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Splitting group divisible designs with block size 2×4

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Abstract

The necessary conditions for the existence of a $(2 \times 4, \lambda)$-splitting GDD of type g^v are $gv \geq 8$, $\lambda g(v-1) \equiv 0 \pmod 4$, $\lambda g^2 v(v-1) \equiv 0 \pmod{32}$. It is proved in this paper that these conditions are also sufficient except for $\lambda \equiv 0 \pmod{16}$ and $(g, v) = (3, 3)$.

1 Introduction

In the study of authentication codes, Ogata et al. [5] found that splitting balanced incomplete block designs (splitting BIBDs) can be used to construct c-splitting A-codes, whose impersonation attack probabilities and substitution attack probabilities all achieve their information-theoretic lower bounds.

Let v, u, c, λ be positive integers such that $v \geq uc$. A $(v, u \times c, \lambda)$-*splitting BIBD* is a pair $(V, \mathcal{B})$ where

1) V is a v-set of elements (called *points*);
2) $\mathcal{B}$ is a collection of $u \times c$ arrays (called *blocks*) with entries from V, such that every point occurs at most once in each block;
3) for every pair of distinct points $x, y \in V$, there are exactly λ blocks in which x and y occur in different rows.

Several authors have studied the existence problem for splitting BIBDs; see [1–6]. In the recursive constructions of splitting BIBDs, splitting group divisible designs (splitting GDDs) play an important role.

A $(u \times c, \lambda)$-*splitting GDD* is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of elements (called *points*), $\mathcal{G}$ is a partition of V into subsets (called *groups*), and $\mathcal{B}$ is a collection of $u \times c$ arrays with entries from V (called *blocks*), such that

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1) every point of V occurs at most once in each block;
2) for every pair of points x, y, where x and y belong to distinct groups, there exist exactly λ blocks of $\mathcal{B}$ in which x and y occur in different rows.

The *group type* (or *type*) of a $(u\times c,\lambda)$-splitting GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We use an exponential notation to describe group type. Thus a splitting GDD of type $g_1^{v_1} \dots g_n^{v_n}$ is one in which there are exactly v_i groups of size g_i, $1 \le i \le n$. Clearly, a $(u \times c, \lambda)$-splitting GDD of type 1^v is equivalent to a $(v, u \times c, \lambda)$-splitting BIBD.

By simple counting arguments, we have the following lemma.

Lemma 1.1 *The necessary conditions for the existence of a $(u\times c,\lambda)$-splitting GDD of type g^v are $gv \ge uc$, $\lambda g(v-1) \equiv 0 \pmod{c(u-1)}$, and $\lambda g^2 v(v-1) \equiv 0 \pmod{c^2u(u-1)}$.*

The necessary conditions for the existence of a $(2 \times 3, \lambda)$-splitting GDD of type g^v have been proved to be sufficient (see [6]), except for $(\lambda, g^v) \in \{(1, 1^{10})\} \cup \{(\lambda, 1^6) : \lambda \equiv 3 \pmod 6\} \cup \{(\lambda, 2^3) : \lambda \equiv 0 \pmod 3\}$, and possibly for $(\lambda, g^v) \in \{(3, 2^4), (6, 2^4)\} \cup \{(1, g^{10}) : g \equiv 1, 5 \pmod 6 \text{ and } g > 1\}$.

In this paper we will study the existence problem for $(2 \times 4, \lambda)$-splitting GDDs of type g^v. We will show that the necessary conditions for such designs are also sufficient except for $\lambda \equiv 0 \pmod{16}$ and $(g, v) = (3, 3)$.

2 Recursive Constructions

In this section we will provide some recursive constructions and related designs.

A *transversal design*, denoted by $\mathrm{TD}_\lambda(k,m)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where

1) V is a set of km elements (called *points*);
2) $\mathcal{G}$ is a partition of V into k subsets (called *groups*), each of size m;
3) $\mathcal{B}$ is a collection of k-subsets of V (called *blocks*);
4) every pair of points from V is contained either in exactly one group or in exactly λ blocks, but not both.

