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Completing partial latin squares: Cropper's question

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Abstract

Hall's condition is a well-known necessary condition for the existence of a proper coloring of a graph from prescribed lists. Completing a partial latin square is a very special kind of graph list-coloring problem. Cropper's question was: is Hall's condition sufficient for the existence of a completion of a partial latin square? The folk belief that the answer must be no is confirmed here, but, also, six theorems giving necessary and sufficient conditions for completion of partial latin squares in different circumstances are recast in the form: when the prescribed cells in a partial latin square form such-and-such a configuration, then not only is Hall's condition sufficient for completion, but, in each of these cases, a small subset of the large set of inequalities constituting Hall's condition suffice.

1 Introduction: Hall's condition for list-colorings

Suppose that G is a finite simple graph, C is an infinite set, and $\mathcal{F}$ is the collection of finite subsets of C. A *list assignment* to G is a function $L : V(G) \to \mathcal{F}$. If L is a list assignment to G, a *proper L-coloring of* G is a function $\varphi : V(G) \to C$ satisfying, for all u, $v \in V(G)$,

(i) $\varphi(u) \in L(u)$ and

(ii) if $uv \in E(G)$ then $\varphi(u) \neq \varphi(v)$.

It is useful to realize that (ii) may be restated as:

(ii)' for each $\sigma \in C$, the preimage $\varphi^{-1}(\sigma) = \{u \in V(G) \mid \varphi(u) = \sigma\}$ is an independent set of vertices of G.

For a graph H, let $\alpha(H)$ denote the vertex independence number of H, the greatest size of an independent set of vertices in H. If L is a list assignment to G, $\sigma \in C$, and H is a subgraph of G, let $H_\sigma = H(\sigma, L)$ be the subgraph of H induced by $\{v \in V(H) \mid \sigma \in L(v)\}$. If φ is a proper L-coloring of G then $\varphi^{-1}(\sigma) \cap V(H)$ is an independent set of vertices in $H(\sigma, L)$, so $|\varphi^{-1}(\sigma) \cap V(H)| \leq \alpha(H(\sigma, L))$. This observation shows that the following condition on G and L is necessary for the existence of a proper L coloring of G.

Hall's condition: For each subgraph H of G,

$$\sum_{\sigma \in C} \alpha(H(\sigma, L)) \geq |V(H)|. \qquad (*)$$

Clearly the Hall inequality $(*)$ holds for every subgraph of G if it holds for every induced subgraph of G. Therefore, on the face of it, verifying Hall's condition amounts to checking $2^n - 1$ inequalities, $n = |V(G)|$, and checking each inequality requires the computation of vertex independence numbers, not an easy task; clearly we are not aiming for computational efficiency in calling attention to Hall's condition! (Although in many cases the checking is not so arduous, because of the necessity of Hall's condition for a proper coloring. For instance, if $G - v$ is properly L-colorable for each $v \in V(G)$, and sometimes it is easy to see this, then $(*)$ need be checked only for $H = G$.)

Our interest in questions involving Hall's condition is purely theoretical, at this point; we are digging for depth in the theory of list-colorings. Our hopes for Hall's condition as some sort of key to previously unnoticed doors arise from its pedigree: when G is a complete graph, any induced subgraph H of G is complete, so

$$\alpha(H(\sigma, L)) = \begin{cases} 1 & \text{if } \sigma \in \bigcup_{v \in V(H)} L(v), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for each } \sigma \in C.$$

Therefore,

$$\sum_{\sigma \in C} \alpha(H(\sigma, L)) = |\bigcup_{v \in V(H)} L(v)|.$$

Then it is easy to see that the satisfaction of the Hall inequality $(*)$ for each induced subgraph H of $G \simeq K_n$ is, in disguise, the condition in Philip Hall's theorem [12] on systems of distinct representatives (SDRs). This theorem guarantees the existence of a system of distinct representatives of the sets $L(v)$, $v \in V(G)$, when Hall's condition is satisfied, and such an SDR is no less nor more than a proper L-coloring of $G \simeq K_n$. That is, Hall's theorem may be restated: when G is complete, Hall's condition on G and L is both sufficient and necessary for the existence of a proper L-coloring of G. (As explained in [14] and [15], it is this view of Hall's theorem as a theorem about list colorings of complete graphs that led to the naming of Hall's condition.)

