AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 247–264

Orthogonal combings of linear sudoku solutions

John Lorch*

Department of Mathematical Sciences
Ball State University
Muncie, IN 47306-0490
U.S.A.
jlorch@bsu.edu

Abstract

We introduce a transversal method for generating orthogonal mates of sudoku solutions. We show that the method produces an orthogonal mate for every linear sudoku solution of parallel type, and we characterize the linear sudoku solutions of non-parallel type for which the method works.

1 Introduction

1.1 Purpose

The purpose of this paper is to determine the effectiveness of a 'combing' method (first proposed in [5]) for producing orthogonal pairs of sudoku solutions. We work entirely in the setting of linear sudoku solutions.

1.2 Background, results, and structure

Recall that a **latin square** of order n is an $n \times n$ array with entries drawn from n distinct symbols in such a way that no symbol is repeated in any row or column. A **sudoku solution** is a latin square of order n^2 with an additional requirement called the **subsquare condition**: Upon partitioning the array into $n \times n$ **subsquares**, each subsquare must contain every symbol. Two order 4 sudoku solutions are shown below.

0	1	3	2
2	3	1	0
3	2	0	1
1	0	2	3

0	3	2	1
2	1	0	3
3	0	1	2
1	2	3	0

* I extend my gratitude to Lisa Mantini for pointing out [5] and for asking me to think about the connection between combing and orthogonality.

Two latin squares are said to be **orthogonal** if, upon superimposition, each ordered pair of entries occurs exactly once. For example, the latin squares above are orthogonal: There is no repetition of ordered pairs upon superimposition, as is indicated in the array below.

00	13	32	21
22	31	10	03
33	20	01	12
11	02	23	30

Questions about orthogonality for latin squares are classical, dating from Euler's thirty-six officers problem [4]. Research in this area continues through the present day, driven by applications to statistical design (see [2] or [10], for example) and by intriguing, long standing open questions, such as the determination of the best possible lower bound for the maximum size of a family of mutually orthogonal latin squares of order n (see [3] for a good survey).

Latin squares have found a new spotlight due to the recent popularity of sudoku. It is natural to transfer questions about latin squares, including those about orthogonality, to the setting of sudoku. For example, in [6] Solomon Golomb asks about the existence of orthogonal pairs of sudoku solutions, a question that has been answered in the affirmative in the course of a number of articles using various methods, such as [7], [2], and [9].

This article concerns a process for producing orthogonal pairs of sudoku solutions. Given a sudoku solution M partitioned by a set of disjoint transversals, we consider the idea of **combing** the transversals of M into rows while fixing the leftmost column of M (the 'scalp'), thus producing a new array $\tilde{M}$ that we hope is a sudoku solution orthogonal to M (see Figure 1 for an illustration). Our work will be done in the setting of **linear sudoku solutions**, originally defined in [2]: Upon viewing the entire set of cell locations as a finite vector space, a sudoku solution is linear if each symbol occupies a set of locations that can be identified with an affine vector subspace. We identify all linear sudoku solutions M (of square prime power order) that possess a certain set of transversals yielding a combed sudoku solution $\tilde{M}$ that is orthogonal to M (Theorems 3.3, 3.5, 4.2, 4.3, and 4.4).

The structure of the paper is as follows: In Section 2 we define linear sudoku solutions and show that they can be described by 2×2 matrices; much of this section is motivated by the remarkable article [2]. In Section 3 we describe the combing process and characterize the parallel linear sudoku solutions that possess orthogonal mates obtainable by combing. In Section 4 we characterize the non-parallel linear sudoku solutions that can be combed to form an orthogonal mate, and we also briefly discuss the relationship between combing and the **Transversal Theorem**, a famous elementary theorem concerning orthogonal latin squares.

$$M = \left|\begin{array}{ccc|ccc|ccc}
\mathbf{0} & \mathtt{1} & \mathit{2} & 5 & 3 & 4 & 8 & 6 & 7 \\
\mathit{3} & \mathbf{4} & \mathtt{5} & 7 & 8 & 6 & 2 & 0 & 1 \\
\mathtt{6} & \mathit{7} & \mathbf{8} & 1 & 2 & 0 & 4 & 5 & 3 \\ \hline
8 & 6 & 7 & \mathtt{0} & \mathit{1} & \mathbf{2} & 5 & 3 & 4 \\
2 & 0 & 1 & \mathbf{3} & \mathtt{4} & \mathit{5} & 7 & 8 & 6 \\
4 & 5 & 3 & \mathit{6} & \mathbf{7} & \mathtt{8} & 1 & 2 & 0 \\ \hline
5 & 3 & 4 & 8 & 6 & 7 & \mathit{0} & \mathbf{1} & \mathtt{2} \\
7 & 8 & 6 & 2 & 0 & 1 & \mathtt{3} & \mathit{4} & \mathbf{5} \\
1 & 2 & 0 & 4 & 5 & 3 & \mathbf{6} & \mathtt{7} & \mathit{8}
\end{array}\right| \qquad
\tilde{M} = \left|\begin{array}{ccc|ccc|ccc}
\mathbf{0} & \mathbf{4} & \mathbf{8} & \mathbf{3} & \mathbf{7} & \mathbf{2} & \mathbf{6} & \mathbf{1} & \mathbf{5} \\
\mathit{3} & \mathit{7} & \mathit{2} & \mathit{6} & \mathit{1} & \mathit{5} & \mathit{0} & \mathit{4} & \mathit{8} \\
\mathtt{6} & \mathtt{1} & \mathtt{5} & \mathtt{0} & \mathtt{4} & \mathtt{8} & \mathtt{3} & \mathtt{7} & \mathtt{2} \\ \hline
8 & 0 & 3 & 2 & 5 & 7 & 4 & 6 & 1 \\
2 & 5 & 7 & 4 & 6 & 1 & 8 & 0 & 3 \\
4 & 6 & 1 & 8 & 0 & 3 & 2 & 5 & 7 \\ \hline
5 & 8 & 0 & 7 & 2 & 4 & 1 & 3 & 6 \\
7 & 2 & 4 & 1 & 3 & 6 & 5 & 8 & 0 \\
1 & 3 & 6 & 4 & 5 & 8 & 7 & 2 & 4
\end{array}\right|$$

Figure 1: Transversal strands in M are combed to rows in $\tilde{M}$ fixing the leftmost column. Three of a set of nine disjoint transversals are illustrated in bold, italic, and teletype.

2 Linear Sudoku Solutions

In this section we introduce the notion of a linear sudoku solution and put it into context. Our sudoku terminology (e.g., location, large row, small row, subsquare, parallel linear) will follow that of [2]. Throughout q will be a prime power, $\mathbb{F}$ the finite field of order q, and $\mathrm{Gr}(2, \mathbb{F}^4)$ the collection of two-dimensional subspaces (referred to hereafter as 2-planes) of $\mathbb{F}^4$. The special case $q = 3$ corresponds to the familiar setting of 9×9 sudoku solutions. Some of the material below may also be found in [2].

