AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 133–143

An enumeration of spherical latin bitrades

Aleš Drápal* Carlo Hämäläinen†

Department of Mathematics
Charles University
Sokolovská 83, 186 75 Praha 8
Czech Republic

Dan Rosendorf

Department of Mathematics
University of Wisconsin, Madison
U.S.A.

Abstract

A *latin bitrade* $(T^{\diamond}, T^{\otimes})$ is a pair of partial latin squares which are disjoint, occupy the same set of non-empty cells, and whose corresponding rows and columns contain the same set of entries. A genus may be associated to a latin bitrade by constructing an embedding of the underlying graph in an oriented surface. We report computational enumeration results on the number of spherical (genus 0) latin bitrades up to size 24.

1 Introduction

A *latin bitrade* $(T^{\diamond}, T^{\otimes})$ is a pair of partial latin squares which are disjoint, occupy the same set of non-empty cells, and whose corresponding rows and columns contain the same set of entries. One of the earliest studies of latin bitrades appeared in [6], where they are referred to as *exchangeable partial groupoids*. Latin bitrades are prominent in the study of *critical sets*, which are minimal defining sets of latin squares ([1],[5],[15]) and the intersections between latin squares ([10]). A genus may be associated to a latin bitrade, and those of genus zero are known as *spherical* latin bitrades. In this paper we report on the enumeration of the number of τ-isomorphism classes of spherical latin bitrades up to size $|T^{\diamond}| = 24$.

* Supported by grant MSM 0021620839
† Supported by Eduard Čech center, grant LC505. Email: `carlo.hamalainen@gmail.com`

For completeness we note that separated spherical latin bitrades are equivalent to cubic 3-connected bipartite planar graphs [4]. Our enumeration method does not consider equivalences where rows, columns, and symbols change their roles (this corresponds to changing the 3-colouring of the related cubic 3-connected bipartite graph), nor does it consider equivalences where $(T^\diamond, T^\otimes)$ and $(T^\otimes, T^\diamond)$ are equivalent (this corresponds to changing the 2-colouring of the related cubic 3-connected bipartite graph). Thus our enumeration method counts a larger class of combinatorial objects compared to the method in [13].

2 Latin bitrades

A *partial latin square* P of order $n > 0$ is an $n \times n$ array where each $e \in \{0, 1, \ldots, n-1\}$ appears at most once in each row, and at most once in each column. A *latin square* L of order $n > 0$ is an $n \times n$ array where each $e \in \{0, 1, \ldots, n-1\}$ appears exactly once in each row, and exactly once in each column. It is convenient to use setwise notation to refer to entries of a (partial) latin square, and we write $(i, j, k) \in P$ if and only if symbol k appears in the intersection of row i and column j of P. In this manner, $P \subseteq A_1 \times A_2 \times A_3$ for finite sets A_i, each of size n. It is also convenient to interpret a (partial) latin square as a multiplication table for a (partial) binary operator $\diamond$, writing $i \diamond j = k$ if and only if $(i, j, k) \in T = T^\diamond$.

Definition 2.1. Let $T^\diamond$, $T^\otimes \subseteq A_1 \times A_2 \times A_3$ be two partial latin squares. Then $(T^\diamond, T^\otimes)$ is a *bitrade* if the following three conditions are satisfied:

(R1) $T^\diamond \cap T^\otimes = \emptyset$;

(R2) for all $(a_1, a_2, a_3) \in T^\diamond$ and all r, $s \in \{1, 2, 3\}$, $r \neq s$, there exists a unique $(b_1, b_2, b_3) \in T^\otimes$ such that $a_r = b_r$ and $a_s = b_s$;

(R3) for all $(a_1, a_2, a_3) \in T^\otimes$ and all r, $s \in \{1, 2, 3\}$, $r \neq s$, there exists a unique $(b_1, b_2, b_3) \in T^\diamond$ such that $a_r = b_r$ and $a_s = b_s$.

