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Large Cayley digraphs of given degree and diameter from sharply t-transitive groups

Mária Ždímalová   Marián Olejár

Department of Mathematics, Faculty of Civil Engineering
Slovak University of Technology
Radlinského 11, 813 68
Bratislava
zdimal@math.sk   olejar@math.sk

Abstract

We show that the large Cayley digraphs of certain given out-degree and diameter resulting from a construction of Faber, Moore and Chen (1993) can be obtained directly from the definition of t-transitive permutation groups, with no reference to the structure and classification of such groups.

1 Introduction

The directed vertex-transitive and Cayley versions of the degree-diameter problems are to determine the largest order of a vertex-transitive and a Cayley digraph, respectively, with a given out-degree and a given diameter. For background, history, and current state of the degree-diameter problem in general we refer to the latest survey paper by Miller and Širáň [11]; here we just sum up the most important facts related to the two special versions of the problem.

The current largest orders of vertex-transitive and Cayley digraphs of degree $\Delta \leq 13$ and diameter $D \leq 11$ are listed in the on-line tables compiled by E. Loz; see [9]. The same source contains references and further particulars about the corresponding digraphs. For large Δ and D the theory has been dominated by three constructions. The first is due to Faber, Moore and Chen [5] and yields vertex-transitive digraphs of order $(\Delta+1)!/(\Delta+1-D)!$, where the two parameters Δ and D satisfy the inequalities $2 \leq D \leq \Delta$. The second construction, discovered by Comellas and Fiol [2] and later improved in special cases by Gómez [8], produces a 'large' vertex-transitive digraph from certain 'small' input digraph. Since the order of the output digraph depends on properties of the input digraph and on two further parameters, a direct comparison with the Faber-Moore-Chen digraphs is not possible and therefore we do not state any formula here. The third record construction, discovered by Gómez [7], yields vertex-transitive digraphs of order $N = (\Delta + \lfloor D - 1/2 \rfloor)!/(\Delta - \lceil D + 1/2 \rceil)!$ for each

$D \geq 3$ and $\Delta \geq \lceil \frac{D+1}{2} \rceil$. All three constructions, although being the best available, give orders that are very far from the theoretically largest such order which, by [1, 12], is $\Delta + \Delta^2 + \ldots + \Delta^D$.

A particularly important subclass of vertex-transitive digraphs are Cayley digraphs, those containing a group of automorphisms acting regularly on the vertex set. Nevertheless, explicit constructions of large Cayley digraphs of given (general) out-degree and diameter have not been considered, perhaps with the exception of Cayley graphs of Abelian groups [4, 6]. To the best of our knowledge, no result to the extent of which of the large vertex-transitive digraphs described above admit a regular action of a group of automorphism on vertices has appeared in the literature prior to submission of this article. Investigation in this direction was initiated by the first author [14] with a characterization of the Faber-Moore-Chen digraphs that are Cayley digraphs. This characterization involves classification of sharply t-transitive groups of a given degree, details of which can be found in [3, 13].

The purpose of this article is to show that the large Cayley digraphs of given degree and diameter described in [14] can be constructed directly from the 'first principles', that is, just with the help of the definition of sharply t-transitive groups for $t \geq 2$, invoking neither any results from the theory of such groups nor any prior knowledge of the Faber-Moore-Chen digraphs.

2 Results

We begin with the case $t = 2$.

Proposition 1 *Let G be a sharply 2-transitive permutation group on a set S with $|S| \geq 2$. Then, there exists a Cayley digraph of the group G of degree $|S| - 1$ and diameter 2.*

Proof. Let us fix two distinct elements $a, b \in S$. Consider the set $X = \{g \in G;\ g(a) = b\}$. The sharp 2-transitivity of G on S implies that $|X| = |S| - 1$. We show that the Cayley digraph $C(G, X)$ has diameter 2. This is equivalent to proving that for any permutation $f \in G$ such that $f(a) = x$ and $f(b) = y$, where $x \neq b, y$ and $(x, y) \neq (a, b)$, there exist $g, h \in X$ such that $f = gh$, the composition being read from the right to the left.

