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A generalization of a result of Segre on permutable polarities

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Abstract

We generalize a result of Segre on permutable orthogonal and unitary polarities of $PG(2, q^2)$ with q odd, by considering non-desarguesian projective planes of odd square order.

1 Introduction

The starting point of this paper is a result of Segre on permutable polarities of $\Pi = PG(2, q^2)$, q odd, see [8, Chap. IX]: Let Ω be a classical oval (the set of all absolute points of a non-degenerate orthogonal polarity ω) and $\mathcal{U}$ a Hermitian unital (the set of all absolute points of a non-degenerate unitary polarity τ); if ω and τ permute, that is, $\omega\tau = \tau\omega$, then $\Omega \cap \mathcal{U}$ is a Baer suboval of Ω.

In the present paper we prove that this result holds true when Π is any finite projective plane of odd square order admitting orthogonal as well as unitary polarities. Our main result is the following theorem.

Theorem 1. *Let π be a finite projective plane of odd square order q^2. Suppose that π has orthogonal and unitary polarities. Let τ denote a unitary polarity and ω an orthogonal polarity of π. Then, if τ and ω permute, the associated unital $\mathcal{U}$ and oval Ω share exactly $q+1$ points and they have the same tangent at each of their common points. Moreover, their intersection pattern is a Baer suboval, that is, it is an oval Ω_0 of a Baer subplane π_0 such that the common tangents of Ω and $\mathcal{U}$ also lie on π_0, being the tangents to Ω_0.*

Our notation and terminology are standard. For generalities on polarities of projective planes the reader is referred to [4, 7].

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2 Definitions and Preliminary Results

An *oval* Ω in a finite projective plane π of order q is a set of $q+1$ points no three of which are collinear. Any line of π meets Ω in 2, 1 or 0 points and the line is accordingly a *secant* or a *tangent* or an *exterior line* of Ω. A polarity of π is called *orthogonal* if the number of its absolute points is $q+1$; these points are collinear when q is even and form an oval when q is odd.

In the case in which π has square order q^2, a *unital* $\mathcal{U}$ in π is defined to be a set of q^3+1 points such that each line of π contains either 1 or $q+1$ points of $\mathcal{U}$. A line of π is called a *tangent* or a *secant* of $\mathcal{U}$ accordingly as its intersection with $\mathcal{U}$ consists of 1 or $q+1$ points. Through each point of $\mathcal{U}$ there pass q^2 secants and one tangent line, whereas through each point $P \notin \mathcal{U}$ there pass $q+1$ tangents and q^2-q secants. A polarity of a projective plane π of square order q^2 is said to be *unitary* if the number of its absolute points is q^3+1; these points form a unital of π.

A polarity of a finite projective plane is called *regular* if every line containing more than one absolute points contains the same number of them. The following theorem will be very useful in our work; see [6].

Theorem 2. *If a regular polarity of a finite projective plane* Π *of square order induces a polarity of a square-root subplane* Π_0 *of* Π*, then the induced polarity is orthogonal.*

3 Main result

Theorem 1. *Let* π *be a finite projective plane of odd square order* q^2*. Suppose that* π *has orthogonal and unitary polarities. Let* τ *denote a unitary polarity and* ω *an orthogonal polarity of* π*. Then, if* τ *and* ω *permute, the associated unital* $\mathcal{U}$ *and oval* Ω *share exactly* $q+1$ *points and they have the same tangent at each of their common points. Moreover, their intersection pattern is a Baer suboval, that is, it is an oval* Ω_0 *of a Baer subplane* π_0 *such that the common tangents of* Ω *and* $\mathcal{U}$ *also lie on* π_0*, being the tangents to* Ω_0*.*

Proof. Let σ denote the following collineation of π

$$\sigma = \tau\omega = \omega\tau. \tag{1}$$

Since $\tau^2 = 1$ and $\omega^2 = 1$, (1) implies $\sigma^2 = 1$,

$$\tau = \sigma\omega = \omega\sigma \tag{2}$$

and

$$\omega = \sigma\tau = \tau\sigma. \tag{3}$$

Let $Fix(\sigma)$ be the set of all points and lines fixed by σ. We first prove that σ satisfies the following two properties:

(a) the collineation σ leaves both Ω and $\mathcal{U}$ invariant;

(b) the intersection points of Ω and $\mathcal{U}$ are fixed by σ; furthermore

$$\Omega \cap \mathcal{U} = \Omega \cap Fix(\sigma) = \mathcal{U} \cap Fix(\sigma)$$

Choose a point P on Ω; this point P is on its polar line P^ω with respect to ω. Since σ preserves the incidences we have that $P^\sigma \in P^{\omega\sigma}$. By (2) σ and ω are permutable and thus it follows that $P^\sigma \in P^{\sigma\omega}$ that is, $P^\sigma \in \Omega$.

Next, consider a point P on the unital $\mathcal{U}$. In order to prove that $P^\sigma \in \mathcal{U}$ it suffices to replace ω by τ in the proof of $\sigma(\Omega) = \Omega$ and thus property (a) follows.

Now, we are going to prove that property (b) is also satisfied. We start by showing that $\Omega \cap \mathcal{U} \subseteq Fix(\sigma)$. Take a point $P \in \Omega \cap \mathcal{U}$. Since P is on $\mathcal{U}$, the point P lies on its polar line P^τ with respect to τ. As σ preserves the incidences we get $P^\sigma \in P^{\tau\sigma}$ and by (3) $P^\sigma \in P^\omega$. Then

$$P^\sigma \in \Omega \cap P^\omega = \{P\}$$

and so $P \in Fix(\sigma)$.

