AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **49** (2011), Pages 39–45

The spectrum of generalized Petersen graphs

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Abstract

In this paper, we completely describe the spectrum of the generalized Petersen graph $P(n, k)$, thus adding to the classes of graphs whose spectrum is known.

1 Introduction and motivation

Let $G = (V(G), E(G))$ be a simple graph. The spectrum of a graph G is the multiset of eigenvalues of the adjacency matrix. The graph spectrum is an important tool one can use to find information about the physical properties of a network, such as robustness, diameter, connectivity [3]. In this research we completely describe the spectrum for the class of graphs, defined below.

The *generalized Petersen graph* (GPG) $P(n, k)$ has vertices, respectively, edges given by

$$\begin{aligned} V(P(n,k)) &= \{a_i, b_i, 0 \le i \le n-1\}, \\ E(P(n,k)) &= \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \,|\, 0 \le i \le n-1\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n ($n \ge 5$), and k is the "skip". Note that $k \le \lfloor \frac{n-1}{2} \rfloor$, because of the obvious isomorphism $P(n, k) \cong P(n, n-k)$. Let $A(n, k)$ (respectively, $B(n, k)$) be the subgraph of $P(n, k)$ consisting of the vertices $\{a_i \,|\, 0 \le i \le n-1\}$ (respectively, $\{b_i \,|\, 0 \le i \le n-1\}$) and edges $\{a_i a_{i+1} \,|\, 0 \le i \le n-1\}$ (respectively, $\{b_i b_{i+k} \,|\, 0 \le i \le n-1\}$). We will call $A(n, k)$ (respectively, $B(n, k)$) the *outer* (respectively, *inner*) subgraph of $P(n, k)$. We display in Figure 1 the graph $P(12, 3)$.

For other graph theoretical terminology the reader could refer to [7].

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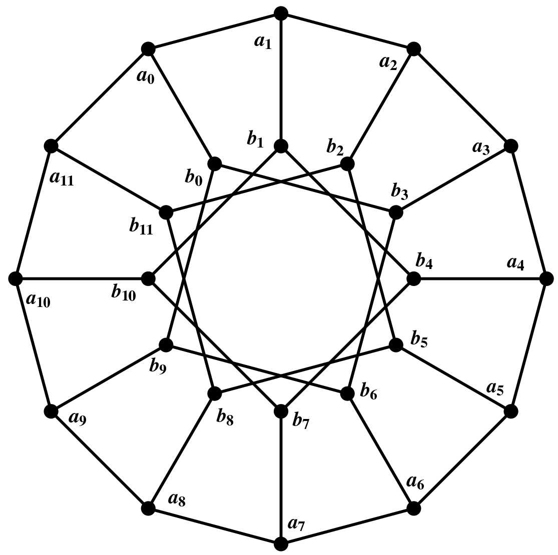

Figure 1: The Generalized Petersen Graph $P(12,3)$

2 Eigenvalues of $P(n,k)$

In this section we find our description for the spectrum of generalized Petersen graphs $P(n,k)$. We denote the adjacency matrix of the GPG $P(n,k)$ by $A(P(n,k))$. Let $\lambda_0 = 3 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n-1}$ be the sequence of eigenvalues of $P(n,k)$.

We call an $n \times n$ matrix *circulant*, and denote it by $circ(a_1, a_2, \ldots, a_n)$ if it is of the form

$$circ(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ \vdots & & & & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix}.$$

Lemma 2.1 *The $(2n) \times (2n)$ adjacency matrix of the GPG $P(n,k)$ has the block form*

$$A(P(n,k)) = \begin{pmatrix} C_k^n & I_n \\ I_n & C^n \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, C^n, C_k^n are circulant matrices, with $C^n = circ(0,1,0,0,\ldots,0,1)$ and $C_k^n = circ(\overbrace{0,\ldots,0}^{k\ times},1,0,\ldots,0,1,\overbrace{0,\ldots,0}^{k-1\ times})$ being the adjacency matrix for $A(n,k)$ and $B(n,k)$, respectively. Thus, C^n is the adjacency matrix *of a cycle graph on n vertices $\mathcal{C}_n$, respectively, C_k^n is the union of d cycle graphs $\mathcal{C}_{n/d}$ on n/d vertices, where $d = \gcd(n, k)$.*