Let f be as above; in what follows we will be tacitly using the fact that G is sharply 2-transitive a number of times. Since $x \neq b$, there is a unique $g \in X$ such that $g(b) = x$. From $x \neq y$ we have $b = g^{-1}(x) \neq g^{-1}(y)$, which implies that there exists a unique $h \in X$ such that $h(b) = g^{-1}(y)$. With these g and h we have $gh(a) = g(b) = x$ and $gh(b) = g(g^{-1}(y)) = y$. But then both gh and f map the ordered pair (a, b) to the ordered pair (x, y) and hence $f = gh$, as required. □

For diameter 2 the only 2-sharply transitive groups are the groups of linear transformations $\{x \longrightarrow ax + b : a, b \in F, a \neq 0\}$ of a finite nearfield F of order $q = p^m$, where p is a prime [13, 3].

We proceed with the case $t = 3$.

Proposition 2 *Let G be a sharply 3-transitive permutation group on a set S with $|S| \geq 3$. Then, there exists a Cayley digraph of the group G of degree $|S| - 1$ and diameter 3.*

Proof. Let us now fix three distinct elements $a, b, c \in S$. This time, consider the set $X = \{g \in G;\ g(a) = b,\ g(b) = c\} \cup \{l \in G; l(a) = a,\ l(b) = c,\ l(c) = b\}$. By the sharp 3-transitivity of G on S we have $|X| = |S| - 2$. We show that the Cayley digraph $C(G, X)$ has diameter 3. This amounts to showing that any non-identity element of G which is not in X can be expressed as a composition of at most three permutations from X.

In what follows, let $f \in G$ be such that $f(a) = x$, $f(b) = y$, and $f(c) = z$ for three *distinct* $x, y, z \in S$, where $(x, y, z) \neq (a, b, c)$. Equivalently, this means that $f \neq id$ and $f \notin X$. We split the analysis into a number of steps and in each we will be making use of the sharp 3-transitivity of G a number of times without explicit mentioning.

Assume first that $x \neq b, c$ and $y \neq c, x$. The first condition implies the existence of a unique $g \in X$ such that $g(c) = x$. From the second condition we have $g^{-1}(y) \neq g^{-1}(c), g^{-1}(x)$, that is, $g^{-1}(y) \neq b, c$. It follows that there is a unique $h \in X$ such that $h(c) = g^{-1}(y)$. Since $z \notin \{x, y\} = \{gh(b), gh(c)\}$, we have $(gh)^{-1}(z) \neq b, c$. Therefore, there is a unique $k \in X$ such that $k(c) = (gh)^{-1}(z)$. It is now easy to check that $ghk(a) = x$, $ghk(b) = y$, and $ghk(c) = z$. By the sharp 3-transitivity of G we conclude that $f = ghk$.

We now turn out the remaining cases. If $x = b$, then $f(a) = b$ and we necessarily have $f(b) = y \neq b, c$ and $f(c) = z \neq y, b$. In the construction of the above paragraph, take $g = l$ and keep h and k unchanged, that is, defined by the conditions $h(c) = l^{-1}(y)$ and $k(c) = (lh)^{-1}(z)$. One can check that our conditions guarantee that h and k are well defined elements of X, and $lhk(a) = lh(b) = l(c) = b$, $lhk(b) = lh(c) = y$, $lhk(c) = z$, that is, $f = lhk$.

The case $x = c$ leads to $f(a) = c$, $f(b) = y \neq c$, and $f(c) = z \neq y, c$. Assume first that $z \neq b$. Then, there is a unique $g \in X$ such that $g(c) = z$. From $g(c) = z \neq y \neq c$ we have $g^{-1}(y) \neq c, b$, which shows that there is a unique $h \in X$ such that $h(c) = g^{-1}(y)$. This shows that $ghl(a) = gh(a) = g(b) = c$, $ghl(b) = gh(c) = y$, and $ghl(c) = gh(b) = g(c) = z$, concluding that $f = ghl$. If $z = b$, we are looking at f such that $f(a) = c$, $f(b) = y \neq c, b$, and $f(c) = b$. If $y = a$, we take $g \in X$ such that $g(c) = a$ and $h \in X$ such that $h(c) = g^{-1}(b)$; these are well defined and give $f = gh$. If $y \neq a$, then take $g \in X$ such that $g(c) = l^{-1}(y) \neq b, c$; it is then easy to check that $f = lgl$.

It remains to handle the case $y = c$, that is, when $f(b) = c$, $f(a) = x \neq b, c$, and $f(c) = z \neq c$. Again, take the unique $g \in X$ for which $g(c) = x$ and the unique $k \in X$ such that $k(c) = (gl)^{-1}(z)$. One may again check that g and k are well defined. Now, $glk(a) = gl(b) = g(c) = x$, $glk(b) = gl(c) = g(b) = c$, and $glk(c) = z$, giving $f = glk$.