Conditions (R2) and (R3) imply that each row (column) of $T^\diamond$ contains the same subset of A_3 as the corresponding row (column) of $T^\otimes$. A bitrade $(T^\diamond, T^\otimes)$ is *indecomposable* if whenever $(U^\diamond, U^\otimes)$ is a bitrade such that $U^\diamond \subseteq T^\diamond$ and $U^\otimes \subseteq T^\otimes$, then $(T^\diamond, T^\otimes) = (U^\diamond, U^\otimes)$. Bijections $A_i \to A_i'$, for $i = 1, 2, 3$, give an *isotopic* bitrade, and permuting each A_i gives an *autotopism*. We refer to the bijections $A_1 \to A_1'$, $A_2 \to A_2'$, $A_3 \to A_3'$ as an *isotopism*.

In [8, 9] there is a representation of bitrades in terms of three permutations τ_i acting on a finite set (see also [12] for another proof). For $r \in \{1, 2, 3\}$, define the map $\beta_r \colon T^\otimes \to T^\diamond$ where $(a_1, a_2, a_3)\beta_r = (b_1, b_2, b_3)$ if and only if $a_r \neq b_r$ and $a_i = b_i$ for $i \neq r$. By Definition 2.1 each β_r is a bijection. Then $\tau_1, \tau_2, \tau_3 \colon T^\diamond \to T^\diamond$ are defined by

$$\tau_1 = \beta_2^{-1}\beta_3, \qquad \tau_2 = \beta_3^{-1}\beta_1, \qquad \tau_3 = \beta_1^{-1}\beta_2. \tag{1}$$

We refer to $[\tau_1, \tau_2, \tau_3]$ as the τ_i *representation*. We write $\mathrm{Mov}(\pi)$ for the set of points that the (finite) permutation π acts on.

Generally we will assume that a bitrade is *separated*, that is, each row, column, and symbol is in bijection with a cycle of τ_1, τ_2, and τ_3, respectively.

Definition 2.2. Let τ_1, τ_2, τ_3 be (finite) permutations and let $\Gamma = \mathrm{Mov}(\tau_1) \cup \mathrm{Mov}(\tau_2) \cup \mathrm{Mov}(\tau_3)$. Define four properties:

(T1) $\tau_1\tau_2\tau_3 = 1$;

(T2) if ρ_i is a cycle of τ_i and ρ_j is a cycle of τ_j then $|\mathrm{Mov}(\rho_i) \cap \mathrm{Mov}(\rho_j)| \leq 1$, for any $1 \leq i < j \leq 3$;

(T3) each τ_i is fixed-point-free;

(T4) the group $\langle \tau_1, \tau_2, \tau_3 \rangle$ is transitive on Γ.

By letting A_i be the set of cycles of τ_i, we have the following theorem, which relates Definition 2.1 and 2.2.

Theorem 2.3 ([8]). *A separated bitrade* $(T^\diamond, T^\otimes)$ *is equivalent (up to isotopism) to three permutations* τ_1, τ_2, τ_3 *acting on a set* Γ *satisfying* (T1), (T2), *and* (T3). *If* (T4) *is also satisfied then the bitrade is indecomposable.*

To construct the τ_i representation for a bitrade we simply evaluate Equation (1). In the reverse direction we have the following construction:

Construction 2.4 (τ_i to bitrade)**.** Let τ_1, τ_2, τ_3 be permutations satisfying Condition (T1), (T2), and (T3). Let $\Gamma = \mathrm{Mov}(\tau_1) \cup \mathrm{Mov}(\tau_2) \cup \mathrm{Mov}(\tau_3)$. Define $A_i = \{\rho \mid \rho \text{ is a cycle of } \tau_i\}$ for $i = 1, 2, 3$. Now define two arrays $T^\diamond$, $T^\otimes$:

$$T^\diamond = \{(\rho_1, \rho_2, \rho_3) \mid \rho_i \in A_i \text{ and } |\mathrm{Mov}(\rho_1) \cap \mathrm{Mov}(\rho_2) \cap \mathrm{Mov}(\rho_3)| \geq 1\}$$
$$T^\otimes = \{(\rho_1, \rho_2, \rho_3) \mid \rho_i \in A_i \text{ and } x, x', x'' \text{ are distinct points of } \Gamma \text{ such that } x\rho_1 = x', x'\rho_2 = x'', x''\rho_3 = x\}.$$

By Theorem 2.3 $(T^\diamond, T^\otimes)$ is a bitrade.