The next step consists in showing that $\Omega \cap \mathcal{U} = \Omega \cap Fix(\sigma)$. Let $P \in Fix(\sigma)$. Then by (3) we get $P^\omega = P^{\sigma\tau}$, and hence

$$P^\omega = P^\tau, \tag{4}$$

that is, P is on P^ω if and only P lies on P^τ. Therefore a point $P \in Fix(\sigma)$ is on the oval Ω if and only if P lies on the unital $\mathcal{U}$ and the assertion follows. Finally, replacing ω by τ in the previous step we also have that $\Omega \cap \mathcal{U} = \mathcal{U} \cap Fix(\sigma)$ and property (b) is proved. Observe that (4) also implies that Ω and $\mathcal{U}$ have the same tangent at each of their common points.

Now, since σ is a collineation of order 2 and q is odd we have that σ is either a homology or a Baer involution, see [7]. We assert that σ is Baer involution. Assume on the contrary that σ is a homology with axis ℓ and center V. From property (b) the axis ℓ must be tangent to the unital $\mathcal{U}$, say at point L. We are going to show that the point V is on the unital.

Suppose that V does not belong to $\mathcal{U}$ and consider the $q+1$ tangents $\ell_1, \dots \ell_{q+1}$ through V to $\mathcal{U}$. Since each of these $q+1$ tangent lines as well as $\mathcal{U}$ are left invariant by σ it follows that the $q+1$ contact points of $\ell_1, \dots \ell_{q+1}$ with $\mathcal{U}$ are fixed points of σ that do not lie on the axis ℓ, a contradiction. Hence the center V is on the unital $\mathcal{U}$.

Next, consider a line s through V different from VL and from the tangent to $\mathcal{U}$. The line s meets the unital in q points $P_1, \dots, P_q$ other than V. Let us restrict the action of σ to s; then the set of q points $\{P_1, \dots, P_q\}$ should split in $< \sigma >$-orbit of length 2 and this is impossible since q is odd.

Therefore σ is a Baer collineation and its fixed structure $Fix(\sigma)$ is a Baer subplane π_0 of π. The orthogonal polarity ω of π induces a polarity of π_0 and hence by Theorem 2 we have that the size of $\Omega \cap \pi_0$ is $q+1$ and thus from property (b) $\Omega \cap \mathcal{U}$ turns out to be a Baer suboval of Ω. □

4 Example

The square order semifield planes coordinatized by Albert twisted field are examples of non-desarguesian planes admiting orthogonal as well as unitary polarities. They were introduced by Albert; see [1, 2]. Here we adopt the model of a commutative twisted field plane introduced in [3]. Let $GF(q^2)$ be a Galois field of order $q^2 = p^{2n}$, p odd prime, containing a subfield $GF(d)$ such that -1 is not a $(d-1)$-th power in $GF(q^2)$. Let $d = p^s$, and put $r = 2n/s$. Then $r \geq 3$ is odd, and every element $x \in GF(q^2)$ can be uniquely expressed as $x = a^d + a$, with $a \in GF(q^2)$. The commutative Albert semifield $\mathcal{S}(+,*)$ of order q^2 may be obtained from $GF(q^2)$ by replacing the multiplication in $GF(q^2)$ with a new one defined by the rule

$$(a^d + a) * (b^d + b) = a^d b + a b^d.$$

Let $\pi = \mathcal{P}(\mathcal{S})$ be the associated projective plane. Precisely, the points of $\mathcal{P}(\mathcal{S})$ are

$$\{(c,d) | c, d \in \mathcal{S}\} \cup \{(m) | m \in \mathcal{S}\} \cup \{(\infty)\}$$

where ∞ is some formal symbol which is not in $\mathcal{S}$. The lines of $\mathcal{P}(\mathcal{S})$ are

$$[m,k] = \{(m)\} \cup \{(c,d) | c, d \in \mathcal{S}, m * c + k = d\}$$

$$[k] = \{(\infty)\} \cup \{(k,d) | d \in \mathcal{S}\}$$

$$[\infty] = \{(\infty)\} \cup \{(m) | m \in \mathcal{S}\}.$$

Let $\theta : x \to x^q$ be the involutory automorphism of $\mathcal{S}(+,*)$ Then, the map ω described as follows

$$(\infty) \leftrightarrow [\infty], \qquad (m)^\omega = [m], \qquad [k]^\omega = (k), \tag{5}$$

$$(c,d)^\omega = [c, -d], \qquad [m,k]^\omega = (m, -k) \tag{6}$$

is an orthogonal polarity of π; see [4]. The set of absolute points of ω form the oval Ω, that is

$$\Omega = \{(t, \delta(t * t)) : t \in \mathcal{S}\} \cup \{\infty\},$$

where the constant δ is the inverse of 2 in $GF(q)$. According to [5, Theorem 8] the following map τ

$$(\infty) \leftrightarrow [\infty], \qquad (m)^\tau = [m^\theta], \qquad [k]^\tau = (k^\theta), \tag{7}$$

$$(c,d)^\tau = [c^\theta, -d^\theta], \qquad [m,k]^\tau = (m^\theta, -k^\theta) \tag{8}$$

is a unitary polarity of π whose absolute points form the unital $\mathcal{U}$, that is

$$\mathcal{U} = \{(u,v) : v + v^\theta - u^\theta * u = 0, u, v \in \mathcal{S}\} \cup \{(\infty)\}.$$

It is simple to verify that ω and τ permute; the collineation $\sigma = \omega\tau = \tau\omega$ turns the point (u,v) into the point (u^q, v^q). Then Ω and $\mathcal{U}$ share exactly $q+1$ points and $\Omega_0 = \Omega \cap \mathcal{U}$ is a Baer suboval of Ω, precisely

$$\Omega_0 = \{(u,v) : 2v = u * u, u, v \in GF(q)\}.$$

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