This shows that the diameter of our digraph is equal to 3 as required. □

The most familiar examples of sharply 3-transitive groups are $PGL_2(q)$, where $q = p^m$ is a prime power. For a complete classification see [3], which also includes the group $P\Delta L_2(q)$ if m is even.

Similar results can also be proved for $t = 4$ and $t = 5$ with careful choices of generating sets. In these two cases, however, there are just two sharply t-transitive groups; namely, the Mathieu group M_{11} for $t = 4$ and the Mathieu group M_{12} for $t = 5$. We therefore omit the details in these two exceptional cases.

3 Isomorphism with the Faber-Moore-Chen digraphs

In this section we show that the digraphs constructed in Propositions 1 and 2 are isomorphic to the Faber-Moore-Chen digraphs for corresponding degree and diameter. The Faber-Moore-Chen digraphs $\Gamma(\Delta, k)$ of degree Δ and diameter k can be described as follows. Vertices of $\Gamma(\Delta, k)$ are k-permutations $x_1x_2\dots x_k$ of an alphabet A where $|A| = \Delta + 1$. Adjacencies are given by:

$$x_1x_2\dots x_k \to \begin{cases} x_2x_3\dots x_kx_{k+1}, & x_{k+1} \neq x_1, x_2, \dots, x_k \\ x_1\dots\widehat{x_i}\dots x_kx_i \end{cases}$$

where $\widehat{x_i}$ denotes omission of the symbol x_i. These digraphs have order $(\Delta+1)_k = (\Delta+1)!/(\Delta-k+1)!$, diameter k and are Δ-regular, meaning that the in- and out- degree of every vertex are equal to Δ. Obviously, any permutation of the set A induces an automorphism of $\Gamma(\Delta, k)$, and hence Faber-Moore-Chen digraphs are vertex-transitive. Note that for $k = 2$ the digraphs $\Gamma(\Delta, 2)$ reduce to the Kautz digraphs [11].

Let the digraphs from Proposition 1 and 2 be denoted Γ_1 and Γ_2. Let $\Gamma(\Delta, 2)$, $\Gamma(\Delta, 3)$ be Faber-Moore-Chen digraphs for diameter 2, and 3. Then for diameter two:

Theorem 1 *The digraph* Γ_1 *is isomorphic with the digraph* $\Gamma(\Delta, 2)$.

Proof. Let Γ_1 be the Cayley digraph from Proposition 1 for a sharply 2-transitive permutation group on a set S with $|S| \geq 2$, with the generating set $X = \{g \in G;\ g(a) = b\}$ where $a, b \in S$ are fixed. Let $\Gamma(\Delta, 2)$ be a Faber-Moore-Chen digraph of diameter 2 whose vertex set V consists of all ordered pairs of type (x_1, x_2), such that $x_1 \neq x_2$ and $x_2 \in A$, where $|A| = \Delta + 1$.

With the help of the fixed pair (a, b) we define a function $j : G \to V$ by letting $j(f) = (f(a), f(b))$. The function j is a bijection, because for any ordered pairs (a, b) and (c, d), where $a \neq b$ and $c \neq d$, there exists a unique permutation $s \in G$ such that $s(a) = c$ and $s(b) = d$.

It remains to show that j preserves arcs.

Given $f, h \in G$, there is an arc in Γ_1 from f to h if and only if $h = f \circ g$ for some $g \in X$. Consider the images $j(f) = (f(a), f(b))$ and $j(h) = (h(a), h(b))$ of the two vertices in $\Gamma(\Delta, 2)$. Observe that $f(b) = f(g(a)) = h(a)$ and $h(b) = f(g(b)) \neq f(b)$. By the adjacency rule for $\Gamma(\Delta, 2)$ we see that there is an arc from $(f(a), f(b))$ to $(h(a), h(b))$.

This proves that j is an isomorphism of the digraphs Γ_1 and $\Gamma(\Delta, 2)$.

□

We have a similar result for diameter 3, with a slightly longer proof.

Theorem 2 *The digraph* Γ_2 *is isomorphic with the digraph* $\Gamma(\Delta, 3)$.

Proof. Let Γ_2 be the Cayley digraph from Proposition 2 for a sharply 3-transitive permutation group on a set S with $|S| \geq 2$, with the generating set $X = X' \cup \{l\}$ where $X' = \{g \in G;\ g(a) = b,\ g(b) = c\}$ and $l \in G$ is determined by $l(a) = a,\ l(b) = c,\ l(c) = b$, where $a, b, c \in S$ are fixed.