2.1 Describing array locations using a rank-4 vector space

We first observe that the set of locations within an **array** of order q^2 (a $q^2 \times q^2$ grid with each of q^2 distinct symbols appearing q^2 times) can be identified with the vector space $\mathbb{F}^4$ over $\mathbb{F}$. Each location has an address (x_1, x_2, x_3, x_4) (denoted $x_1x_2x_3x_4$ hereafter), where x_1 and x_3 denote the large row and column of the location, respectively, while x_2 and x_4 denote the small row and column of the location, respectively. Rows (both large and small) are labeled in increasing lexicographic order from top to bottom starting from zero, while columns (both large and small) are labeled in increasing lexicographic order from left to right (see Figure 2).

2.2 Linear arrays: parallel and non-parallel

We say that an array is **linear** if the collection of locations housing any given symbol is a coset of some two dimensional vector subspace of $\mathbb{F}^4$. Linear arrays come in two flavors: If every such coset originates from a *single* two dimensional subspace, then the array is of **parallel type**; otherwise the array is of **non-parallel type**.

A linear array of parallel type is characterized (up to labeling) by a single 2-plane g. This is illustrated in Figure 3 in the case $q = 3$ and $g = \langle 1000, 0100 \rangle \subset \mathbb{Z}_3^4$.

A linear array of non-parallel type is characterized (up to relabeling) by

0	1	2	1	2	0	8	6	7
3	4	5	7	8	6	2	0	*
6	7	8	4	5	3	5	3	4
1	2	0	5	3	4	2	0	1
4	5	3	6	7	8	0	1	2
7	8	6	8	6	7	3	4	5
2	0	1	3	4	5	7	8	6
5	3	4	6	7	8	1	2	0
8	6	7	0	1	2	4	5	3

Figure 2: An array of order 9 with asterisk in location with address 0122.

0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8

Figure 3: Linear array of parallel type corresponding to $g = \langle 1000, 0100\rangle$.

- a pair g_1, g_2 of 2-planes with rank-1 intersection, and
- a partition of $\mathbb{F}^4$ by cosets of g_1 and g_2, where we require that each of g_1 and g_2 be used in constructing the partition.

There are $2^q - 2$ such partitions, constructed as follows: Put $Y = \langle g_1, g_2\rangle$ and choose $a_1, \dots, a_{q-1} \in \mathbb{F}^4$ such that $\mathbb{F}^4 = Y \cup (a_1 + Y) \cup \cdots \cup (a_{q-1} + Y)$ is a partition. Each of $Y, a_1 + Y, \dots, a_{q-1} + Y$ can be partitioned by either g_1 or g_2, so there are 2^q ways to partition $\mathbb{F}^4$ via g_1 and g_2, but we discard two partitions on the grounds that they don't use *both* g_1 and g_2. A non-parallel array is illustrated in Figure 4 in case $q = 3$, $g_1 = \langle 1010, 0111\rangle$, and $g_2 = \langle 1010, 0212\rangle$.

Finally, to put this notion of linearity into context we state the following:

Proposition 2.1 *Let M be an array of order q^2.*

(a) M is linear of parallel type if and only if there exists a surjective linear transformation $L : \mathbb{F}^4 \to \mathbb{F}^2$ such that the corresponding array (determined by L(location) = symbol) is a relabeling of M.

(b) If q is prime then M is linear of parallel type if and only if M is a Keedwell linear array[1].

[1] Let K denote the upper left subsquare of an array M, and let α, β permute an array by shifting

0	1	2	8	5	7	4	6	3
3	4	5	2	0	6	7	8	1
6	7	8	1	3	4	5	2	0
4	6	3	0	1	2	8	5	7
7	8	1	3	4	5	2	0	6
5	2	0	6	7	8	1	3	4
8	5	7	4	6	3	0	1	2
2	0	6	7	8	1	3	4	5
1	3	4	5	2	0	6	7	8

Figure 4: Linear array of non-parallel type corresponding to $g_1 = \langle 1010, 0111 \rangle$ and $g_2 = \langle 1010, 0212 \rangle$. The symbols 1, 6, and 5 are located in cosets of g_2; the remaining symbols lie in cosets of g_1. The repeating subsquare pattern occurring in this example is not characteristic of all non-parallel linear arrays.

2.3 From arrays to sudoku solutions

Figures 2 and 3 indicate that linear arrays need not be sudoku solutions. In fact, they need not even be latin squares. To remedy this we present geometric conditions characterizing linear latin squares and linear sudoku solutions.

Lemma 2.2 *A pair of 2-planes in* $\mathrm{Gr}(2, \mathbb{F}^4)$ *intersect trivially if and only if any two cosets of these planes intersect in a single vector in* $\mathbb{F}^4$*.*

Proof. Let $g_1, g_2 \in \mathrm{Gr}(2, \mathbb{F}^4)$ and $v, w \in \mathbb{F}^4$. Suppose that z_1, z_2 are two distinct members of $(v + g_1) \cap (w + g_2)$. Then $z_1 - z_2$ is a *nonzero* member of $g_1 \cap g_2$, so the planes g_1, g_2 intersect nontrivially. On the other hand, suppose $(v+g_1) \cap (w+g_2) = \emptyset$. Then $v - w$ is nonzero and fails to be a member of $g_1 + g_2$. By the dimension formula we conclude that the rank of $g_1 + g_2$ is less than four, so the rank of $g_1 \cap g_2$ is at least one. Conclude again that g_1 and g_2 intersect nontrivially.

Now suppose that g_1, g_2 intersect nontrivially. Then by the dimension formula the rank of $g_1 + g_2$ as a subspace of $\mathbb{F}^4$ is 3 or less. Pick a nonzero vector $w \in \mathbb{F}^4$ not lying in $g_1 + g_2$. Then the cosets $0 + g_1$ and $w + g_2$ have no vector in common, so it is not true that the intersection of any two cosets consists of a single vector in $\mathbb{F}^4$. □

Proposition 2.3 *Let M_g be a linear array of parallel type generated by a 2-plane* $g \in \mathrm{Gr}(2, \mathbb{F}^4)$*.*

(i) M_g is a latin square if and only if g has trivial intersection with both

$$g_c = \langle 1000, 0100 \rangle \text{ and } g_r = \langle 0010, 0001 \rangle.$$

the rows up by one row and the columns left by one column, respectively. The array M is said to be *Keedwell* if each subsquare of M has the form $\alpha^i \beta^j K$ for some $(i, j) \in \mathbb{Z}_q^2$. Locations of subsquares in M can be identified with $\mathbb{Z}_q^2$, and if the mapping $\mathbb{Z}_q^2 \to \mathbb{Z}_q^2$ that carries a subsquare location to the corresponding ordered pair of exponents (i, j) is a $\mathbb{Z}_q$-homomorphism then we say that M is *Keedwell linear*. See [7] and [9] for further information.