The following lemma is well known.

Lemma 2.1 *There exists a $TD_\lambda(2,m)$ for any $\lambda \ge 1$ and $m \ge 1$.*

The following construction is a powerful tool in constructing splitting GDDs.

Construction 2.2 *Suppose there exist a $(u\times c,\lambda_1)$-splitting GDD of type $g_1^{v_1} \cdots g_n^{v_n}$ and a $TD_{\lambda_2}(u,m)$. Then there exists a $(u\times c,\lambda_1\lambda_2)$-splitting GDD of type $(mg_1)^{v_1} \cdots (mg_n)^{v_n}$.*

Proof. Let $(V_1, \mathcal{G}_1, \mathcal{B}_1)$ be a $(u \times c, \lambda_1)$-splitting GDD of type $g_1^{v_1} \cdots g_n^{v_n}$. Let $(V_2, \mathcal{G}_2, \mathcal{B}_2)$ be a $\mathrm{TD}_{\lambda_2}(u, m)$ where $V_2 = \{1, 2, \ldots, u\} \times M$, $\mathcal{G}_2 = \big\{\{i\} \times M : i = 1, 2, \ldots, u\big\}$, and $M = \{1, 2, \ldots, m\}$. For each block $B \in \mathcal{B}_1$, suppose

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1c} \\ b_{21} & b_{22} & \cdots & b_{2c} \\ \cdots & \cdots & \cdots & \cdots \\ b_{u1} & b_{u2} & \cdots & b_{uc} \end{pmatrix}.$$

Now let

$$\mathcal{A}_B = \left\{ \begin{pmatrix} (b_{11}, j_1) & (b_{12}, j_1) & \cdots & (b_{1c}, j_1) \\ (b_{21}, j_2) & (b_{22}, j_2) & \cdots & (b_{2c}, j_2) \\ \cdots & \cdots & \cdots & \cdots \\ (b_{u1}, j_u) & (b_{u2}, j_u) & \cdots & (b_{uc}, j_u) \end{pmatrix} : \{(1, j_1), (2, j_2), \ldots, (u, j_u)\} \in \mathcal{B}_2 \right\}.$$

Then it is easy to check that $(V, \mathcal{G}, \mathcal{B})$ is the desired design, where $V = V_1 \times M$, $\mathcal{G} = \{G \times M : G \in \mathcal{G}_1\}$, and $\mathcal{B} = \bigcup_{B \in \mathcal{B}_1} \mathcal{A}_B$. □

Combining Construction 2.2 and Lemma 2.1 gives the following construction.

Construction 2.3 *Let $\lambda_2 \geq 1$ and $m \geq 1$. Suppose there exists a $(2 \times c, \lambda_1)$-splitting GDD of type $g_1^{v_1} \cdots g_n^{v_n}$. Then there exists a $(2 \times c, \lambda_1\lambda_2)$-splitting GDD of type $(mg_1)^{v_1} \cdots (mg_n)^{v_n}$.*

The following results can be found in [4].

Lemma 2.4 [4] *There exists a $(v, 2 \times 4, \lambda)$-splitting BIBD if and only if $\lambda(v-1) \equiv 0 \pmod 4$, $\lambda v(v-1) \equiv 0 \pmod{32}$.*

Lemma 2.5 [4] *There exists a $(2 \times 4, 1)$-splitting GDD of type 4^v for any $v \geq 2$.*

Applying Construction 2.3 to Lemma 2.5 gives the following lemma.

Lemma 2.6 *There exists a $(2 \times 4, \lambda)$-splitting GDD of type g^v for any $\lambda \geq 1$, $g \equiv 0 \pmod 4$, and $v \geq 2$.*

The following lemma is obvious.