With the help of the fixed triple (a, b, c) we define a function $j : G \to V$ by letting $j(f) = (f(a), f(b), f(c))$. The function j is a bijection, because for any ordered triples (a, b, c) and (e, f, g), where $a \neq b \neq c$ and $e \neq f \neq g$ there exists a unique permutation $s \in G$ such that $s(a) = e, s(b) = f$, and $s(c) = g$.

We show that j preserves arcs. Consider an arc in Γ_2 from f to h, which exists if and only if $h = f \circ g$ for some $g \in X$. We will distinguish two cases: $g \in X_1$ and $g = l$.

If $g \in X_1$, there is an arc in Γ_2 from f to h if and only if $h = f \circ g$. Consider the images $j(f) = (f(a), f(b), f(c))$, $j(h) = (h(a), h(b), h(c))$ of the two vertices in $\Gamma(\Delta, 3)$. Observe that $h(a) = fg(a) = f(b)$, $h(b) = fg(b) = f(c)$ and $h(c) = fg(c) \neq f(b), f(c)$. By the adjacency rule for $\Gamma(\Delta, 2)$ we see that there is an arc from $(f(a), f(b), f(c))$ to $(h(a), h(b), h(c))$.

If $g = l$, there is an arc in Γ_2 from f to h, if and only if $h = f \circ g$. Now consider the images $j(f) = (f(a), f(b), f(c))$, $j(k) = (k(a), k(b), k(c))$ of the two vertices in $\Gamma(\Delta, 3)$. Observe $k(a) = fl(a) = f(a)$, $k(b) = fl(b) = f(c)$ and $k(c) = fl(c) = f(b)$. We can again see that there is an arc from $(f(a), f(b), f(c))$ to $(k(a), k(b), k(c))$ in $\Gamma(\Delta, 2)$.

This proves that j is an isomorphism of the digraphs Γ_2 and $\Gamma(\Delta, 3)$.

□

4 Remarks

By ignoring directions of edges in the constructions of Faber-Moore-Chen and Comellas-Fiol, one obtains large vertex-transitive *undirected* graphs of given degree and diameter. In the undirected case there is also the recent construction of Macbeth, Šiagiová, Širáň and Vetrík, giving explicit Cayley graphs for a much wider range of degrees and diameters [10].

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References

[1] W. G. Bridges and S. Toueg, On the impossibility of directed Moore graphs, *J. Combin. Theory Ser. B* 29 (1980), 339–341.

[2] F. Comellas and M. A. Fiol, Vertex-symmetric digraphs with small diameter, *Discrete Appl. Math.* 58 (1995), 1–11.

[3] J. Dixon and B. Mortimer, *Permutation groups*, Springer, New York, (1996).

[4] R. Dougherty and V. Faber, The Degree-Diameter Problem for Several Varieties of Cayley Graphs I: The Abelian Case, *SIAM J. Discr. Math.* 17 (3), (2004), 478–519.

[5] V. Faber, J. W. Moore and W. Y. C. Chen, Cycle prefix digraphs for symmetric interconnection networks, *Networks* 23 (1993), 641–649.

[6] C. Garcia and C. Peyrat, Large Cayley graphs on an abelian group, *Discrete Appl. Math.* 75 No. 2 (1997), 125–133.

[7] J. Gómez, Large vertex-symmetric digraphs, *Networks* 50 (2007), 421–250.

[8] J. Gómez, On large vertex-symmetric digraphs, *Discrete Math.* 309 (6) (2009), 1213–1221.

[9] E. Loz, www.eyal.com.au/wiki/The_Degree/Diameter_Problem

[10] H. Macbeth, J. Šiagiová, J. Širáň and T. Vetrík, Large Cayley graphs and vertex-transitive non-Cayley graphs of given degree and diameter, *J. Graph Theory* (in press).

[11] M. Miller and J. Širáň, Moore graphs and beyond: A survey, *Electron. J. Combin.*, Dynamic Survey DS 14 (2005), 61 pp.

[12] J. Plesník and Š. Znám, Strongly geodetic directed graphs, Acta F.R.N. Univ. Comen.-Mathematica XXIX, (1974), 29-34.

[13] H. Zassenhaus, Über endliche Fastkörper, Abh. Math. Sem. Hamburg 11 (1936), 187-220.

[14] M. Ždímalová, Ľ. Staneková, Which Faber-Moore-Chen digraphs are Cayley digraphs?, submitted to Discrete Mathematics, August 2009.

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