Example 2.5. The smallest bitrade $(T^\diamond, T^\otimes)$ is the *intercalate*, which has four entries. The bitrade is shown below:

$$T^\diamond = \begin{array}{c|cc} \diamond & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad T^\otimes = \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

The τ_i representation is $\tau_1 = (000, 011)(101, 110)$, $\tau_2 = (000, 101)(011, 110)$, $\tau_3 = (000, 110)(011, 101)$, where we have written ijk for $(i, j, k) \in T^\diamond$ to make the presentation of the τ_i permutations clearer. By Construction 2.4 with $\Gamma = \{000, 011, 101, 110\}$

we can convert the τ_i representation to a bitrade $(U^\diamond,\, U^\otimes)$:

$U^\diamond =$

$\diamond$	(000, 101)	(011, 110)
(000, 011)	(000, 110)	(011, 101)
(101, 110)	(011, 101)	(000, 110)

$U^\otimes =$

$\otimes$	(000, 101)	(011, 110)
(000, 011)	(011, 101)	(000, 110)
(101, 110)	(000, 110)	(011, 101)

In this way we see that row 0 of $T^\diamond$ corresponds to row $(000, 011)$ of $U^\diamond$, which is the cycle $(000, 011)$ of τ_1, and so on for the columns and symbols.

Example 2.6. The following bitrade is spherical:

$T^\diamond =$

$\diamond$	0	1	2	3	4
0	0		2		4
1				4	2
2	1	3	0	2	
3	4	1		3	
4					

$T^\otimes =$

$\otimes$	0	1	2	3	4
0	4		0		2
1				2	4
2	0	1	2	3	
3	1	3		4	
4					

Here, the τ_i representation is

$$\tau_1 = (000, 022, 044)(134, 142)(201, 213, 232, 220)(304, 333, 311)$$
$$\tau_2 = (000, 304, 201)(213, 311)(022, 220)(134, 232, 333)(044, 142)$$
$$\tau_3 = (000, 220)(201, 311)(022, 232, 142)(213, 333)(044, 134, 304)$$

If $Z = [\tau_1,\, \tau_2,\, \tau_3]$ then the inverse is denoted by $Z^{-1} = [\tau_1^{-1},\, \tau_2^{-1},\, \tau_2\tau_1]$. This is equivalent to exchanging $(T^\diamond,\, T^\otimes)$ for $(T^\otimes,\, T^\diamond)$ and relabelling entries as the following lemma shows:

Lemma 2.7. *Let $(T^\diamond,\, T^\otimes)$ be a separated bitrade with representation $Z = [\tau_1,\, \tau_2,\, \tau_3]$. Then the inverse bitrade $(T^\otimes,\, T^\diamond)$ has representation, denoted Z^{-1}. This representation is isomorphic to $[\tau_1^{-1},\, \tau_2^{-1},\, \tau_2\tau_1]$.*

Proof. Observe that Definition 2.1 does not specify the order of the two partial latin squares in a bitrade so the bitrade $(T^\otimes,\, T^\diamond)$ is well-defined. For this bitrade we have permutation representation $[\nu_1,\, \nu_2,\, \nu_3]$. Using Equation (1) we find that $\nu_1 = \beta_2\beta_3^{-1}$ and $\nu_2 = \beta_3\beta_1^{-1}$. Taking the conjugate of ν_r by β_3^{-1} gives us a representation that acts on the set $T^\diamond$. In particular, $\beta_3^{-1}\nu_1\beta_3 = \beta_3^{-1}\beta_2\beta_3^{-1}\beta_3 = \beta_3^{-1}\beta_2 = (\beta_2^{-1}\beta_3)^{-1} = \tau_1^{-1}$ and $\beta_3^{-1}\nu_2\beta_3 = \beta_3^{-1}\beta_3\beta_1^{-1}\beta_3 = \beta_1^{-1}\beta_3 = (\beta_3\beta_1^{-1})^{-1} = \tau_2^{-1}$. This shows that the representation $[\nu_1,\, \nu_2,\, \nu_3]$ is isomorphic to $[\tau_1^{-1},\, \tau_2^{-1},\, \tau_2\tau_1]$. ☐

Let D be the graph with vertices Γ and directed edges (x, y) where $x\tau_i = y$ for some i. A rotation scheme can be imposed on this graph, by ordering the edges as shown in Figure 1. This turns D into an oriented combinatorial surface. Define

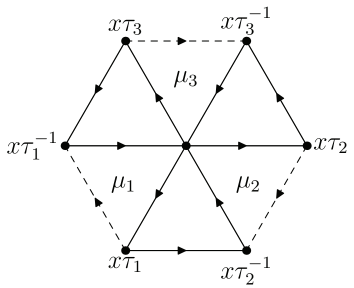

Figure 1: Rotation scheme for the combinatorial surface. Dashed lines represent one or more edges, where μ_i is a cycle of τ_i. The vertex in the centre is x.

