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On the domination subdivision numbers of trees

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Abstract

A set D of vertices of a graph G is a *dominating set* if every vertex in $V \setminus D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The domination subdivision number of G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. Arumugam has shown that for any tree, the domination subdivision number always lies between one and three inclusive. In this paper, we provide a constructive characterization of trees whose domination subdivision number is exactly two.

1 Introduction

Let $G = (V, E)$ be a graph. A set D of vertices is a dominating set if every vertex in $V \setminus D$ is adjacent to some vertex in D. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A $\gamma(G)$-set is a dominating set of cardinality $\gamma(G)$. An edge uv is said to be subdivided if the edge uv is deleted, but a new vertex w is added, along with two new edges uw and wv. The vertex w is called a subdivision vertex. The domination subdivision number of a graph G, denoted $sd_\gamma(G)$, is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. Domination and its parameters are well studied in graph theory. For a survey on this subject one can go through the two books by Haynes et al. [5], [6].

In applications, for instance in networks, it is not only important to study a parameter of interest; also the effect that modifications of the graph have on the parameter matters. It is desirable to consider the number of modifications which influence the change of the parameter. The minimum number of edges whose removal increases the domination number was defined by Bauer et al. [2]. Fink et al. [4] called this number the bondage number of a graph. In network design, deleting a vertex or an edge represents a component failure.

Arumugam and Velammal [8] considered a different type of graph modification and defined the domination subdivision number, which has been studied in [3]. A variation of domination subdivision number, namely the total domination subdivision number is studied in [7]. The trees, whose domination (respectively total domination) subdivision number is three, are characterized in [1] (respectively in [7]). In this paper, we give a constructive characterization of trees whose domination subdivision number is exactly two. Since the domination number of K_2 does not change when its only edge is subdivided, we consider graphs of order at least three.

All graphs considered in this paper are finite and simple. For definitions not given here and notations not presented see [6]. A tree is an acyclic graph. A *leaf* of a tree is a vertex of degree 1. A *support vertex* is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex that is adjacent to more than one leaf.

2 Main results

The following results guide the main results of this paper.

Proposition 1 [8]. *For a path P_n on $n \geq 3$ vertices,*

$$sd_\gamma(P_n) = \begin{cases} 1 & if \quad n \equiv 0 \ (mod\ 3) \\ 3 & if \quad n \equiv 1 \ (mod\ 3) \\ 2 & if \quad n \equiv 2 \ (mod\ 3). \end{cases}$$

Proposition 2 [1]. *If G is a graph with a strong support vertex, then $sd_\gamma(G) = 1$.*

Lemma 3 [1]. *If G is a graph obtained from a graph G' containing a pendant edge ya or a pendant path ybc by adding a path xz and an edge joining x to the vertex y in G', then $\gamma(G) = \gamma(G') + 1$. Moreover, if y is a support vertex of G' and G' has order at least 3, then $sd_\gamma(G) \leq sd_\gamma(G')$.*

Theorem 4 [8]. *For any tree T of order at least 3, $1 \leq sd_\gamma(T) \leq 3$.*

Trees are classified as Class 1, Class 2 or Class 3 depending on whether their domination subdivision number is 1, 2 or 3 respectively, using Theorem 4.

The main objective of this paper is to provide a constructive characterization of trees in Class 2. We accomplish this by defining a family of labeled trees which are in Class 2 as follows:

Let $\mathcal{F} = \{T_n\}_{n\geq 1}$ be the family of trees constructed inductively such that T_1 is a path P_5 and $T_n = T$, a tree. If $n \geq 1$, T_{n+1} can be obtained recursively from T_n by one of the two operations $\mathcal{F}_1$, $\mathcal{F}_2$ listed below.

We define the status of a vertex v, denoted sta(v), to be A, B or C. Initially if $T_1 = P_5$, then sta$(v) = C$ for every leaf of P_5, sta$(v) = B$ for the two support vertices and sta$(v) = A$ for the central vertex. Once a vertex is assigned a status, this status remains unchanged as the tree is constructed.

Operation $\mathcal{F}_1$: Assume $y \in T_n$ and sta$(y) \in \{B, C\}$. The tree T_{n+1} is obtained from T_n by adding a path x, w, v and the edge xy. Let sta$(x) = A$, sta$(w) = B$ and sta$(v) = C$.

Operation $\mathcal{F}_2$: Assume $y \in T_n$ and sta$(y) \in \{A, B\}$. The tree T_{n+1} is obtained from T_n by adding a path x, w and the edge xy. Let sta$(x) = B$ and sta$(w) = C$.

$\mathcal{F}$ is closed under the two operations $\mathcal{F}_1$ and $\mathcal{F}_2$. For $T \in \mathcal{F}$, let $A(T)$, $B(T)$ and $C(T)$ be the sets of vertices of status A, B and C, respectively. We have the following observations, which follow directly from the construction of $\mathcal{F}$.