(ii) M_g *is a sudoku solution if and only if* g *has trivial intersection with*

$$\begin{aligned} g_c &= \langle 1000, 0100\rangle, \\ g_r &= \langle 0010, 0001\rangle, \ \textit{and} \\ g_{ss} &= \langle 0100, 0001\rangle. \end{aligned}$$

Proof. Assume the conditions in (i). Observe that the cosets of g_c correspond to the columns of an array. Since g and g_c intersect trivially, we know by Lemma 2.2 that any coset of g meets any coset of g_c in a single vector, corresponding to a location in an array. Therefore every coset of g meets the columns of the array in a single location, and, since each coset of g defines the locations of one particular symbol for M_g, we conclude that each symbol of M_g is contained in each column in a single location. Similarly, since g and g_r intersect trivially, each symbol of M_g is contained in each row in a single location. Therefore M_g is a latin square. Conversely, if M_g is a latin square, each coset of g must intersect each coset of g_c, and each coset of g_r, in exactly one vector. It follows from Lemma 2.2 that g has trivial intersection with both g_c and g_r, so the conditions hold.

Assume the conditions in (ii). In addition to the latin square conditions we are adding the condition that each coset of g intersect each subsquare in exactly one location (cosets of g_{ss} correspond to subsquares in an array). Conclude that each symbol of M_g is contained exactly once in each subsquare, so M_g is a sudoku solution. Likewise, if M_g is a sudoku solution then it is a latin square, so g has trivial intersection with both g_c and g_r by part (i). Observe that g must also have trivial intersection with g_{ss} because of the sudoku subsquare condition. □

The ideas in the proof of Proposition 2.3 extend to linear arrays of non-parallel type:

Corollary 2.4 *Let* M_{g_1,g_2} *be a linear array of non-parallel type generated by* 2*-planes* $g_1, g_2 \in \mathrm{Gr}(2, \mathbb{F}^4)$ *and a corresponding partition of* $\mathbb{F}^4$.

(i) M_{g_1,g_2} *is a latin square if and only if both* g_1 *and* g_2 *have trivial intersection with*

$$g_c = \langle 1000, 0100\rangle \ \textit{and} \ g_r = \langle 0010, 0001\rangle.$$

(ii) M_{g_1,g_2} *is a sudoku solution if and only if both* g_1 *and* g_2 *have trivial intersection with*

$$\begin{aligned} g_c &= \langle 1000, 0100\rangle, \\ g_r &= \langle 0010, 0001\rangle, \ \textit{and} \\ g_{ss} &= \langle 0100, 0001\rangle. \end{aligned}$$

We present an example illustrating Proposition 2.3. Put $q = 3$ and $g = \langle 1002, 0212\rangle$. Observe that g has trivial intersection with g_c, g_r, and g_{ss} as given in Proposition 2.3, so we expect g to generate a sudoku solution. We form a linear array of parallel type corresponding to g by placing the the number $0 \leq j \leq 8$ in the locations determined by $0k_j0l_j + g$, where k_jl_j is the base three representation of j. The resulting sudoku solution is shown in Figure 5.

0	1	2	4	5	3	8	6	7
3	4	5	7	8	6	2	0	1
6	7	8	1	2	0	5	3	4
1	2	0	5	3	4	6	7	8
4	5	3	8	6	7	0	1	2
7	8	6	2	0	1	3	4	5
2	0	1	3	4	5	7	8	6
5	3	4	6	7	8	1	2	0
8	6	7	0	1	2	4	5	3

Figure 5: A linear sudoku solution of parallel type generated by $g = \langle 1002, 0212 \rangle$.

2.4 Matrix representations of linear sudoku solutions

In this section we will see that all linear sudoku solutions can be represented by 2×2 matrices. If $A, B \in M^{2\times 2}(\mathbb{F})$ we let $\begin{bmatrix} A \\ B \end{bmatrix}$ denote the subspace of $\mathbb{F}^4$ spanned by the columns of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Also let I denote the 2×2 identity matrix.

Proposition 2.5 *Let $g \in \mathrm{Gr}(2, \mathbb{F}^4)$.*

(a) The 2-plane g generates a linear latin square of parallel type if and only if there exists an invertible 2×2 matrix $C \in M^{2\times 2}(\mathbb{F})$ such that $g = \begin{bmatrix} I \\ C \end{bmatrix}$.

(b) The 2-plane g generates a linear sudoku solution of parallel type if and only if there exists a non-lower triangular matrix C satisfying the conditions of part (a).

Proof. Let $A, B \in M^{2\times 2}(\mathbb{F})$ such that $g = \begin{bmatrix} A \\ B \end{bmatrix}$, and assume that g generates a latin square of parallel type. Then A and B must be invertible to guarantee that g has trivial intersection with both g_r and g_c, respectively (see Proposition 2.3). Therefore

$$g = \begin{bmatrix} A \\ B \end{bmatrix} = \left[\begin{pmatrix} A \\ B \end{pmatrix} A^{-1} \right] = \begin{bmatrix} I \\ BA^{-1} \end{bmatrix},$$

and we choose $C = BA^{-1}$. The matrix C is invertible, and further, if g generates a sudoku solution C also be non-lower triangular or else the second column of $\begin{pmatrix} I \\ C \end{pmatrix}$

will lie in $g \cap g_{ss}$, contradicting the fact that g and g_{ss} must have trivial intersection (Proposition 2.3).

Conversely, given an invertible matrix C, the plane $g = \begin{bmatrix} I \\ C \end{bmatrix}$ satisfies the conditions of Proposition 2.3, so g generates a linear latin square of parallel type. Also, g will generate a sudoku solution if in addition C is non-lower triangular. □

Corollary 2.6 *Let g_1 and g_2 be distinct members of $\mathrm{Gr}(2, \mathbb{F}^4)$ that have nontrivial intersection.*

(a) The 2-planes g_1, g_2 together with an associated partition of $\mathbb{F}^4$ generate a linear latin square of non-parallel type if and only if there exists a pair C_1, C_2 of distinct, invertible 2×2 matrices in $M^{2\times 2}(\mathbb{F})$ such that $C_1 - C_2$ is singular, $g_1 = \begin{bmatrix} I \\ C_1 \end{bmatrix}$, and $g_2 = \begin{bmatrix} I \\ C_2 \end{bmatrix}$.

(b) The 2-planes g_1, g_2 together with an associated partition of $\mathbb{F}^4$ generate a linear sudoku solution of non-parallel type if and only if there exists a pair of non-lower triangular matrices C_1, C_2 satisfying the conditions of part (a).