Proof. The outer subgraph (whose adjacency matrix is C^n) of $P(n, k)$ is the cycle graph $\mathcal{C}_n$ and the inner subgraph (whose adjacency matrix is C_k^n) has d connected components each isomorphic to $\mathcal{C}_{n/d}$. Also, the adjacency matrix (which depends on the labeling) has the claimed form where the labels used on the outer subgraph are consecutively $1, 2, \ldots, n$, and on the inner subgraph the adjacent labels are $i, i+k, i+2k, \ldots$ (where $i + sk$ is understood as $1 + (i - 1 + sk) \pmod n$). Note that b_0 is adjacent to vertex b_k in the subgraph and to vertex labeled b_{n-k} in $B(n, k)$, and so $C_k^n = circ(\overbrace{0, \ldots, 0}^{k \ times}, 1, 0, \ldots, 0, 1, \overbrace{0, \ldots, 0}^{k-1 \ times})$ □

We recall the Chebyshev's polynomial of the first kind [5], defined by the identity $T_n(\cos\theta) = \cos(n\theta)$, with the generating function $\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2}$. We now present the eigenvectors and eigenvalues for $\mathcal{C}_n$ (see [2, p. 53 and pp. 72–73]). Let $\mathbf{v}^t$ denote the transpose of $\mathbf{v}$.

Lemma 2.2 *The eigenvalues of the cycle graph $\mathcal{C}_n$ on n vertices are*

$$\alpha_j = 2\cos\left(\frac{2\pi j}{n}\right)$$

with a corresponding eigenvector

$$\mathbf{v}_j = (1, \zeta^j, \zeta^{2j}, \ldots, \zeta^{(n-1)j})^t,$$

$0 \le j \le n-1$. The characteristic polynomial of the cycle $\mathcal{C}_n$ is $2T_n(x/2) - 2$ where T_n is the Chebyshev's polynomial of the first kind.

Corollary 2.3 *The eigenvalues corresponding to the circulant C in the adjacency matrix $A(P(n, k))$ are $\alpha_j = 2\cos\left(\frac{2\pi j}{n}\right)$ $(0 \le j \le n-1)$, and the eigenvalues corresponding to C_k are $\beta_j = 2\cos\left(\frac{2\pi jk}{n}\right)$ $(0 \le j \le n-1)$.*

We now state our main theorem which adds to the class of graphs whose spectrum is now known.

Theorem 2.4 *The eigenvalues of $P(n, k)$, say $\delta_{2j}, \delta_{2j+1}$, are all roots of the quadratic equation*

$$\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0, \tag{1}$$

where $\alpha_j = 2\cos\left(\frac{2\pi j}{n}\right)$, $\beta_j = 2\cos\left(\frac{2\pi jk}{n}\right) = 2T_k(\alpha_j/2)$ $(0 \le j \le n-1)$ are the eigenvalues of C, respectively C_k.

Proof. We first consider the case of $d = \gcd(n, k) = 1$. Since $d = 1$, then C_k is the adjacency matrix of a cycle graph isomorphic to $\mathcal{C}_n$, and so it is similar to C, that is, there exists a permutation matrix P, such that $P^{-1}C_kP = C$. This implies that the two matrices will have the same eigenvalues and eigenvectors. Then α_j, β_j are eigenvalues corresponding to the same eigenvector, say $\mathbf{v}_j = (1, \zeta_n^j, \ldots, \zeta_n^{(n-1)j})^t$. We are looking for an eigenvector for $A(P(n,k))$ of the form $\mathbf{w}_j = (a_j\mathbf{v}_j, \mathbf{v}_j)^t$, where a_j will be determined later. If two distinct values for a_j are to be found, for any $0 \le j \le n-1$, then we are done with our search for the eigenvectors/eigenvalues.