$\mathrm{order}(\tau_i) = z(\tau_1) + z(\tau_2) + z(\tau_3)$, the total number of cycles, and $\mathrm{size}(\tau_i) = |\Gamma|$. By some basic counting arguments we find that there are $\mathrm{size}(\tau_i)$ vertices, $3 \cdot \mathrm{size}(\tau_i)$ edges, and $\mathrm{order}(\tau_i) + \mathrm{size}(\tau_i)$ faces. Then Euler's formula $V - E + F = 2 - 2g$ gives

$$\mathrm{order}(\tau_i) = \mathrm{size}(\tau_i) + 2 - 2g \tag{2}$$

where g is the genus of the combinatorial surface. We say that the bitrade T has genus g. If $g = 0$ then we say that the bitrade is *spherical*.

3 Slide expansion

Let $[\tau_1, \tau_2, \tau_3]$ be a spherical bitrade on the set Γ. Choose some $x \in \Gamma$ and fix a direction $j \in \{1, 2, 3\}$. Set $k = j + 1$ and $l = k + 1$ (calculating modulo 3). Suppose that (1) the j-cycle at x has length at least 3 and (2) the k-cycle through x and the l-cycle through $w = x\tau_j$ do not meet in any common point. Then the *slide expansion* at $x \in \Gamma$ in direction j is the following augmentation of the cycles of τ_1, τ_2, and τ_3 by a new point $u \notin \Gamma$:

$$\begin{aligned}
\tau_j &\colon (a, x, w, b, \dots) \mapsto (a, u, b, \dots) \text{ and } (x, w) \\
\tau_k &\colon (\dots, x, z, \dots) \mapsto (\dots, x, u, z, \dots) \\
\tau_l &\colon (\dots, y, w, \dots) \mapsto (\dots, y, u, w, \dots).
\end{aligned}$$

We denote the result of the slide expansion on $X = [\tau_1, \tau_2, \tau_3]$ at vertex x in direction j by $\mathrm{slide}(X, x, j)$.

A separated bitrade $[\tau_1, \tau_2, \tau_3]$ is *bicyclic* if there exists j such that τ_j consists of only two cycles. These bicyclic bitrades form the root nodes of our enumeration procedure due to the following result:

Theorem 3.1 ([7]). *Let $Z = [\tau_1, \tau_2, \tau_3]$ be a spherical bitrade. Then there exists a sequence $W_0, W_1, \dots, W_\ell = Z$ where W_0 is a bicyclic bitrade and W_{i+1} is the result of a slide expansion on either W_i or W_i^{-1}, for $0 \le i < \ell$.*

The opposite of a slide expansion is a *slide contraction*. For a spherical bitrade $[\tau_1, \tau_2, \tau_3]$ on the set Γ, choose some $u \in \Gamma$ and fix a direction $j \in \{1, 2, 3\}$. Set $k = j + 1$ and $l = k + 1$ (calculating modulo 3). The slide contraction at u in direction j modifies the cycles of τ_1, τ_2, and τ_3 as follows:

$$\begin{aligned} \tau_j &\colon (a, u, b, \dots) \text{ and } (x, w) \mapsto (a, x, w, b, \dots) \\ \tau_k &\colon (\dots, x, u, z, \dots) \mapsto (\dots, x, z, \dots) \\ \tau_l &\colon (\dots, y, u, w, \dots) \mapsto (\dots, y, w, \dots). \end{aligned}$$

A necessary condition for a slide contraction to be valid is that the u-cycle in τ_k and τ_l has length at least 3 (this avoids the creation of a fixed point when u is removed). In general it is necessary to check the bitrade conditions (T1), (T2), and (T3) to ensure that a slide contraction is valid.