Proof. We prove part (b). Assume g_1 and g_2 generated a non-parallel sudoku solution. Proposition 2.3 and Corollary 2.4 indicate that each of g_1 and g_2 generates a sudoku solution of parallel type, so by Proposition 2.5 there exist matrices C_1 and C_2 meeting all of the conditions listed in part (b), save the singularity of $C_1 - C_2$. For this singularity, recall that $\langle g_1, g_2 \rangle$ is three dimensional in $\mathbb{F}^4$ so $g_1 \cap g_2$ is non-trivial in $\mathbb{F}^4$. This implies that $C_1 - C_2$ must be singular.

On the other hand, if g_1 and g_2 meet the conditions in part (b), then they meet the conditions in Corollary 2.4, so they generate a sudoku solution of non-parallel type. □

3 Combing and Orthogonality

In this section we introduce a 'combing' method for generating orthogonal mates of linear sudoku solutions. We show that the method produces an orthogonal mate for every linear sudoku solution of parallel type, and we characterize the linear sudoku solutions of non-parallel type for which the method works.

3.1 Orthogonality of linear sudoku solutions

Two arrays are said to be **orthogonal** if, upon superimposition, each ordered pair of symbols occurs exactly once (see Section 1.2). There are simple geometric conditions that characterize orthogonality of linear arrays:

Lemma 3.1 *Let $g, h \in \mathrm{Gr}(2, \mathbb{F}^4)$ and M_g, M_h be linear arrays of parallel type generated g and h, respectively. The two arrays are orthogonal if and only if g and h have trivial intersection.*

Proof. This is much like the proof of Proposition 2.3. Let a, b be any two symbols used in the arrays. Since the arrays are linear of parallel type there are cosets $x + g_1$ and $y + g_2$ (affine 2-dimensional subspaces of $\mathbb{F}^4$) of g_1 and g_2 whose elements define the locations of the symbol a in M_{g_1} and of b in M_{g_2}, respectively. If g_1 and g_2 have trivial intersection we know from Lemma 2.2 that $(x + g_1) \cap (y + g_2)$ consists of a single vector. Therefore there is exactly one location that contains both a in M_{g_1} and b in M_{g_2}, so when the two arrays are superimposed there is precisely one location housing the ordered pair (a, b). Therefore the arrays are orthogonal. Likewise, if the arrays are orthogonal then no two cosets of g_1 and g_2 can meet in anything else but a single vector, or else an ordered pair of symbols (a, b) will either appear more than once or not at all. We conclude from Lemma 2.2 that g_1 and g_2 have trivial intersection. □

Similarly we have

Lemma 3.2 *Let M_{g_1,g_2} and M_{h_1,h_2} be linear arrays of non-parallel type generated by 2-planes g_1, g_2, h_1, h_2 and some associated partitions of $\mathbb{F}^4$. The two arrays are orthogonal if and only if g_i and h_j have trivial intersection for all $i, j \in \{1, 2\}$.*

We emphasize that the result of Lemma 3.2 does not depend on choices of partitions of $\mathbb{F}^4$: one of $2^q - 2$ choices for g_1, g_2, and likewise for h_1, h_2 (see Section 2.2).

3.2 Orthogonal combings: parallel case

To each $Z \in M^{2\times 2}(\mathbb{F})$ we associate a linear transformation $T_Z : \mathbb{F}^4 \to \mathbb{F}^4$, called a **combing transformation**, whose (block) matrix with respect to the standard basis of $\mathbb{F}^4$ is $\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}$. Let M_g be a linear sudoku solution of parallel type. When $M_{T_Z(g)}$ is an orthogonal sudoku mate for M_g we call $M_{T_Z(g)}$ an **orthogonal combing** of M_g. We will show that each linear sudoku solution of parallel type possesses an orthogonal combing.

Before proceeding with our argument we briefly provide motivation for the 'combing' terminology used above. Imagine a linear sudoku solution M_g of parallel type and $h \subset \mathrm{Gr}(2, \mathbb{F}^4)$ that has trivial intersection with g_c. Cosets of h meet cosets of g_c in a single vector, so it is possible to permute the columns of M so that locations in h are taken by these column permutations to locations in g_r. In general cosets of h (strands of hair) are taken to cosets of g_r by these permutations (the 'comb') in such a way that g_c remains fixed (the scalp). This process yields a new array $\tilde{M}_g$. We would like to determine whether h can be chosen so that $\tilde{M}_g$ is both a

$$M_g = \begin{array}{|ccc|ccc|ccc|}\hline \mathbf{0} & 1 & 2 & 4 & 5 & 3 & 8 & \mathit{6} & 7 \\ \mathit{3} & 4 & 5 & 7 & 8 & \mathbf{6} & 2 & 0 & 1 \\ 6 & 7 & 8 & 1 & 2 & \mathit{0} & 5 & \mathbf{3} & 4 \\ \hline 1 & \mathbf{2} & 0 & 5 & 3 & 4 & 6 & 7 & \mathit{8} \\ 4 & \mathit{5} & 3 & \mathbf{8} & 6 & 7 & 0 & 1 & 2 \\ 7 & 8 & 6 & \mathit{2} & 0 & 1 & 3 & 4 & \mathbf{5} \\ \hline 2 & 0 & \mathbf{1} & 3 & 4 & 5 & \mathit{7} & 8 & 6 \\ 5 & 3 & \mathit{4} & 6 & \mathbf{7} & 8 & 1 & 2 & 0 \\ 8 & 6 & 7 & 0 & \mathit{1} & 2 & \mathbf{4} & 5 & 3 \\ \hline \end{array} \qquad \tilde{M}_g = \begin{array}{|ccc|ccc|ccc|}\hline \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{8} & \mathbf{7} & \mathbf{6} & \mathbf{4} & \mathbf{3} & \mathbf{5} \\ \mathit{3} & \mathit{5} & \mathit{4} & \mathit{2} & \mathit{1} & \mathit{0} & \mathit{7} & \mathit{6} & \mathit{8} \\ 6 & 8 & 7 & 5 & 4 & 3 & 1 & 0 & 2 \\ \hline 1 & 0 & 2 & 6 & 8 & 7 & 5 & 4 & 3 \\ 4 & 3 & 5 & 0 & 2 & 1 & 8 & 7 & 6 \\ 7 & 6 & 8 & 3 & 5 & 4 & 2 & 1 & 0 \\ \hline 2 & 1 & 0 & 7 & 6 & 8 & 3 & 5 & 4 \\ 5 & 4 & 3 & 1 & 0 & 2 & 6 & 8 & 7 \\ 8 & 7 & 6 & 4 & 3 & 5 & 0 & 2 & 1 \\ \hline \end{array}$$

Figure 6: An orthogonal combing of a parallel linear sudoku solution: Transversal strands in M_g are combed to rows in $\tilde{M}_g$ fixing the leftmost column (the scalp). Here $g = \langle 1002, 0212\rangle$ and the transversals are cosets of $h = \langle 1110, 1001\rangle$. The subspace h and one of its cosets (original in M_g and combed in $\tilde{M}_g$) are illustrated in bold and italic, respectively.

sudoku solution and orthogonal to M_g. For a given h observe that $\tilde{M}_g = M_{T_Z(g)}$ up to relabeling, where Z is chosen so that $h = T_Z^{-1}(g_r)$ (see Figure 6).