Lemma 2.7 *If there exist a $(u \times c, \lambda_1)$-splitting GDD of type g^v and a $(u \times c, \lambda_2)$-splitting GDD of type g^v, then there exists a $(u \times c, \lambda_1 + \lambda_2)$-splitting GDD of type g^v.*

3 Direct Constructions

Lemma 3.1 *For each $t \geq 1$, there exists a $(2 \times 4, 1)$-splitting GDD of type 2^{8t+1}.*

Proof. Let $V = Z_{8t+1} \times \{1,2\}$, $\mathcal{G} = \{\{i_1, i_2\} : i \in Z_{8t+1}\}$, and develop the following base blocks mod $8t+1$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+4i)_1 & (4+4i)_1 & (2+4i)_2 & (4+4i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \dots, t-1. \qquad \square$$

Lemma 3.2 *For each $t \geq 1$, there exists a $(2 \times 4, 2)$-splitting GDD of type 2^{4t}.*

Proof. Let $V = (Z_{4t-1} \cup \{\infty\}) \times \{1,2\}$, $\mathcal{G} = \{\{i_1, i_2\} : i \in Z_{4t-1}\} \cup \{\{\infty_1, \infty_2\}\}$, and develop the following base blocks mod $4t-1$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ 2_1 & 2_2 & \infty_1 & \infty_2 \end{pmatrix},$$
$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+2i)_1 & (4+2i)_1 & (2+2i)_2 & (4+2i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \dots, t-2. \qquad \square$$

Lemma 3.3 *For each $t \geq 1$, there exists a $(2 \times 4, 2)$-splitting GDD of type 2^{4t+1}.*

Proof. Let $V = Z_{4t+1} \times \{1,2\}$, $\mathcal{G} = \{\{i_1, i_2\} : i \in Z_{4t+1}\}$, and develop the following base blocks mod $4t+1$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+4i)_1 & (4+4i)_1 & (2+4i)_2 & (4+4i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \dots, t-1. \qquad \square$$

Lemma 3.4 *For each $t \geq 1$, there exists a $(2 \times 4, 4)$-splitting GDD of type $(4t+2)^2$.*

Proof. Let $V = Z_{2t+1} \times \{1,2,3,4\}$, $\mathcal{G} = \{G_1, G_2\}$, where

$$G_1 = \{0_1, 1_1, \dots, (2t)_1, 0_2, 1_2, \dots, (2t)_2\},$$

$$G_2 = \{0_3, 1_3, \dots, (2t)_3, 0_4, 1_4, \dots, (2t)_4\}.$$

The blocks can be obtained by developing the following base blocks mod $2t+1$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ i_3 & (i+1)_3 & i_4 & (i+1)_4 \end{pmatrix}, \quad i = 0, 1, 2, \dots, 2t. \qquad \square$$

Lemma 3.5 *For each $t \geq 1$, there exists a $(2 \times 4, 4)$-splitting GDD of type $(4t+2)^3$.*

Proof. Let $V = Z_{6t+3} \times \{1,2\}$, $\mathcal{G} = \{G_1, G_2, G_3\}$, where

$$G_{i+1} = \{i_1, (i+3)_1, \dots, (i+6t)_1, i_2, (i+3)_2, \dots, (i+6t)_2\}, \; i = 0, 1, 2.$$

The blocks can be obtained by developing the following base blocks mod $6t+3$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+3i)_1 & (5+3i)_1 & (2+3i)_2 & (5+3i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \dots, 2t. \qquad \square$$

Lemma 3.6 *For each $t \geq 1$, there exists a $(2 \times 4, 4)$-splitting GDD of type 2^{4t+2}.*