4 Enumeration

To enumerate separated spherical bitrades we must fix the notion of τ-isomorphism. Two bitrades $[\tau_1, \tau_2, \tau_3]$ and $[\nu_1, \nu_2, \nu_3]$ on the same set Γ are τ-isomorphic if there is a permutation $\theta \in \mathrm{Sym}(\Gamma)$ such that $\tau_i^\theta = \nu_i$ for all $i \in \{1, 2, 3\}$. A τ-automorphism of a bitrade is defined similarly.

We follow the presentation in [14]. In the context of an algorithm for enumerating combinatorial objects, the *domain* of a search is the finite set Ω that contains all objects considered by the search. The search space is conveniently modelled by a rooted tree, with nodes corresponding to elements of Ω, and nodes joined by an edge if they are related by one search step. The root node is the starting node of the search. We write $C(X)$ for the set of all child nodes of X, and $p(X)$ for the parent node of X.

To reduce the search time we use the method of *canonical augmentation* [16], following the presentation in [14]. In terms of the search tree, the ordered pair $(X, p(X))$ characterises the augmentation used to generate X from the node $p(X)$ during the search. In our case, this augmentation $(X, p(X))$ is the result of a slide expansion on $p(X)$. The goal of the canonical augmentation algorithm is to choose one of the possible parents of X to be the canonical parent. Formally, for any nonroot node X, the *canonical parent* $m(X) \in \Omega$ must satisfy:

(C1) for all nonroot objects X, Y it holds that $X \cong Y$ implies $(X, m(X)) \cong (Y, m(Y))$.

The next property captures the canonical augmentations in the search tree:

(C2) a node Z occurring in the search tree is generated by canonical augmentation if $(Z, p(Z)) \cong (Z, m(Z))$.

Algorithm 1 gives pseudo-code for the general canonical augmentation search algorithm. Algorithm 2 is a more explicit version for enumerating separated spherical latin bitrades. The correctness of Algorithm 1 relies on two properties:

(I1) for all nodes X, Y it holds that if $X \cong Y$, then for every $Z \in C(X)$ there exists a $W \in C(Y)$ such that $(Z, X) \cong (W, Y)$.

(I2) for every nonroot node X, there exists a node Y such that $X \cong Y$ and $(Y, m(Y)) \cong (Y, p(Y))$.

Condition (I1) says that isomorphic nodes have isomorphic children. Condition (I2) says that every nonroot node is generated by some canonical augmentation.

Algorithm 1 CANAUG-TRAVERSE(X)

report X (if applicable)
for all $\mathcal{Z} \in \{C(X) \cap \{aZ : a \in \mathrm{Aut}(X)\} : Z \in C(X)\}$ **do**
  select any $Z \in \mathcal{Z}$
  if $(Z, p(Z)) \cong (Z, m(Z))$ **then**
    CANAUG-TRAVERSE(Z)
  end if
end for

Algorithm 2 CANAUG-SPHERICAL(X)

report X (if applicable)
for all $\mathcal{Z} \in \{C(X) \cap \{aZ : a \in \mathrm{Aut}(X)\} : Z \in C(X)\}$ **do**
  select any $Z \in \mathcal{Z}$
  if $(Z, p(Z)) \cong (Z, m(Z))$ **then**
    CANAUG-SPHERICAL(Z)
    if Z^{-1} has no parents **then**
      CANAUG-SPHERICAL(Z^{-1})
    end if
  end if
end for
if $(|X| + 1) \equiv 0 \pmod 2$ **then**
  for all bicyclic Z with $|Z| = |X| + 1$ **do**
    CANAUG-SPHERICAL(Z)
  end for
end if

Theorem 4.1 ([14]). *When implemented on a search tree satisfying* (I1) *and* (I2)*, the algorithm* CANAUG-TRAVERSE *reports exactly one node from every isomorphism class of nodes.*

4.1 Canonical form

The canonical form $\hat{Z}$ is a relabelling of Z such that $\hat{Z} = \hat{Y}$ if and only if $Z \cong Y$. We need to be able to efficiently compute the canonical form in order to construct the canonical parent of a bitrade.