Observations 5 *Let $T \in \mathcal{F}$ and $v \in V(T)$.*

1. *If sta$(v) = A$, then v is adjacent to at least one vertex of $B(T)$.*
2. *If sta$(v) = B$, then v is adjacent to exactly one vertex of $C(T)$.*
3. *If sta$(v) = C$, then v is adjacent to exactly one vertex of $B(T)$.*
4. *If v is a support vertex, then sta$(v) = B$.*
5. *If v is a leaf, then sta$(v) = C$.*
6. *$|B(T)| = |C(T)|$ (a direct consequence of operations $\mathcal{F}_1$ and $\mathcal{F}_2$).*

Lemma 6 *If $T \in \mathcal{F}$, then $B(T)$ is a $\gamma(T)$-set. Moreover if T is obtained from $T' \in \mathcal{F}$ using operation $\mathcal{F}_1$ or $\mathcal{F}_2$, then $\gamma(T) = \gamma(T') + 1$.*

Proof. By the Observations 5, it is clear that $B(T)$ is a dominating set. Assume that $B(T)$ is a $\gamma(T)$-set. If $T \in \mathcal{F}$ is obtained from T' using operation $\mathcal{F}_1$ or $\mathcal{F}_2$, then T can have exactly one more vertex with status B than T'. Since $\gamma(T) = |B(T)|$ and $\gamma(T') = |B(T')|$, it follows that $\gamma(T) = \gamma(T') + 1$.

Now we prove that $B(T)$ is a $\gamma(T)$-set. We proceed by induction on the length n of the sequence of trees needed to construct the tree T. Suppose $n = 1$; then $T = P_5$ belongs to $\mathcal{F}$. Let the vertices of P_5 be labeled as a, b, c, d, e. Then $B(P_5) = \{b, d\}$ and is a $\gamma(P_5)$-set. This establishes the base case. Assume then that the result holds for all trees in $\mathcal{F}$ that can be constructed from a sequence of fewer than n trees where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_n$ of n trees, where $T' = T_{n-1}$ and $T = T_n$. By our inductive hypothesis, $B(T')$ is a $\gamma(T')$-set.

We now consider two possibilities depending on whether T is obtained from T' by operation $\mathcal{F}_1$ or $\mathcal{F}_2$.

Case 1: T is obtained from T' by operation $\mathcal{F}_1$.

Suppose T is obtained from T' by adding a path y, x, w, v of length 3 to the attacher vertex $y \in V(T')$. Any $\gamma(T')$-set can be extended to a $\gamma(T)$-set by adding to it the vertex w, which is of status B. Hence $B(T) = B(T') \cup \{w\}$ is a $\gamma(T)$-set.

Case 2: T is obtained from T' by operation $\mathcal{F}_2$.

The proof is very similar to Case 1. □

Lemma 7 *If $T \in \mathcal{F}$ and T^* is obtained from T by subdividing one edge, then $\gamma(T^*) = \gamma(T)$.*

Proof. Let e be the edge of T that is subdivided to form T^*. Clearly, $\gamma(T) \leq \gamma(T^*)$. We proceed by induction on the length n of the sequence of trees needed to construct the tree T.

When $n = 1$, then $T = P_5$ and $T^* = P_6$. Hence $\gamma(T^*) = \gamma(T) = 2$. Therefore the lemma is true for the base case.

Assume that the result holds for all trees in $\mathcal{F}$ that can be constructed from a sequence of fewer than n trees, where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_n$ of n trees. We denote T_{n-1} by T'. By our inductive hypothesis, if T'' is obtained from T' by subdividing one edge, then $\gamma(T'') = \gamma(T')$. We now consider two possibilities depending on whether T is obtained from T' by operation $\mathcal{F}_1$ or $\mathcal{F}_2$.

Case 1: T is obtained from T' by operation $\mathcal{F}_1$.

Suppose T is obtained from T' by adding a path y, x, w, v of length 3 to the attacher vertex $y \in V(T')$. Then $\text{sta}(y) = B$ or C, $\text{sta}(x) = A$, $\text{sta}(w) = B$ and $\text{sta}(v) = C$.

Case 1.1: $e \in E(T')$.

Let T'' be obtained from T' by subdividing the edge e. Thus T^* is obtained from T'' by adding the path y, x, w, v to the vertex $y \in V(T'')$. Any $\gamma(T'')$-set can be extended to a dominating set of T^* by adding w and so $\gamma(T^*) \leq \gamma(T'') + 1$. By the inductive hypothesis, $\gamma(T'') = \gamma(T')$ and by Lemma 6, $\gamma(T) = \gamma(T') + 1$, i.e., $\gamma(T') = \gamma(T) - 1$. Hence $\gamma(T^*) - 1 \leq \gamma(T'') = \gamma(T') = \gamma(T) - 1 \leq \gamma(T^*) - 1$. Consequently, we must have equality throughout this inequality chain. Hence, $\gamma(T^*) = \gamma(T)$.