The class of graphs that share this property of complete graphs is small.

Theorem HJW ([14], [15]) *Suppose G is a finite simple graph. The following are equivalent:*

(a) G has a proper L-coloring whenever G and L satisfy Hall's condition;

(b) every block of G is a clique;

(c) G contains no induced cycle C_n, $n \geq 4$, nor an induced copy of K_4-minus-an-edge.

In other investigations, Hall's condition has been considered in conjunction with other requirements—for instance, $|L(v)| \geq 2$ for all $v \in V(G)$—and the question becomes: which graphs G are properly L-colorable whenever G and L satisfy the full list of requirements, including Hall's condition? See [10] and [15].

2 Partial latin squares and Cropper's question

Surprisingly, in view of the context in which Hall's condition was first stumbled upon [13], the following kind of restriction on L has not been considered in conjunction with Hall's condition until recently. A *partial proper m-coloring* of G is a coloring of a subset $V_0 \subseteq V(G)$ with colors from $\{1, \ldots, m\}$ so that adjacent vertices in V_0 are assigned different colors. A *completion* of a partial proper m-coloring $\varphi : V_0 \to \{1, \ldots, m\}$ is an extension $\hat{\varphi} : V(G) \to \{1, \ldots, m\}$ of φ to a proper coloring of G. Such an extension is never possible unless $m \geq \chi(G)$, and examples abound in which no such extension is possible for some partial m-colorings even when m is much greater than $\chi(G)$. For examples, let $1 \leq n < m$ and take $G = K_n \vee \bar{K}_m$, with "$\vee$" denoting the join operation. (To form the join of two graphs G_1 and G_2 take the union of vertex-disjoint copies of the graphs with the complete bipartite graph whose parts are $V(G_1)$ and $V(G_2)$.) Let φ assign $1, \ldots, m$ to the m vertices of $\bar{K}_m$. This partial m-coloring is proper, no completion is possible, and m may be arbitrarily large, while $\chi(G) = n + 1$.

Every partial proper m-coloring φ of G defines a list assignment $L = L_\varphi$ to $V(G)$ in a natural way: if $v \in V_0$, $L(v) = \{\varphi(v)\}$, and if $v \in V(G) \backslash V_0$ then $L(v) = \{1, \ldots, m\} \backslash (\varphi(N_G(v) \cap V_0))$; that is, each vertex without a prescribed color is endowed with the list of all colors among $1, \ldots, m$ that do not appear on its neighbors in the set V_0 of vertices with prescribed colors. Clearly φ has a completion if and only if G has a proper L_φ-coloring. We declare G to be *Hall m-completable* if and only if every partial proper m-coloring φ of G such that G and L_φ satisfy Hall's condition has a completion.

In all that follows, when L is a list assignment to a graph G, and H is a subgraph of G, the restriction of L to H, sometimes denoted $L|_H$ or $L|_{V(H)}$, will be denoted simply by L. The reader will be able to discern which L is meant by the context.

Lemma 1 *Suppose that $\varphi : V_0 \to \{1, \ldots, m\}$ is a partial proper m-coloring of G, and G' is the subgraph of G induced by $V(G) \backslash V_0$. There is a proper L_φ-coloring of G if and only if there is a proper L_φ-coloring of G'. Also, G and L_φ satisfy Hall's condition if and only G' and L_φ satisfy Hall's condition.*

PROOF: The claim about the L_φ-colorability of G and G' is easy to see.