To compute the canonical form of a bitrade $Z = [\tau_1, \tau_2, \tau_3]$, we fix a starting point $x \in \Gamma$. We then perform a breadth-first traversal of the bitrade's underlying directed graph. Vertices are labelled from the set $\{1, \dots, |Z|\}$ in the order that they are visited. At a vertex v we first visit the τ_1 cycle at v, then the τ_2 cycle at v, and then the τ_3 cycle at v (only if the cycle has not been visited before). As we traverse a cycle we label unseen vertices and append the cycle (using the new vertex labels) to an array C. After traversing a cycle we append a marker (here, -1) to denote the end of the cycle. While traversing a τ_i cycle at the vertex v, we check if the neighbouring τ_j and τ_k cycle at v has been visited (j, $k \neq i$). If it has not, we push (j, v) and (k, v) onto the queue, making sure that $j < k$. The lexicographically least $C = C(x)$ is the canonical form $\hat{Z}$ of Z. Since $\hat{Z}$ encodes just the cycle structure of the τ_i permutations, it follows that $\hat{Z} = \hat{Z}'$ if and only if $Z \cong Z'$.

The canonical parent $m(Z)$ is now computed as follows. First we find the canonical form $\hat{Z}$ of Z. In our case, we choose the lexicographically maximal pair (j, u) such that a slide contraction may be performed on $\hat{Z}$ at u in direction j. Let Z_0 be the result of this contraction. Then $m(Z) = \kappa^{-1}(Z_0)$ where $\kappa\colon Z \mapsto \hat{Z}$ is the canonical relabelling map.

4.2 Correctness

Condition (I1) holds since a slide expansion $\mathrm{slide}(X, x, j)$ is equivalent to $\mathrm{slide}(Y, y, j)$ where $X \cong Y$ and y is the image of x under the isomorphism. In other words, the slide expansion relies only on the cycle structure of the bitrade, not its specific labelling. Condition (I2) holds because the canonical parent $m(Z)$ is chosen only based on the canonical form of Z and so is independent of isomorphism.

Algorithm 2 produces bicyclic bitrades at each step where even-sized bitrades are being generated. The following lemma and corollary are necessary to show that no τ-isomorphism class of bicyclic bitrades is produced more than once.

Lemma 4.2 (Lemma 10.1 [7]). *Let $[\tau_1, \tau_2, \tau_3]$ be a bitrade such that τ_j, for some $j \in \{1, 2, 3\}$, consists of exactly two cycles. Let one of the cycles be $(x_0, \dots, x_{n-1})$. Then the other cycle of τ_j can be expressed as $(x'_{n-1}, \dots, x'_0)$ in such a way that τ_{j+1} consists of cycles (x_i, x'_i) and τ_{j-1} consists of cycles (x_{i+1}, x'_i), for $0 \leq i < n$.*

Corollary 4.3. *Let Z be a bicyclic bitrade. Then there is no X such that $\mathrm{slide}(X, x, j) = Z$ or $\mathrm{slide}(X, x, j) = Z^{-1}$ for any x, j.*

Proof. The slide contraction shortens the cycle at some point u in τ_{j+1} and τ_{j-1}. By Lemma 4.2 at least one of the cycles in τ_{j+1} or τ_{j-1} has length 2, so the slide contraction would introduce a fixed point, contradicting (T3). ☐

It can be shown that for any bicyclic bitrade Z we have $Z \cong Z^{-1}$. Corollary 4.3 shows that lines 11–15 of Algorithm 2 are correct.

5 Implementation and relation to other work

See [11] for C++ computer code. The MPI library is used to allow parallel processing of the search tree on a Linux cluster. The size of the automorphism group $\mathrm{Aut}(Z)$ is bounded by the size of Γ. Further, the number of homogeneous bitrades is quite small, so usually the automorphism group is even smaller. We just compute the elements of $\mathrm{Aut}(Z)$ explicitly.

Cavenagh and Lisonek [4] showed that spherical bitrades are equivalent to planar Eulerian triangulations. To verify our results (for small n) we used plantri [2] to generate planar Eulerian triangulations. We then applied a simultaneous vertex 3-colouring and face 2-colouring, giving a bitrade $(T^{\diamond}, T^{\otimes})$. Then each conjugate of $(T^{\diamond}, T^{\otimes})$ was produced, and finally isomorphic copies removed.

