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Largest 6-regular toroidal graphs for a given diameter

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Abstract

We show that a 6-regular graph of diameter d embedded on the torus can have at most $3d^2+3d+1$ vertices having exhibited a graph of this order for each $d \geq 1$.

1 6-regular Toroidal Graphs

In [1] it is proven that all 6-regular toroidal graphs can have their vertices arranged in a rectangular based grid structure with parallel diagonals across every sub-rectangle in the grid. A special case of this structure is for graphs we shall refer to as $H(n,k)$. They are constructed by taking a cycle with n vertices consecutively labelled 0, $\dots, n-1$ and adding edges from each vertex to the vertices with labels k and $k+1$ (modulo n) around the cycle. We note that $H(n,k)$ is vertex transitive and hence one can determine the diameter simply by calculating the eccentricity of one vertex.

2 A Graph of diameter d with order $3d^2+3d+1$

We define the graph $H_d := H(3d^2+3d+1, 3d+1)$ with the $n := 3d^2+3d+1$ vertices numbered consecutively clockwise from 0 to $3d^2+3d$. Note that this implies that vertex i also has edges from vertices $n-(3d+2+i) \equiv 3d^2-1-i$ and $n-(3d+1+i) \equiv 3d^2-i$. This implies that H_d is 6-regular since none of -1, 1, $3d+1$, $3d+2$, $3d^2-1$ and $3d^2$ are equal for any positive integer d. The edges between vertices with a label difference of 1 will be referred to as *exterior* edges and all other edges will be *jump* edges.

Theorem 2.1 *The diameter of H_d is exactly d.*

Proof:
We will now describe how to travel from vertex 0 to every other vertex using at most d edges, using a combination of the types of edges defined in the graph. Because of the symmetry in the graph structure we can choose to first use the jump edges j times and then use the exterior edges up to $d-j$ times for our walk of length at most d. We also suppose the vertices are labelled clockwise.

Each of the j clockwise jump edges can be chosen to add any combination of the two possible values $3d+1$ or $3d+2$ and thus we can reach any vertex between $(3d+1)j$ and $(3d+2)j$ using j jumps. The remaining $d-j$ exterior edges can then be used to reach to the vertices numbered between $(3d+1)j-(d-j)=(3j-1)d+2j$ (moving anticlockwise up to $d-j$ times from the lower end of the jump range) and $(3d+2)j+(d-j)=3dj+d+j=(3j+1)d+j$ (moving on clockwise from the maximum jump distance). The minimum and maximum values for the reachable ranges are summarised in the left side of Table 1.

	j jumps clockwise from 0		$d-j$ jumps anti-clockwise from 0	
j	minimum	maximum	minimum	maximum
0	$n-d$	d	$d+1$	$2d+1$
1	$2d+2$	$4d+1$	$4d+2$	$5d+3$
2	$5d+4$	$7d+2$	$7d+3$	$8d+5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$(3j-1)d+2j$	$(3j+1)d+j$	$(3j+1)d+j+1$	$(3j+2)d+2j+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
d	$3d^2+d$	$3d^2+2d$	$3d^2+2d+1=n-d$	$3d^2+4d+1=n+d$

Table 1: Vertices reachable after combinations of jumps

Alternatively, we can travel anti-clockwise using the jump edges $d-j$ times and the exterior edges j times, starting at vertex 0 which is congruent to $n \equiv 3d^2+3d+1$. This will enable us to reach any vertices with labels between these:

$$\begin{aligned} n-(d-j)(3d+2)-j &= 3d+1-2d+3dj+2j-j \\ &= d+1+(3d+1)j \\ &= (3j+1)d+j+1 \end{aligned}$$

and

$$\begin{aligned} n-(d-j)(3d+1)+j &= 3d+1-d+(3d+1)j+j \\ &= 2d+1+(3d+2)j \\ &= (3j+2)d+2j+1 \end{aligned}$$

The right hand side of Table 1 tabulates these values.