Case 1.2: $e \in E(T) \setminus E(T')$.

Choose a $\gamma(T')$-set S' containing y. Suppose $e \neq vw$. Then $S' \cup \{w\}$ is a dominating set of T^*, and so $\gamma(T^*) \leq \gamma(T') + 1$. Hence, $\gamma(T) \leq \gamma(T^*) \leq \gamma(T') + 1 = \gamma(T)$. Consequently, $\gamma(T^*) = \gamma(T)$.

Suppose $e = vw$. Let c denote the resulting vertex of degree 2 when the edge e is subdivided. Then $S' \cup \{c\}$ is a dominating set of T^* and as before $\gamma(T^*) = \gamma(T)$.

Case 2: T is obtained from T' by operation $\mathcal{F}_2$.

The proof is similar to that of Case 1. □

Corollary 8 *If $T \in \mathcal{F}$, then $sd_\gamma(T) \geq 2$.*

Lemma 9 *If $T \in \mathcal{F}$, then T is in Class 2.*

Proof. By Corollary 8, $sd_\gamma(T) \geq 2$. To show that $sd_\gamma(T) = 2$, we need to show that $sd_\gamma(T) \leq 2$, that is, there exist two edges whose subdivision increases the domination number of T. We proceed by induction on the length n of the sequence of trees needed to construct the tree T. Suppose $n = 1$; then $T = P_5 \in \mathcal{F}$ and also T is in Class 2. This establishes the base case. Assume then that the result holds for all trees in $\mathcal{F}$ that can be constructed from a sequence of fewer than n trees where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_n$ of n trees. For notational convenience, we denote T_{n-1} simply by T'. By our inductive hypothesis T' is in Class 2.

By Lemma 6, $\gamma(T) = \gamma(T') + 1$. Let T be obtained from T' by adding the path y, x, w, v in Operation $\mathcal{F}_1$ or the path y, x, w in Operation $\mathcal{F}_2$. Let T^* be obtained from T by subdividing two edges of T'. To show that T is in Class 2, it suffices to show that $\gamma(T^*) > \gamma(T)$. Let T'' be the tree obtained from T' by subdividing two edges. Any $\gamma(T'')$-set can be extended to a dominating set of T^* by adding the vertex w, so $\gamma(T^*) \leq \gamma(T'') + 1$. Among all $\gamma(T^*)$-sets, select S to have the minimum number of vertices from $V(T) \setminus V(T'')$. Clearly, S contains exactly one vertex of $V(T) \setminus V(T'')$. Hence $\gamma(T'') \leq |S| - 1 = \gamma(T^*) - 1$. Thus, $\gamma(T^*) = \gamma(T'') + 1$. Since $T' \in \mathcal{F}$, by our inductive hypothesis, T' is in Class 2 and so $\gamma(T') < \gamma(T'')$. Hence $\gamma(T) = \gamma(T') + 1 < \gamma(T'') + 1 = \gamma(T^*)$, that is, $\gamma(T) < \gamma(T^*)$, as desired. □

Proposition 10 *If G is a graph obtained from a graph G' by adding a path x, w, v and an edge joining x to the vertex $y \in G'$, then $\gamma(G) = \gamma(G') + 1$. Moreover if G' has order at least 3, then $sd_\gamma(G) \leq sd_\gamma(G')$.*

Proof. If S' is a $\gamma(G')$-set, then $S' \cup \{w\}$ is a dominating set of G and so $\gamma(G) \leq \gamma(G') + 1$. Conversely, if S is a $\gamma(G)$-set then $S \setminus \{w\}$ is a dominating set of G' and so $\gamma(G') \geq \gamma(G) - 1$. Hence $\gamma(G) = \gamma(G') + 1$.

Let $sd_\gamma(G') = k$. Consider k edges $e_j \in E(G')$ such that their subdivision yields a graph G'^* satisfying $\gamma(G'^*) > \gamma(G')$. Let G^* be obtained from G by subdividing the k edges e_j and let S' be a $\gamma(G'^*)$-set. Then $S' \cup \{v\}$ is a $\gamma(G^*)$-set and so $\gamma(G^*) = \gamma(G'^*) + 1 > \gamma(G') + 1 = \gamma(G)$. Hence $sd_\gamma(G) \leq k$, i.e., $sd_\gamma(G) \leq sd_\gamma(G')$. □

Theorem 11 *A tree T of order $n \geq 5$ is in Class 2 if and only if $T \in \mathcal{F}$.*

Proof. By Lemma 9, it is sufficient to prove that the condition is necessary. We proceed by induction on the order n of a tree T in Class 2. For $n = 5$, $T = P_5$ is in Class 2 and also it belongs to the family $\mathcal{F}$. Assume that $n \geq 6$ and all trees in Class 2 with order less than n belong to $\mathcal{F}$. Let T be a tree of order n in Class 2. T has no strong support vertices; otherwise by Proposition 2, T is in Class 1. Let $P : v_1, v_2, \ldots, v_k$ be a longest path in T. Obviously $\deg(v_1) = \deg(v_k) = 1$ and $\deg(v_2) = \deg(v_{k-1}) = 2$ and $k \geq 5$. We consider two possibilities depending on the degree of v_3 in T.