This 'linear' combing is a specific instance of a more general combing scheme proposed by Fincher [5], wherein a sudoku solution (not necessarily linear) possessing a collection of disjoint transversals is combed preserving the left-most column of locations.

Theorem 3.3 *If M_g is a linear sudoku solution of parallel type then there is an orthogonal combing of M_g.*

Proof. We first consider the case $q > 2$. Let $\lambda \in \mathbb{F}$ with $\lambda(\lambda+1) \neq 0$. Put $g = \begin{bmatrix} I \\ C \end{bmatrix}$ as in Proposition 2.5 . We claim $M_{T_{\lambda C^{-1}}(g)}$ is an orthogonal combing of M_g.
Calculate

$$T_{\lambda C^{-1}}(g) = \left[\begin{pmatrix} I & \lambda C^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} I \\ C \end{pmatrix}\right] = \begin{bmatrix} (\lambda+1)I \\ C \end{bmatrix} = \begin{bmatrix} I \\ (\lambda+1)^{-1}C \end{bmatrix}.$$

Since C satisfies the conditions in part (b) of Proposition 2.5, so does $(\lambda+1)^{-1}C$. It follows from Proposition 2.5 that $M_{T_{\lambda C^{-1}}(g)}$ is a sudoku solution.

Observe further that

$$\begin{bmatrix} I \\ (\lambda+1)^{-1}C \end{bmatrix} = \begin{bmatrix} (\lambda+1)C^{-1} \\ I \end{bmatrix}$$

and that

$$\det\begin{pmatrix} (\lambda+1)C^{-1} & I \\ I & C \end{pmatrix} = \det((\lambda+1)I - I) = \det(\lambda I) \neq 0. \tag{1}$$

Since the leftmost determinant in (1) is nonzero we know that g and $T_{\lambda C^{-1}}(g)$ have trivial intersection, so M_g and $M_{T_{\lambda C^{-1}}(g)}$ are orthogonal by Lemma 3.1.

For the case $q = 2$, observe that if $C \in M^{2\times 2}(\mathbb{F})$ satisfies the conditions of Proposition 2.5 then

$$C \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

One can use Proposition 2.5 and Lemma 3.1 (much as we did above for $q > 2$) to check that $M_{T_Z(g)}$ is an orthogonal combing of M_g, where

$$Z = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. □

3.3 Orthogonal combings: non-parallel case

Let M_{g_1,g_2} be a linear sudoku solution of non-parallel type generated by $g_1, g_2 \in \mathrm{Gr}(2, \mathbb{F}^4)$ and one of the $2^q - 2$ associated partitions of $\mathbb{F}^4$, and let T_Z be a combing transformation as defined in Section 3.2. Whenever $M_{T_Z(g_1),T_Z(g_2)}$ (using any one of the $2^q - 2$ partitions of $\mathbb{F}^4$ determined by $T_Z(g_1), T_Z(g_2)$) is an orthogonal sudoku mate for M_{g_1,g_2} we call $M_{T_Z(g_1),T_Z(g_2)}$ an **orthogonal combing** of M_{g_1,g_2} (see Figure 7). In this section we begin to determine which non-parallel sudoku solutions possess orthogonal combings.

$M_{g_1,g_2} =$

0	1	2	*5*	6	7	4	8	3
3	4	5	8	0	1	**7**	2	6
6	7	8	**2**	3	4	*1*	5	0
4	*8*	3	0	1	2	5	**6**	7
7	2	6	3	**4**	5	8	*0*	1
1	**5**	0	6	*7*	8	2	3	4
5	6	7	4	8	**3**	0	1	*2*
8	0	**1**	7	2	*6*	3	4	5
2	3	*4*	1	5	0	6	7	**8**

$M_{T_Z(g_1),T_Z(g_2)} =$

0	**5**	**1**	**2**	**4**	**3**	**7**	**6**	**8**
3	*8*	*4*	*5*	*7*	*6*	*1*	*0*	*2*
6	2	7	8	1	0	4	3	5
4	3	5	6	2	7	8	1	0
7	6	8	0	5	1	2	4	3
1	0	2	3	8	4	5	7	6
5	7	6	1	0	2	3	8	4
8	1	0	4	3	5	6	2	7
2	4	3	7	6	8	0	5	1

Figure 7: An orthogonal combing of a non-parallel linear sudoku solution: The process is identical to that for linear parallel sudoku solutions. Here $g_1 = \langle 1010, 0111 \rangle$, $g_2 = \langle 1010, 0122 \rangle$, $Z = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, and transversals illustrated in the leftmost figure are cosets of $T_Z^{-1}(g_r) = \langle 0210, 1201 \rangle$.

Before stating results, we compile conditions on Z that will ensure an orthogonal combing of M_{g_1,g_2}. If $g_1 = \begin{bmatrix} I \\ C_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} I \\ C_2 \end{bmatrix}$, then for $i \in \{1, 2\}$ we have

$$T_Z(g_i) = \left[\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ C_i \end{pmatrix} \right] = \begin{bmatrix} I + ZC_i \\ C_i \end{bmatrix} = \begin{bmatrix} C_i^{-1} + Z \\ I \end{bmatrix}. \quad (2)$$

In consideration of Equation (2), Corollary 2.6, and Lemma 3.2 we see:

Lemma 3.4 *A matrix $Z \in M^{2\times 2}(\mathbb{F})$ yields an orthogonal combing for M_{g_1,g_2} if and only if*

(i) $C_i^{-1} + Z$ *is nonsingular,*

(ii) $C_i^{-1} + Z$ *is not lower triangular, and*

(iii) $\det \begin{pmatrix} C_i^{-1} + Z & I \\ I & C_j \end{pmatrix} \neq 0 \quad$ (*equivalently* $\det(C_i^{-1} - C_j^{-1} + Z) \neq 0$)

for all $i, j \in \{1, 2\}$.