With this value for $\mathbf{w}_j$, we need δ (dependent on j) such that

$$\begin{pmatrix} C_k & I_n \\ I_n & C \end{pmatrix} \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix} = \delta \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix}$$

and so, we get the system

$$\left\{ \begin{array}{lclcl} a_jC_k\mathbf{v}_j & + & \mathbf{v}_j & = & \delta a_j\mathbf{v}_j \\ a_j\mathbf{v}_j & + & C\mathbf{v}_j & = & \delta\mathbf{v}_j. \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{lclcl} a_j\beta_j\mathbf{v}_j & + & \mathbf{v}_j & = & \delta a_j\mathbf{v}_j \\ a_j\mathbf{v}_j & + & \alpha_j\mathbf{v}_j & = & \delta\mathbf{v}_j, \end{array} \right.$$

which implies

$$\left\{ \begin{array}{lcl} a_j(\delta - \beta_j)\mathbf{v}_j & = & \mathbf{v}_j \\ (\delta - \alpha_j)\mathbf{v}_j & = & a_j\mathbf{v}_j, \end{array} \right.$$

and so, $(\delta - \beta_j)(\delta - \alpha_j) = 1$, which renders the claim for this case, that is, δ must satisfy the equation $\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0$.

The case of $d > 1$ is treated similarly. The eigenvectors $\mathbf{w}_j$ must have the form $\mathbf{w}_j = (a_1\mathbf{v}'_j,\ a_2\mathbf{v}'_j,\ \ldots,\ a_d\mathbf{v}'_j,\ \mathbf{v}_j)$, with $\mathbf{v}_j$ as before and $\mathbf{v}'_j = (1, \zeta_n^j, \ldots, \zeta_n^{(n'-1)j})^t$, $n' = n/d$, for some appropriate multipliers a_i. A similar system to the one for $d = 1$ case will be obtained and, interestingly enough, the same polynomial whose roots are the eigenvalues λ_i will be found. The theorem is proved. □

Using the quadratic formula in (1) and simplifying we get the following corollary.

Corollary 2.5 *The eigenvalues of $P(n,k)$ are given by*

$$\cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi jk}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi jk}{n}\right)\right)^2 + 1},\ 0 \le j \le n-1.$$

The largest eigenvalue of $P(n,k)$, $\lambda_0 = 3$, is one of the two values obtained for $j = 0$ in the previous corollary. It is known (see [2, Thm. 3.11]) that if a graph is bipartite, then its spectrum is symmetric with respect to 0. In our case, we have the following result.

Corollary 2.6 *If n is even and k is odd, then the eigenvalues of the bipartite graph $P(n,k)$ are given by ± 3 and*

$$\cos(2j\pi/n) + \cos(2jk\pi/n) \pm \sqrt{(\cos(2j\pi/n) - \cos(2jk\pi/n))^2 + 1}$$

$$-\cos(2j\pi/n) + (-1)^k\cos(2jk\pi/n) \mp \sqrt{(\cos(2j\pi/n) + (-1)^k\cos(2jk\pi/n))^2 + 1},$$

for $0 \le j < n/2$.

3 Bounds on the eigenvalues of $P(n,2)$

In the previous section we found the complete set of eigenvalues of $P(n,k)$ under no restrictions on n and k. Here, we would like to find some bounds on some eigenvalues. Eigenvalue interlacing techniques (see the great survey by Haemers [4] on the topic) will not work easily since there is no visible connection between the various $P(n,k)$, and moreover, the technique is not sensitive enough for our purpose. We shall use a different method.

Here, we will take $k = 2$ and consider $P(n,2)$ (this includes the case of the classical Petersen graph $P(5,2)$). Since the second Chebyshev polynomial of the first kind is $T_2(x) = 2x^2 - 1$, we immediately obtain the following:

Theorem 3.1 *The eigenvalues of* $P(n,2)$ *are (for* $0 \le j \le n-1$*)*

$$2\cos^2(2j\pi/n) + \cos(2j\pi/n) - 1 \pm \sqrt{(2\cos^2(2j\pi/n) - \cos(2j\pi/n) - 1)^2 + 1}.$$