Case 1: $\deg(v_3) > 2$.

Let $T' = T \setminus \{v_1, v_2\}$. By Lemma 3, $\gamma(T) = \gamma(T') + 1$ and $sd_\gamma(T) \leq sd_\gamma(T')$. Since T is in Class 2, $2 \leq sd_\gamma(T')$.

Claim: T' is in Class 2.

Suppose T' is not in Class 2. Then, on subdividing any two edges of T' we get a tree T'^* such that $\gamma(T') = \gamma(T'^*)$. Subdivide two edges e, f of T to form T^*. If $e, f \in T'$ or $|\{e, f\} \cap T'| = 1$, then any $\gamma(T'^*)$-set can be extended to a $\gamma(T^*)$-set by adding v_1. So, $\gamma(T^*) = \gamma(T'^*) + 1 = \gamma(T') + 1 = \gamma(T)$. If $e, f \in E(T) \setminus E(T')$, then also we can prove that $\gamma(T) = \gamma(T^*)$. Hence T is not in Class 2, a contradiction. This proves the claim.

The order of T' is less than n and so it belongs to $\mathcal{F}$ by the inductive hypothesis. Note that $\deg(v_3) \geq 2$ in T'. The vertex v_3 can not be a leaf of T'; otherwise $\deg(v_3) = 1$ in T'. Hence either v_3 must be a support vertex or a vertex adjacent to a support vertex. By Observations 5, $\mathrm{sta}(v_3) \in \{A(T'), B(T')\}$. Thus, T can be obtained from T' by the operation $\mathcal{F}_2$ and hence T belongs to $\mathcal{F}$.

Case 2: $\deg(v_3) = 2$.

Let $T' = T \setminus \{v_1, v_2, v_3\}$. Thus T is obtained from T' by adding a path v_1, v_2, v_3 of length 3 to the vertex $v_4 \in T'$. By Proposition 10, $sd_\gamma(T) \leq sd_\gamma(T')$. As in Case 1, we can prove that T' is in Class 2.

We claim that either $\deg(v_4) = 2$ and v_4 is a leaf of T' or $\deg(v_4) > 2$ and v_4 is a support vertex of T'. Suppose v_4 has a neighbor which is a support vertex, let T^* (T'^* respectively) be obtained from T (T' respectively) by subdividing one edge. Let S be a $\gamma(T)$-set and S^* be a $\gamma(T^*)$-set. The set S^* contains v_4 or one of its neighbors in T' and does not contain v_3. Hence, $|S^* \cap V(T'^*)| = |S \cap V(T')| + 1$ and $|S^* \cap V(T')| = |S \cap V(T')|$ yielding, $|S^* \cap V(T^*)| > |S \cap V(T)|$. Therefore $\gamma(T^*) > \gamma(T)$. Hence $sd_\gamma(T) = 1$, in contradiction to $sd_\gamma(T) = 2$. Hence either $\deg(v_4) = 2$ and v_4 is a leaf of T' or $\deg(v_4) > 2$ and v_4 is a support vertex of T'. By Observations 5, $\mathrm{sta}(v_4) \in \{B(T'), C(T')\}$. Thus, T can be obtained from T' by the operation $\mathcal{F}_1$ and hence T belongs to $\mathcal{F}$. □

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References

[1] H. Aram, S. M. Sheikholeslami and O. Favaron, Domination subdivision numbers of trees, *Discrete Math.* 309 (2009), 622–628.

[2] D. Bauer, F. Harary, J. Nieminen and C. L. Suffel, Domination alteration sets in graphs, *Discrete Math.* 47 (1983), 153–161.

[3] O. Favaron, T. W. Haynes and S. T. Hedetniemi, Domination Subdivision Numbers in Graphs, *Util. Math.* 66 (2004), 195–209.

[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.* 86 (1990), 47–57.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[7] T. W. Haynes, M. A. Henning and L. S. Hopkins, Total domination subdivision numbers of trees, *Discrete Math.* 286 (2004), 195–202.

[8] S. Velammal, *Studies in Graph Theory: Covering, Independence, Domination and Related Topics*, Ph.D. Thesis, Manomanian Sundarnar University, Tirunelvelli, 1997.

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