Theorem 3.5 *If $q \geq 8$ then every linear non-parallel sudoku solution of order q^2 possesses an orthogonal combing.*

Proof. Let M_{g_1,g_2} be a linear non-parallel sudoku solution with $g_1 = \begin{bmatrix} I \\ C_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} I \\ C_2 \end{bmatrix}$. Note C_1^{-1} and C_2^{-1} have at most two nonzero eigenvalues in $\mathbb{F}$. Further observe $C_1 - C_2$ is singular by Corollary 2.6, hence $C_1^{-1} - C_2^{-1}$ is singular and thus possesses at most one nonzero eigenvalue in $\mathbb{F}$. All together the matrices in the set $S = \{-C_1^{-1}, -C_2^{-1}, \pm(C_1^{-1} - C_2^{-1})\}$ possess at most six nonzero eigenvalues in $\mathbb{F}$. Since $q \geq 8$, there exists a nonzero $\lambda \in \mathbb{F}$ which is not an eigenvalue for any of the matrices in S. It follows that $Z = \lambda I$ satisfies the conditions of Lemma 3.4 so there is an orthogonal combing of M_{g_1,g_2}. □

The following result is useful for constructing orthogonal combings without having to compute eigenvalues.

Theorem 3.6 *Suppose M_{g_1,g_2} is a non-parallel linear sudoku solution of order q^2 with $g_1 = \begin{bmatrix} I \\ C_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} I \\ C_2 \end{bmatrix}$, and put $Z = -C_1^{-1} - C_2^{-1}$. If $\mathbb{F}$ is not of characteristic 2 (i.e., q is not a power of 2) and if $C_1 + C_2$ is invertible in $M^{2\times 2}(\mathbb{F})$ then $M_{T_Z(g_1),T_Z(g_2)}$ is an orthogonal combing for M_{g_1,g_2}.*

Proof. It suffices to check that $Z = -C_1^{-1} - C_2^{-1}$ meets the conditions of Lemma 3.4. Note $C_1^{-1} + Z = -C_2^{-1}$ while $C_2^{-1} + Z = -C_1^{-1}$; both are nonsingular and non-lower triangular, so conditions (i) and (ii) are satisfied. For condition (iii), observe that $C_i^{-1} - B_j^{-1} + Z \in \{Z, -2C_2^{-1}, -2C_1^{-1}\}$. By hypothesis and Corollary 2.6 all three of these matrices are nonsingular. Thus all of the conditions of Lemma 3.4 are satisfied. □

4 Combing Non-parallel Sudoku Solutions over Small Fields

If $C_1, C_2 \in M^{2\times 2}(\mathbb{F})$ satisfy the conditions of Corollary 2.6 part (b), the collection of open conditions in Lemma 3.4 would clearly be satisfied for some $Z \in M^{2\times 2}(\mathbb{F})$ if $\mathbb{F}$ were a 'large' field like $\mathbb{R}$ or $\mathbb{C}$. We therefore expect that the conditions are satisfied provided that q is sufficiently large; this is borne out in Theorem 3.5. For smaller fields the situation is a bit more difficult. Given $q \leq 7$ we determine which non-parallel linear sudoku solutions of order q^2 possess orthogonal combings.

4.1 Group actions

We begin by recalling some matrix group structure theory and introducing notation. Let $\mathrm{GL}(2, q)$ denote the invertible elements of $M^{2\times 2}(\mathbb{F})$, $\mathcal{B}$ the lower triangular elements of $\mathrm{GL}(2, q)$, and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By the Bruhat decomposition (see [1]) we have $\mathrm{GL}(2, q) = \mathcal{B} \cup \mathcal{B}w\mathcal{B}$ as a disjoint union. Observe that $\mathcal{B}w\mathcal{B}$ is precisely the set of matrices satisfying the conditions in part (b) of Proposition 2.5 and that $\mathcal{B} \times \mathcal{B}$ acts transitively on $\mathcal{B}w\mathcal{B}$ by

$$(b_1, b_2).b_3 w b_4 = b_1 b_3 w b_4 b_2^{-1}, \tag{3}$$

with $b_1, b_2, b_3, b_4 \in \mathcal{B}$. Likewise $\mathcal{B} \times \mathcal{B}$ acts on $\mathcal{B}w\mathcal{B} \times \mathcal{B}w\mathcal{B}$ diagonally according to (3). Finally let

- $\mathcal{X}_{\mathrm{np}}$ denote the set of pairs $(C_1, C_2) \in \mathcal{B}w\mathcal{B} \times \mathcal{B}w\mathcal{B}$ that satisfy the conditions of Corollary 2.6 part (b), and let
- $\mathcal{X}_{\mathrm{orth}}$ denote the set of pairs $(C_1, C_2) \in \mathcal{X}_{\mathrm{np}}$ that possess a $Z \in M^{2\times 2}(\mathbb{F})$ satisfying the conditions of Lemma 3.4.

To put this notation into the context of sudoku solutions, let M_{g_1,g_2} be a non-parallel linear array, and suppose $g_1 = \begin{bmatrix} I \\ C_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} I \\ C_2 \end{bmatrix}$. Then M_{g_1,g_2} is a non-parallel sudoku solution if and only if $(C_1, C_2) \in \mathcal{X}_{\mathrm{np}}$, and M_{g_1,g_2} is a non-parallel sudoku solution possessing an orthogonal combing if and only if $(C_1, C_2) \in \mathcal{X}_{\mathrm{orth}}$.

Lemma 4.1 *The action of $\mathcal{B} \times \mathcal{B}$ on $\mathcal{B}w\mathcal{B} \times \mathcal{B}w\mathcal{B}$ restricts to an action of $\mathcal{B} \times \mathcal{B}$ on $\mathcal{X}_{\mathrm{np}}$. The action further restricts to an action on $\mathcal{X}_{\mathrm{orth}}$.*

Proof. Let $(b_1, b_2) \in \mathcal{B} \times \mathcal{B}$ and suppose $(C_1, C_2) \in \mathcal{X}_{\mathrm{np}}$. Appealing to part (b) of Corollary 2.6 and the fact that $\mathcal{B} \times \mathcal{B}$ acts on $\mathcal{B}w\mathcal{B}$, we have $(b_1, b_2).(C_1, C_2) = (b_1 C_1 b_2^{-1}, b_1 C_2 b_2^{-1})$ where $b_1 C_j b_2^{-1}$ is invertible and non-lower triangular for $i \in \{1, 2\}$, and $b_1 C_1 b_2^{-1} - b_1 C_2 b_2^{-1} = b_1(C_1 - C_2) b_2^{-1}$ is singular. We conclude that $(b_1, b_2).(C_1, C_2) \in \mathcal{X}_{\mathrm{np}}$.