Proof. Let $V = (Z_{4t+1} \cup \{\infty\}) \times \{1, 2\}$, $\mathcal{G} = \{\{i_1, i_2\} : i \in Z_{4t+1}\} \cup \{\{\infty_1, \infty_2\}\}$, and develop the following base blocks mod $4t + 1$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ 2_1 & 2_2 & \infty_1 & \infty_2 \end{pmatrix}, \quad \begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ 2_1 & 2_2 & \infty_1 & \infty_2 \end{pmatrix},$$
$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+2i)_1 & (4+2i)_1 & (2+2i)_2 & (4+2i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \ldots, t-2,$$
$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+2i)_1 & (4+2i)_1 & (2+2i)_2 & (4+2i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \ldots, t-1.$$

□

Lemma 3.7 *For each $t \geq 1$, there exists a $(2 \times 4, 4)$-splitting GDD of type 2^{4t+3}.*

Proof. Let $V = Z_{4t+3} \times \{1, 2\}$, $\mathcal{G} = \{\{i_1, i_2\} : i \in Z_{4t+3}\}$, and develop the following base blocks mod $4t + 3$.

$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ (2+2i)_1 & (4+2i)_1 & (2+2i)_2 & (4+2i)_2 \end{pmatrix}, \quad i = 0, 1, 2, \ldots, 2t-1,$$
$$\begin{pmatrix} 0_1 & 1_1 & 0_2 & 1_2 \\ 2_1 & (4t+2)_1 & 2_2 & (4t+2)_2 \end{pmatrix}.$$

□

Lemma 3.8 *For each $t \geq 1$, there exists a $(2 \times 4, 8)$-splitting GDD of type $(2t+1)^4$.*

Proof. Let $V = Z_{6t+3} \cup \{\infty_i : i \in Z_{2t+1}\}$, $\mathcal{G} = \{G_1, G_2, G_3, G_4\}$, where

$$G_{i+1} = \{i, i+3, \ldots, i+6t\}, \ i = 0, 1, 2,$$

$$G_4 = \{\infty_i : i \in Z_{2t+1}\}.$$

The blocks can be obtained by developing the following base blocks mod $6t + 3$.

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 2+3i & 5+3i & \infty_i & \infty_{1+i} \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.$$

□

Lemma 3.9 *For each $t \geq 1$, there exists a $(2 \times 4, 8)$-splitting GDD of type $(2t+1)^5$.*

Proof. Let $V = Z_{10t+5}$, $\mathcal{G} = \{\{0, 5, 10, \ldots, 10t\} + i : i = 0, 1, 2, 3, 4\}$, and develop the following base blocks mod $10t + 5$.

$$t = 1: \quad \begin{pmatrix} 0 & 1 & 2 & 5 \\ 3 & 4 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 5 \\ 3 & 4 & 8 & 13 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 5 & 10 \\ 1 & 6 & 8 & 11 \end{pmatrix}.$$
$$t > 1: \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & 14+5i & 19+5i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.$$

□

Lemma 3.10 *For each $t \geq 2$, there exists a $(2 \times 4, 16)$-splitting GDD of type $(2t+1)^2$.*

Proof. Let $V = Z_{2t+1} \cup \{\infty_i : i \in Z_{2t+1}\}$, $\mathcal{G} = \{G_1, G_2\}$, where

$$G_1 = Z_{2t+1}, \quad G_2 = \{\infty_i : i \in Z_{2t+1}\}.$$

The blocks can be obtained by developing the following base blocks mod $2t+1$.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ \infty_i & \infty_{1+i} & \infty_{2+i} & \infty_{3+i} \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.$$

□

Lemma 3.11 *For each $t \geq 2$, there exists a $(2{\times}4, 16)$-splitting GDD of type $(2t{+}1)^3$.*

Proof. Let $V = Z_{6t+3}$, $\mathcal{G} = \{\{0, 3, 6, \ldots, 6t\} + i : i = 0, 1, 2\}$, and develop the following base blocks mod $6t+3$.