The number of τ-isomorphism classes of spherical bitrades with $|T^{\diamond}| = n$ is given below:

n	# iso classes	$e_3(n_v)$
4	1	1
6	3	1
7	1	1
8	6	2
9	9	2
10	30	8
11	51	8
12	198	32
13	470	57
14	1623	185
15	4830	466
16	16070	1543
17	51948	4583
18	175047	15374
19	588120	50116
20	2015226	171168
21	6933048	582603
22	24123941	2024119
23	84428820	7057472
24	297753519	24873248

The third column is the value $e_3(n_v)$, which is the number of isomorphism classes of eulerian plane triangulations with connectivity at least 3 (see Table 6 in [3] for values $n_f = 2n$ in their first column and also [17]). The enumeration resulting in $e_3(n_v)$ considers isomorphic any two graphs that are related by a permutation of the face 2-colouring, changing the orientation of the permutations, and permuting the cyclic ordering. So for large values of n there will be at most $2 \cdot 2 \cdot 3 = 12$ ways to relabel one of their graphs to obtain one of our bitrades. For example, with $n = 24$ we have $e_3(n_v) = 24873248$ and $12 \cdot 24873248 = 298478976 \approx 297753519$. Also, Wanless [18] enumerated spherical bitrades up to size 19, under various various equivalences, but not τ-isomorphism as studied in this paper.

References

[1] J. A. Bate and G. H. J. van Rees, Minimal and near-minimal critical sets in back-circulant Latin squares, *Australas. J. Combin.* **27** (2003), 47–61.

[2] G. Brinkmann and B. D. McKay, `plantri` (software), `http://cs.anu.edu.au/~bdm/plantri`.

[3] Gunnar Brinkmann and Brendan D. McKay, *Fast generation of planar graphs*, MATCH Commun. Math. Comput. Chem. **58** (2007), no. 2, 323–357.

[4] Nicholas J. Cavenagh and Petr Lisonek, *Planar eulerian triangulations are equivalent to spherical latin bitrades*, J. Comb. Theory, Ser. A **115** (2008), no. 1, 193–197.

[5] Joan Cooper, Diane Donovan, and Jennifer Seberry, *Latin squares and critical sets of minimal size*, Australas. J. Combin. **4** (1991), 113–120, Combinatorial mathematics and combinatorial computing (Palmerston North, 1990).

[6] A. Drápal and T. Kepka, *Exchangeable partial groupoids. I*, Acta Univ. Carolin. Math. Phys. **24** (1983), no. 2, 57–72.

[7] Aleš Drápal, *On elementary moves that generate all spherical latin trades*, *Comment Math. Univ. Car.* (to appear).

[8] Aleš Drápal, Geometry of latin trades, Manuscript, Loops conf., Prague, 2003.

[9] Aleš Drápal, Geometrical structure and construction of latin trades, *Advances in Geometry* **9** (2009), no. 3, 311–348.

[10] H-L. Fu, *On the construction of certain type of latin squares with prescribed intersections*, Ph.D. thesis, Auburn University, 1980.

[11] Carlo Hämäläinen, Spherical bitrade enumeration code, `http://bitbucket.org/carlohamalainen/spherical`.

[12] Carlo Hämäläinen, Latin bitrades and related structures, PhD University of Queensland, 2007, http://carlo-hamalainen.net/phd/hamalainen-20071025.pdf.

[13] Derek A. Holton, Bennet Manvel and Brendan D. McKay, Hamiltonian cycles in cubic 3-connected bipartite planar graphs, *J. Combin. Theory Ser. B* **38** (1985), 279–297.

[14] Petteri Kaski and Patric R. J. Östergård, Classification algorithms for codes and designs, *Algorithms and Computation in Mathematics*, no. 15, Springer-Verlag, Berlin Heidelberg, 2006.

[15] A. D. Keedwell, Critical sets in latin squares and related matters: an update, *Util. Math.* **65** (2004), 97–131.

[16] Brendan D. McKay, Isomorph-free exhaustive generation, *J. Algorithms* **26** (1998), no. 2, 306–324.

[17] On-Line Encyclopedia of Integer Sequences, *Number of unlabeled trivalent 3-connected bipartite planar graphs with 2n nodes*, `http://www.research.att.com/~njas/sequences/A000108`.

[18] Ian M. Wanless, A computer enumeration of small Latin trades, *Australas. J. Combin.* **39** (2007), 247–258.

(Received 8 July 2009; revised 21 Sep 2009)