Now suppose $(C_1, C_2) \in \mathcal{X}_{\mathrm{orth}}$ and let $Z \in M^{2\times 2}(\mathbb{F})$ satisfy the conditions of Lemma 3.4. We claim that $b_2 Z b_1^{-1}$ satisfies the conditions of Lemma 3.4 for the pair $(b_1 C_1 b_2^{-1},$

$b_1C_2b_2^{-1}$). Again, since $\mathcal{B}\times\mathcal{B}$ acts on $\mathcal{B}w\mathcal{B}$, we have that

$$(b_1C_ib_2^{-1})^{-1}+b_2Zb_1^{-1}=b_2(C_i^{-1}+Z)b_1^{-1}$$

is invertible and non-lower triangular for $i\in\{1,2\}$. Further

$$\det((b_1C_ib_2^{-1})^{-1}-(b_1C_jb_2^{-1})^{-1}++b_2Zb_1^{-1})=\det(b_2)\cdot\det(C_i-C_j+Z)\cdot\det(b_1^{-1})\neq 0.$$

Therefore $(b_1C_1b_2^{-1},b_1C_2b_2^{-1})\in\mathcal{X}_{\text{orth}}$. □

4.2 Non-parallel case: small fields

We now determine which non-parallel linear sudoku solutions of order q^2 possess orthogonal combings in cases where $q\leq 8$. Throughout let M_{g_1,g_2} be a linear sudoku solution of non-parallel type with $g_1=\begin{bmatrix} I \\ C_1\end{bmatrix}$ and $g_2=\begin{bmatrix} I \\ C_2\end{bmatrix}$. We keep in mind that M_{g_1,g_2} possesses an orthogonal combing if and only if $(C_1,C_2)\in\mathcal{X}_{\text{orth}}$.

Theorem 4.2 *If $q=2$ then there is no orthogonal combing of M_{g_1,g_2}.*

Proof. Suppose, for a contradiction, that there is a $Z\in M^{2\times 2}(\mathbb{F}_2)$ satisfying the conditions of Lemma 3.4. Then $h=T_Z^{-1}(g_r)$ (see Section 3.2) and its cosets form a collection of four disjoint transversals of M_{g_1,g_2} (that is, each coset of h meets each row, column, and symbol of M_{g_1,g_2} exactly once). There are two orbits of the sudoku solutions of order 4 under the sudoku group; these orbits are characterized by whether their members possess four disjoint transversals. Order four sudoku solutions of non-parallel type lie in the orbit whose solutions do not possess four disjoint transversals (see [8]), contradicting the existence of the h above. We conclude that there is no $Z\in M^{2\times 2}(\mathbb{F}_2)$ satisfying the conditions of Lemma 3.4, so M_{g_1,g_2} does not possess an orthogonal combing. □

Theorem 4.3 *If $q=3$ then M_{g_1,g_2} possesses an orthogonal combing if and only if C_1-C_2 is not strictly lower-triangular.*

Proof. By Lemma 4.1 and the fact that $\mathcal{B}\times\mathcal{B}$ acts transitively on $\mathcal{B}w\mathcal{B}$, it suffices to determine which pairs $(w,C)\in\mathcal{X}_{\text{np}}$ also lie in $\mathcal{X}_{\text{orth}}$. Recall $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and put $C^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. The following tables indicates which pairs $(w,C)\in\mathcal{X}_{\text{np}}$ actually lie in $\mathcal{X}_{\text{orth}}$. In cases where a pair does lie in $\mathcal{X}_{\text{orth}}$, the table gives a four-tuple $(z_{11},z_{12},z_{21},z_{22})$ indicating the entries of a matrix Z satisfying the conditions of Lemma 3.4. One can check directly that the matrices Z satisfy Lemma 3.4. Rows of the tables are indexed by possible values of b.

	$a \neq 0, d = 0, c = 1$	$a = 0, d \neq 0, c = 1$	$a \neq 0, d = 0, c = 2$	$a = 0, d \neq 0, c = 2$
$b = 1$	$(a, 1, 1, 0)$	$(0, 1, 1, a)$	$(a, 1, 0, a^{-1})$	$(d, 1, 0, d^{-1})$
	$a = d = 0, c = 1$	$a \neq 0, d = 0, c = 1$	$a = 0, d \neq 0, c = 1$	$ad = 2, c = 0$
$b = 2$	$(w, C) \notin \mathcal{X}_{\text{orth}}$	$(a, 0, 1, a)$	$(d, 0, 1, d)$	$(d, 0, 2, a)$

We claim (see table) that when $C^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, the pair (w, C) is the only pair not lying in $\mathcal{X}_{\text{orth}}$ (i.e., the only pair such that a corresponding M_{g_1,g_2} does not admit an orthogonal combing). In this case the conditions of Lemma 3.4 reduce to

$$z_{11}z_{22} \neq 0, z_{11}z_{22} - z_{21} \neq 0, z_{11}z_{22} + z_{21} \neq 0, z_{11}z_{22} - z_{21} - 1 \neq 0, z_{11}z_{22} + z_{21} + 1 \neq 0, \quad (4)$$

and given that $z_{11}z_{22} \neq 0$ there is no choice of z_{21} that will satisfy the rest of the conditions in (4). Further observe that this pair $(w, C) \in \mathcal{X}_{\text{np}}$ is the only pair such that $C - w$ is strictly lower triangular. Since strict lower triangularity is preserved by the action of $\mathcal{B} \times \mathcal{B}$, it follows that a pair $(C_1, C_2) \in \mathcal{X}_{\text{np}}$ is a member of $\mathcal{X}_{\text{orth}}$ if and only if $C_2 - C_1$ is *not* strictly lower triangular. (For an example see Figure 8.) $\square$

0	1	2	6	8	7	3	4	5
3	4	5	0	1	2	6	8	7
6	8	7	3	4	5	0	1	2
2	0	1	8	7	6	5	3	4
5	3	4	2	0	1	8	7	6
8	7	6	5	3	4	2	0	1
1	2	0	7	6	8	4	5	3
4	5	3	1	2	0	7	6	8
7	6	8	4	5	3	1	2	0

Figure 8: A linear sudoku solution of non-parallel type that does not possess an orthogonal combing. Here $g_1 = \langle 1001, 0110 \rangle$ and $g_2 = \langle 1002, 0110 \rangle$, and the resulting $C_1 - C_2$ is strictly lower triangular.

Theorem 4.4 *If $q = 4$, 5, or 7 then M_{g_1,g_2} possesses an orthogonal combing.*

Proof. We first present a proof for $q = 7$. Pick $(C_1, C_2) \in \mathcal{X}_{\text{np}}$. It suffices to show that $\mathcal{X}_{\text{np}} \subset \mathcal{X}_{\text{orth}}$. Since $\mathcal{B} \times \mathcal{B}$ acts transitively on $\mathcal{B}w\mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$ acts on both $\mathcal{X}_{\text{np}}$ and $\mathcal{X}_{\text{orth}}$ (Lemma 4.1) we may assume that C_1 is any member of $\mathcal{B}w\mathcal{B}$ of our choice; we select $C_1 = \begin{pmatrix} 0 & 1 \\ r^{-1} & 0 \end{pmatrix}$ where r is not a square in $\mathbb{F}$. As a consequence of our choice of r we find that $C_1^{-1} = \begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix}$ has no eigenvalues in $\mathbb{F}$. Therefore the collection

$\{-C_1^{-1}, -C_2^{-1}, \pm(C_1^{-1} - C_2^{-1})\}$ has at most four distinct nonzero eigenvalues in $\mathbb{F}$ (see proof of Theorem 3.5). Therefore we may pick a nonzero $\lambda \in \mathbb{F}$ such that $Z = \lambda I$ satisfies Lemma 3.4, and so $(C_1, C_2) \in \mathcal{X}_{\text{orth}}$.

