph G as above with $(x,y)\in E$, if $Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C})<(n-2)(n-5)-2|\mathcal{C}|$, then $|E|\le\binom{n-2}{2}$.*

We are now prepared to prove Proposition 15.

Proof of Proposition 15. Given G as above with $(x,y)\in E$, note that for any vertex $a\in\mathcal{A}$, there exist $a'\in\mathcal{A}$ and $b'\in\mathcal{B}$ such that $a',b'\notin N[a]$ (if not, then either $\{a,y\}$ or $\{x,a\}$ would be a γ-set). Thus $Id(a)\leq n-5$ for all $a\in\mathcal{A}$. Similarly, $Id(b)\leq n-5$ for all $b\in\mathcal{B}$.

First assume that $\mathcal{C}=\emptyset$. If all vertices in $\mathcal{A}\cup\mathcal{B}$ have interior degree $n-5$, then for any vertex $a\in\mathcal{A}$, there exists a vertex $a'\in\mathcal{A}$ and a vertex $b'\in\mathcal{B}$ such that $a',b'\notin N[a]$ and $N[a,G_{xy}]=(\mathcal{A}\cup\mathcal{B})\setminus\{a',b'\}$. However $Id(b')=n-5$ so there exists a $b''\in\mathcal{B}$ such that $N[b',G_{xy}]=(\mathcal{A}\cup\mathcal{B})\setminus\{a,b''\}$. But then $\{a,b'\}$ would be a γ-set which is a contradiction. This implies that there is a $v\in\mathcal{A}\cup\mathcal{B}$ such that $Id(v)\leq n-6$. Hence

$$Id(\mathcal{A})+Id(\mathcal{B})\leq(n-3)(n-5)+(n-6)$$

and by Lemma 16 we have our result for the case $|\mathcal{C}|=0$.

Assume now that $\mathcal{C}\neq\emptyset$. By Lemma 5, for each $c\in\mathcal{C}$, $Id(c)\leq n-5$ and there exist $a\in\mathcal{A}$ and $b\in\mathcal{B}$ such that $a,b\notin N(c)$. This restriction on edges between $\mathcal{C}$ and $\mathcal{A}\cup\mathcal{B}$ is not accounted for in our original bound on $Id(\mathcal{A})$ and $Id(\mathcal{B})$, so we have the following:

$$Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C})\leq(n-2)(n-5)-2|\mathcal{C}|.$$

If there is a $c\in\mathcal{C}$ with $Id(c)\leq n-6$, then we have

$$Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C})\leq(n-3)(n-5)+(n-6)-2|\mathcal{C}|$$

and by Lemma 16 we have our result for this case.

If all elements of $\mathcal{C}$ have degree $n-5$, then fix a $c\in\mathcal{C}$. There exist $a\in\mathcal{A}$ and $b\in\mathcal{B}$ such that $N[c,G_{xy}]=(\mathcal{A}\cup\mathcal{B}\cup\mathcal{C})\setminus\{a,b\}$. As noted above, there must be $a'\in\mathcal{A}$ and $b'\in\mathcal{B}$ such that $a'\notin N[a]$ and $b'\notin N[b]$. By assumption, $Id(c)=n-5$, so $a',b'\in N[c]$. Since $\{a',b'\}$ is not a γ-set there exists a vertex $v\notin N[a']\cup N[b']$. The vertex v cannot be an element of $\mathcal{C}$ since if it was it would have $Id(v)=n-5$ by assumption, and so $a,b\in N[v]$ making $\{c,v\}$ a γ-set. If $v\in\mathcal{A}$ then we have $a,v\notin N[a']$ and so we see $Id(a')\leq n-6$ (since by the above arguments there is also a $b''\in\mathcal{B}$ with $b''\notin N[a']$). Similarly, if $v\in\mathcal{B}$ then $Id(b')\leq n-6$.

Thus we have an element of $\mathcal{A}\cup\mathcal{B}$ whose interior degree is bounded by $n-6$ before considering the restrictions on edges between $\mathcal{C}$ and $\mathcal{A}\cup\mathcal{B}$, so

$$Id(\mathcal{A})+Id(\mathcal{B})+Id(\mathcal{C})\leq(n-3)(n-5)+(n-6)-2|\mathcal{C}|$$

and by Lemma 16 we have our result for the final case. □

Propositions 9 and 15 together provide the proof of our main result, Theorem 1. Fischermann, Rautenbach, and Volkmann [2] proved their conjecture for the case $\gamma=1$, and $n=3\gamma$. Corollary 4 proves the conjecture for $\gamma=2$; however, these techniques do not immediately generalize for $\gamma>2$.

References

[1] Miranca Fischermann, Block graphs with unique minimum dominating sets, *Discrete Math.* **240** (2001), 247–251.

[2] Miranca Fischermann, Dieter Rautenbach and Lutz Volkmann, Maximum graphs with a unique minimum dominating set, *Discrete Math.* **260** (2003), 197–203.

[3] Miranca Fischermann and Lutz Volkmann, Unique minimum domination in trees, *Australas. J. Combin.* **25** (2002), 117–124.

[4] Miranca Fischermann, Lutz Volkmann and Igor Zverovich, Unique irredundance, domination and independent domination in graphs, *Discrete Math.* **305** (2005), 190–200.

[5] G. Gunther, B. Hartnell, L.R. Markus and D. Rall, Graphs with unique minimum dominating sets, *Congressus Numerantium* **101** (1994), 55–63.

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