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 2+3i & 5+3i & 8+3i & 11+3i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.$$

□

Lemma 3.12 *For each $t \geq 1$, there exists a $(2{\times}4, 16)$-splitting GDD of type $(2t{+}1)^6$.*

Proof. Let $V = Z_{10t+5} \cup \{\infty_i : i \in Z_{2t+1}\}$, $\mathcal{G} = \{G_1, G_2, G_3, G_4, G_5, G_6\}$, where

$$G_{i+1} = \{i, i+5, \ldots, i+10t\}, \ i = 0, 1, 2, 3, 4; \quad G_6 = \{\infty_i : i \in Z_{2t+1}\}.$$

The blocks can be obtained by developing the following base blocks mod $10t+5$.

$$\begin{array}{ll}
t = 1: & \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & \infty_i & \infty_{1+i} \end{pmatrix}, \ i = 0, 1, 2, \\
& \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & \infty_i & \infty_{1+i} \end{pmatrix}, \ i = 0, 1, 2, \\
& \begin{pmatrix} 0 & 1 & 2 & 5 \\ 3 & 4 & 8 & 9 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 2 & 5 \\ 3 & 4 & 8 & 13 \end{pmatrix}, \ \begin{pmatrix} 0 & 2 & 5 & 10 \\ 1 & 6 & 8 & 11 \end{pmatrix}. \\
t > 1: & \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & \infty_i & \infty_{1+i} \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t, \\
& \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & \infty_i & \infty_{1+i} \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t, \\
& \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4+5i & 9+5i & 14+5i & 19+5i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.
\end{array}$$

□

Lemma 3.13 *For each $t \geq 1$, there exists a $(2{\times}4, 16)$-splitting GDD of type $(2t{+}1)^7$.*

Proof. Let $V = Z_{14t+7}$, $\mathcal{G} = \{\{0, 7, \ldots, 14t\} + i : i = 0, 1, \ldots, 6\}$, and develop the following base blocks mod $14t+7$.

$$\begin{array}{ll}
t = 1: & \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 11 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 11 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 11 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 11 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 11 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 2 & 4 \\ 3 & 6 & 10 & 12 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 3 & 4 \\ 2 & 9 & 12 & 16 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 & 4 & 11 \\ 10 & 13 & 16 & 20 \end{pmatrix}, \ \begin{pmatrix} 0 & 2 & 7 & 14 \\ 1 & 8 & 13 & 15 \end{pmatrix}. \\
t > 1: & \begin{pmatrix} 0 & 1 & 4 & 5 \\ 6+7i & 13+7i & 20+7i & 27+7i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t, \\
& \begin{pmatrix} 0 & 1 & 4 & 5 \\ 6+7i & 13+7i & 20+7i & 27+7i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t, \\
& \begin{pmatrix} 0 & 1 & 7 & 8 \\ 4+7i & 11+7i & 18+7i & 25+7i \end{pmatrix}, \ i = 0, 1, 2, \ldots, 2t.
\end{array}$$

□

4 Main Result

Now we are in a position to prove the main theorem.

Theorem 4.1 *There exists a* $(2 \times 4, \lambda)$*-splitting GDD if and only if* $gv \geq 8$, $\lambda g(v-1) \equiv 0 \pmod 4$, $\lambda g^2 v(v-1) \equiv 0 \pmod{32}$, *and* $(\lambda, g, v) \notin \{(\lambda, 3, 3) : \lambda \equiv 0 \pmod{16}\}$.

Proof. From Lemma 1.1 we have the following necessary conditions for the existence of a $(2 \times 4, \lambda)$-splitting GDD of type g^v (see Table 1):