The cases $q = 4, 5$ seem to require painful special cases. We provide summary information below, indicating choices of $Z \in M^{2\times 2}(\mathbb{F})$ that satisfy Lemma 3.4. Observe that if $(C_1, C_2) \in \mathcal{X}_{\text{np}}$ then by Lemma 4.1 we may assume $C_1 = w$. Also we put $C_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For $q = 5$, if $b + c = 1$ then $(w, C_2) \notin \mathcal{X}_{\text{np}}$, so $b + c \in \{0, 2, 3, 4\}$ and

(i) If $b + c = 2$ then $Z = \lambda C_2^{-1}$ works for some $\lambda \in \{2, 3\}$.

(ii) If $b+c = 3$ then $Z = C_2^{-1}$ works provided that $b \neq 4$. If $b = 4$ then put $Z = 2w$.

(iii) If $b+c = 4$ then $Z = 2C_2^{-1}$ works provided that $b \neq 2$. If $b = 2$ then put $Z = w$.

(iv) If $b + c = 0$ then we have the following subcases:

- If $b^2 = 4$ then put $Z = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$.
- If $b^2 = 1$, $a = 0$, and $d \neq 0$ then there exists $\lambda \in \{1, 4\}$ such that $Z = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda - d \end{pmatrix}$ works.
- If $b^2 = 1$, $a \neq 0$, and $d = 0$ then there exists $\lambda \in \{1, 4\}$ such that $Z = \begin{pmatrix} \lambda - a & 0 \\ 0 & \lambda \end{pmatrix}$ works.
- If $b^2 = 1$ and $a = d = 0$ then put $Z = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ when $b = 4$ and put $Z = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ when $b = 1$.

For $q = 4$, the following tables give the field operations in $\mathbb{F}$:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

If $b + c = 1$ then $(w, C_2) \notin \mathcal{X}_{\text{np}}$, so $b + c \in \{0, 2, 3\}$ and

(i) If $b + c = 0$ then $Z = \lambda C_2^{-1}$ works for some $\lambda \in \{2, 3\}$.

(ii) If $b + c = 2$ or $b + c = 3$ then

- If $b, c \neq 1$ then $Z = \begin{pmatrix} 0 & b + c + 1 \\ b + c & 0 \end{pmatrix}$ works.

– If $b = 1$ or $c = 1$ then $C_2^{-1} \in$

$$\left\{\begin{pmatrix}0 & 1\\3 & d\end{pmatrix}, \begin{pmatrix}a & 1\\3 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\2 & d\end{pmatrix}, \begin{pmatrix}a & 1\\2 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\1 & d\end{pmatrix}, \begin{pmatrix}a & 2\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 3\\1 & d\end{pmatrix}, \begin{pmatrix}a & 3\\1 & 0\end{pmatrix}\right\},$$

among which $\begin{pmatrix}0 & 1\\3 & d\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\2 & d\end{pmatrix}$ are representative cases. For the first of these matrices, when $d \neq 2$ there exists $\lambda \in \{2, 3\}$ such that $Z = \lambda I$ works; when $d = 2$ choose $Z = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$. For the second of these matrices, when $d \neq 3$ there exists $\lambda \in \{2, 3\}$ such that $Z = \lambda I$ works; when $d = 3$ choose $Z = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$.

□

Theorem 4.4 combined with Theorem 3.5 indicates:

Corollary 4.5 *If $q \geq 4$ then each non-parallel linear sudoku solution of order q^2 possesses an orthogonal combing.*

4.3 Concluding remarks

Let $q = 3$ and M be a non-parallel linear sudoku solution of order q^2 that does not possess an orthogonal combing (see Theorem 4.3). One can obtain a sudoku solution orthogonal to M by rotating M by 90° counter clockwise and then combing. (Equivalently, one changes the 'scalp' from g_c to g_r.) Allowing for this variation we can conclude that every non-parallel sudoku solution with $q \geq 3$ possesses an orthogonal combing.

The **Transversal Theorem** states that a latin square of order n possesses an orthogonal mate if and only if it possesses n disjoint **transversals**. (A transversal is a collection of locations in which each row, column, and symbol is represented exactly once.) A proof of the theorem generates a method for constructing an orthogonal latin mate—one wipes out every symbol in a given transversal and replaces them with a single symbol. Unfortunately the Transveral Theorem method does not restrict to sudoku because the resulting latin square may fail the subsquare condition. (For example, see M and the choice of transversals given in Figure 1.) However, our results imply that every linear sudoku solution possesses a set of disjoint transversals for which the Transversal Theorem method can be applied successfully to obtain an orthogonal sudoku mate, except for the non-parallel solutions of order 4. (It is not necessary to use the results of this paper to show this.) Further, whether one applies the Transversal Theorem method or the combing method depends on the choice of transversals: combing is the best option for the sudoku solution M and accompanying transversals given in Figure 1, while the Transversal Theorem method is the best option for the sudoku solution given in Figure 8.

References

[1] J.L. Alperin and R.B. Bell, *Groups and Representations*, Springer-Verlag, New York, 1995.

[2] R.A. Bailey, P.J. Cameron, and R. Connelly, *Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes*, Amer. Math. Monthly **115** (2008), 383-403.

[3] C.J. Colbourn and J.H. Dinitz, 'Mutually orthogonal latin squares: a brief survey of constructions', *J. Statist. Plann. Inference* **95** (2001), 9-48.

[4] L. Euler, 'Recherches sur une nouvelle espèce de quarrés magiques', Opera Omnia Series I volume VII, Teubner', Leipzig and Berlin (1923), 291-392.

[5] C. Fincher, *Symmetry and Sudoku*, Senior Honors Thesis, Oklahoma State University, Fall 2006.

[6] S.W. Golomb, 'Problem 11214', in *Amer. Math. Monthly* (Problem section eds: G. Edgar, D. Hensley, and D. West) **113** no. 3 (2006), 268.

[7] A.D. Keedwell, 'On sudoku squares', *Bull. Inst. Combin. Appl.* **50** (2007), 52-60.

[8] J. Lorch and C. Lorch, *Enumerating small sudoku solutions in a first algebra course*, PRIMUS, 2008.

[9] J. Lorch, *Mutually orthogonal families of linear sudoku solutions*, J. Austral. Math. Soc., **87** (2009), 409-420.

[10] F.S. Roberts, 'Applied Combinatorics', Prentice Hall, New Jersey, 1984.

(Received 30 Oct 2009; revised 25 Jan 2010)