Together, the ranges in Table 1 will cover all vertices in the graph since $(3j+1)d+j$ is clearly one less than $(3j+1)d+j+1$ and $(3j+2)d+2j+1$ is one more than

$(3(j+1)-1)d+2j$. The final row shows that all vertices can be reached by using d or fewer jumps.

Thus H_d has diameter at most d and, for instance, we cannot reach the vertex numbered d using fewer than d edges of either type so the diameter of H_d is exactly d. ◇

3 Toroidal Embedding

The graph H_d defined in Section 2 can be embedded in the torus (as in [1]) in the following way: Firstly create two cycles each with $n := 3d^2+3d+1$ vertices. Form C_1 consisting of a sequence of vertices labelled $0, 1, 2, \ldots, 3d^2-1, 3d^2, \ldots, 3d^2+3d$ and the other, C_2, with the sequence of vertices $3d+1, 3d+2, 3d+3, \ldots, 3d^2+3d, 0, \ldots, 3d$. Adding edges from each vertex k in C_1 to the vertices in C_2 labelled $k+3d+1$ and $k+3d+2 \pmod{n}$ will create a cylinder consisting of rectangles with diagonals. Upon identifying the vertices with the same labels from C_1 and C_2 a torus will be formed with H_d embedded on it.

Theorem 3.1 *The graph H_d is the largest toroidal 6-regular graph of diameter d.*

Proof:
The universal covering surface of the torus is the plane and an embedding of any 6-regular toroidal graph (including H_d) in the torus determines a 6-regular tiling of the plane by triangles. Choose any vertex v in a 6-regular tiling of the plane by triangles. There are 6 vertices at distance one from v. There are 12 vertices at distance two; 18 at distance three, $\ldots, 6i$ vertices at distance i from v.

The number of distinct vertices at distance at most d from v is therefore not more than

$$6(1+2+\cdots+d) = 3d(d+1) = 3d^2+3d.$$

It follows that a 6-regular toroidal graph of diameter d has at most $3d^2+3d+1$ vertices. ◇

4 Applications

It is straightforward to generate straight line embeddings of the graphs described in Section 2 as shown in Figure 1. Other embeddings than those described in Section 3 are possible, and this one shows the underlying hexagonal structure.

There exists a 5-regular planar graph of diameter seven with 254 vertices [2] and it can obviously be embedded on the torus, but we have proven that H_7 (the largest 6-regular toroidal graph of diameter seven) has 169 vertices. Although, for a given diameter, a higher valency in a regular graph usually implies that more vertices are

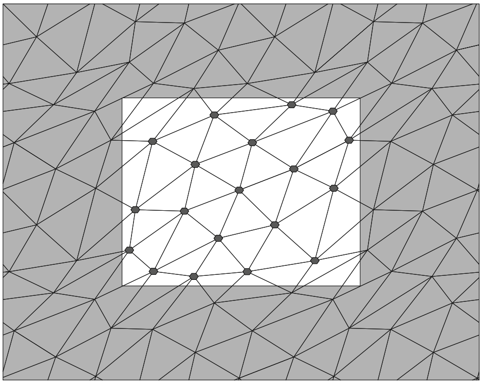

Figure 1: The maximal toroidal 6-regular graph of diameter two; $H(19, 7)$

Table 2: The valency-diameter table for toroidal graphs

	Diameter					
Valency	2	3	4	5	6	7
3	**10**	14	22	34	52	68
4	13	21	39	58	82	134
5	16	26	46	66	124	254
6	**19**	**37**	**61**	**91**	**127**	**169**

possible, the restricted nature of face sizes starts to affect the number of vertices possible in a 6-regular graph from this point onwards.

We tabulate these results for the largest known graph of a given order and valency as shown in Table 2. The entries shown in bold are proven to be the best possible in this paper and the Petersen graph is the (3,5)-Moore graph which is necessarily maximal. All other entries result from constructions of graphs and their images are linked to on the website [3].

Acknowledgements

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References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* 4 1973, 201–217.

[2] J. Preen, http://moorebound.indstate.edu/index.php/The_Degree_Diameter_Problem_for_Planar_Graphs

[3] J. Preen, http://faculty.cbu.ca/jpreen/torvaldiam.html

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