	λ	g	v
Case 1a	$\equiv 1 \pmod 2$	$\equiv 1 \pmod 2$	$\equiv 1 \pmod{32}$
Case 1b		$\equiv 2 \pmod 4$	$\equiv 1 \pmod 8$
Case 1c		$\equiv 0 \pmod 4$	$gv \geq 8$
Case 2a	$\equiv 2 \pmod 4$	$\equiv 1 \pmod 2$	$\equiv 1 \pmod{16}$
Case 2b		$\equiv 2 \pmod 4$	$\equiv 0, 1 \pmod 4$
Case 2c		$\equiv 0 \pmod 4$	$gv \geq 8$
Case 3a	$\equiv 4 \pmod 8$	$\equiv 1 \pmod 2$	$\equiv 0, 1 \pmod 8$
Case 3b		$\equiv 0 \pmod 2$	$gv \geq 8$
Case 4a	$\equiv 8 \pmod{16}$	$\equiv 1 \pmod 2$	$\equiv 0, 1 \pmod 4$
Case 4b		$\equiv 0 \pmod 2$	$gv \geq 8$
Case 5	$\equiv 0 \pmod{16}$	≥ 1	$gv \geq 8$

Table 1. Necessary conditions for the existence of a $(2 \times 4, \lambda)$-splitting GDD of type g^v

Now we consider the sufficiency.

Case 1a. Apply Construction 2.3 to a $(2 \times 4, 1)$-splitting GDD of type 1^v (see Lemma 2.4).

Case 1b. Apply Construction 2.3 to a $(2 \times 4, 1)$-splitting GDD of type 2^v (see Lemma 3.1).

Case 1c. See Lemma 2.6.

Case 2a. Apply Construction 2.3 to a $(2 \times 4, 2)$-splitting GDD of type 1^v (see Lemma 2.4).

Case 2b. Apply Construction 2.3 to a $(2 \times 4, 2)$-splitting GDD of type 2^v (see Lemmas 3.2 and 3.3).

Case 2c. See Lemma 2.6.

Case 3a. Apply Construction 2.3 to a $(2 \times 4, 4)$-splitting GDD of type 1^v (see Lemma 2.4).

Case 3b. For $g \equiv 0 \pmod 4$, see Lemma 2.6. For $g \equiv 2 \pmod 4$ and $v \equiv 0, 1 \pmod 4$, apply Lemma 2.7 to a $(2 \times 4, 2)$-splitting GDD of type g^v (see Case 2b). For $g \equiv 2 \pmod 4$, $v \equiv 2, 3 \pmod 4$, and $v \geq 6$, apply Construction 2.3 to a $(2 \times 4, 4)$-splitting GDD of type 2^v (see Lemmas 3.6 and 3.7). For $g \equiv 2 \pmod 4$ and $v = 2, 3$, apply Lemma 2.7 to a $(2 \times 4, 4)$-splitting GDD of type g^v (see Lemmas 3.4 and 3.5).

Case 4a. For $v \geq 8$, apply Construction 2.3 to a $(2\times 4, 8)$-splitting GDD of type 1^v (see Lemma 2.4). For $v = 4, 5$, apply Lemma 2.7 to a $(2\times 4, 8)$-splitting GDD of type g^v (see Lemmas 3.8 and 3.9).

Case 4b. Apply Lemma 2.7 to a $(2\times 4, 4)$-splitting GDD of type g^v (see Case 3b).

Case 5. For $g \equiv 1 \pmod 2$ and $v \geq 8$, apply Construction 2.3 to a $(2\times 4, 16)$-splitting GDD of type 1^v (see Lemma 2.4). For $g \equiv 1 \pmod 2$ and $v = 4, 5$, apply Lemma 2.7 to a $(2\times 4, 8)$-splitting GDD of type g^v (see Lemmas 3.8 and 3.9). For $g \equiv 1 \pmod 2$ and $v = 2, 3, 6, 7$, apply Lemma 2.7 to a $(2\times 4, 16)$-splitting GDD of type g^v (see Lemmas 3.10, 3.11, 3.12, and 3.13). For $g \equiv 0 \pmod 2$, apply Lemma 2.7 to a $(2\times 4, 8)$-splitting GDD of type g^v (see Case 4b).

It is obvious that there does not exist a $(2\times 4, \lambda)$-splitting GDD of type 3^3. This completes the proof. □

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