To find good bounds on the eigenvalues in this case, we look for the extreme points of the two functions

$$f_\pm(x) = 2x^2 + x - 1 \pm \sqrt{(2x^2 - x - 1)^2 + 1}, \tag{2}$$

in the interval $-1 \le x \le 1$. Certainly, we cannot expect exact or even tight results, in general, since the sequence $\frac{2j\pi}{n}$, $0 \le j < n-1$, is finite and therefore, $\cos\left(\frac{2j\pi}{n}\right)$ is not dense in this interval. However, we will have lower and upper bounds, which is what we are interested in. Since any differentiable function in a compact domain attains its extreme points at either the critical points or on the boundary, we proceed by studying first the functions' critical points:

$$f'_\pm(x) = 4x + 1 \pm \frac{(2x^2 - x - 1)(4x - 1)}{\sqrt{(2x^2 - x - 1)^2 + 1}} = 0,$$

has solutions (computed by Mathematica[1]) at $x_1 \sim -0.41100$ (for f_+) and $x_2 \sim -0.65041, x_3 \sim -0.04610$ (for f_-). The values of the corresponding $f_\pm$ at these critical points are

$$\begin{aligned} f_+(-0.41100) &= -0.04210\ldots \\ f_-(-0.65041) &= -1.92081\ldots \\ f_-(-0.04610) &= -2.42092\ldots. \end{aligned}$$

Further, we look at the values of $f_\pm$ at $|x| = 1$. Thus, $f_+(1) = 3, f_+(-1) = \sqrt{5}$, and $f_-(1) = 1, f_-(-1) = -\sqrt{5}$. Certainly, the maximum value is 3, and the minimum value is approximately -2.42092. We sketch in Figure 2 the two functions $f_\pm$, to visualize our analysis from above:

[1] A Trademark of Wolfram Research

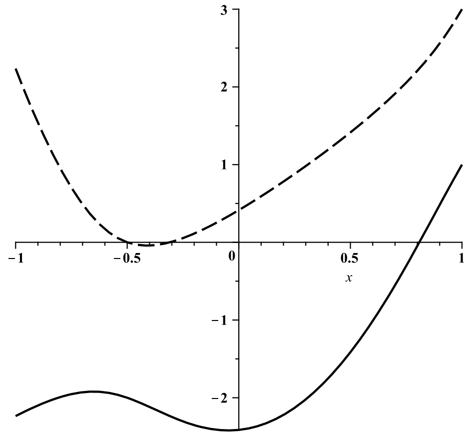

Figure 2: The top function is f_+ and the bottom function is f_-

Every value of f_+ is above every value of f_-, and so the minimum is attained by f_- and the maximum is attained by f_+. Furthermore, we see that the second largest eigenvalue of $P(n, 2)$ is

$$\begin{aligned}\lambda_1 &= f_+\left(\cos\left(\frac{2\pi}{n}\right)\right)\\ &= \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \sqrt{4\left(2\cos\left(\frac{2\pi}{n}\right)+1\right)^2 \sin^4\left(\frac{\pi}{n}\right)+1},\end{aligned} \tag{3}$$

which increases as n increases (shown simply by using Calculus techniques). For instance, for $3 \le n \le 20$ the sequence $\lambda_1 = \lambda_1(n)$ is

$$0, 0.41421, 1., 1.41421, 1.71083, 1.93185, 2.10199, 2.23607, 2.34356, 2.43091,$$
$$2.50268, 2.56224, 2.61211, 2.65421, 2.69002, 2.7207, 2.74716, 2.77011.$$

Since $\lim_{n\to\infty} \cos\left(\frac{2\pi}{n}\right) = 1$, we obtain the next result.

Theorem 3.2 *The eigenvalues of* $P(n, 2)$ *are*

$$\lambda_0 = 3 > \lambda_1 \ge \ldots \ge \lambda_{2n-1} \ge -2.42092.$$

Moreover, the second largest eigenvalue satisfies $\lim_{n\to\infty} \lambda_1(n) = 3$.

4 Further comments

All of our results for $P(n, 2)$ can be certainly extended to $P(n, 3)$, $P(n, 4)$, etc., but to find sensitive bounds on eigenvalues for arbitrary GPG $P(n, k)$ does not seem to be easy, since the sequence of the involved Chebyshev's polynomials of the first kind does not have a "controllable" behavior in $|x| \leq 1$.

Also, it would be interesting to investigate the number of and distinct values among the eigenvalues of $P(n, k)$, and that is presumably doable. We suspect that the methods of this paper can be also applied to the I-graphs of [1] or the *supergeneralized* Petersen graphs of [6].

Acknowledgement

The authors are grateful to the referee whose suggestions have led to an improvement in the presentation of this paper. During the preparation of this paper, the authors were partially supported by a RIP grant from NPS.

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(Received 12 Oct 2009; revised 5 Sep 2010)