If G and L_φ satisfy Hall's condition then G' and L_φ satisfy Hall's condition, just because G' is an induced subgraph of G, which implies that every induced subgraph of G' is an induced subgraph of G. Suppose that H is an induced subgraph of G and $v \in V_0 \backslash V(H)$. Let H_1 be the subgraph of G induced by $V(H) \cup \{v\}$. Then $|V(H_1)| = |V(H)| + 1$ and $\sum_{\sigma \in C} \alpha(H_1(\sigma, L_\varphi)) = \sum_{\sigma \in C} \alpha(H(\sigma, L_\varphi)) + 1$ because $\alpha(H_1(\varphi(v), L_\varphi)) = \alpha(H(\varphi(v), L_\varphi)) + 1$ and $\alpha(H_1(\sigma, L_\varphi)) = \alpha(H(\sigma, L_\varphi))$ for all $\sigma \in C \backslash \{\varphi(v)\}$. That is, adding a vertex of V_0 to H increases both sides of $(*)$ by 1. Therefore, if $(*)$ holds for every induced subgraph of G' then it holds for every induced subgraph of G. $\square$

		7	3	5		
				6	1	5
	6					3
6						
4						
2		1				
3	4	2				

Figure 1: A 7×7 partial latin square, an early candidate for answering Cropper's question in the negative.

If G is Hall $\chi(G)$-completable we will sometimes say that G is Hall chromatic completable. A *partial latin square* (p.l.s.) of order n is the graph $K_n \Box K_n$ together with a partial proper n-coloring of it ($\Box$ denotes the Cartesian product). Matt Cropper's question, advanced persistently since about 1999, is: if the list assignment L_φ associated with a partial latin square satisfies Hall's condition, with $K_n \Box K_n$, does the partial latin square necessarily have a completion? In other words, since $n = \chi(K_n \Box K_n)$, is $K_n \Box K_n$ Hall chromatic completable? This question was ignored until 2004, or thereabouts, when an attempt was made to put an end to this discussion by producing an example to show that $K_n \Box K_n$ is not Hall chromatically completable. This turned out to be not so easy! In Figure 1, for instance, is a 7×7 partial latin square, a modification by the third author of an example provided by Ron Aharoni. What follows is a brief account of efforts to show that this square dismisses Cropper's question, leading to the tragic discovery that it does not.

Per convention, the vertices of $K_n \Box K_n$ are the "cells" in an $n \times n$ array, with each cell adjacent to all and only the cells in its row and column. Throughout, the cell in the i^{th} row, j^{th} column of the array representing $K_n \Box K_n$ will be denoted $v(i, j)$.

The incompletability of this partial latin square can be easily seen just by looking at the upper left 3×3 subsquare. The third author, with the assistance of the first, expended $3 - 10$ months in trying to see that the list assignment induced by the prescribed cells satisfies Hall's condition, thereby settling Cropper's question. In view of the Lemma, verifying Hall's condition by brute force would have meant verifying $2^{34} - 1$ inequalities; the third author brought to bear a fiendishly clever program to reduce these to a couple of thousand. This program led to the dashing of all hopes: if H is the subgraph of $K_7 \Box K_7$ induced by

$$\begin{aligned} S = \{v(i,j) |\ 1 \le i,j \le 2 \quad & \text{or } 2 \le i \le 5 \text{ and } j = 3, \\ & \text{or } 4 \le i \le 5 \text{ and } j \ge 2, \\ & \text{or } i = 6 \text{ and } j = 2,4,5,6, \text{ or } 7, \\ & \text{or } i = 7 \text{ and } 4 \le j \le 7\}, \end{aligned}$$

then $\sum_{i=1}^{7} \alpha(H(i, L_\varphi)) = 26 < 27 = |V(H)|$

The third author believes that he can prove, with some assistance from the first

$\{1\}$	$\{1,2\}$					
$\{7\}$	$\{2,3,7\}$	$\{3,4\}$				
		$\{4,5\}$				
	$\{1,2,3,5,7\}$	$\{3,4,5\}$	$\{1,2,4,5,7\}$	$\{1,2,3,4,7\}$	$\{2,3,4,5,7\}$	$\{1,2,4,7\}$
	$\{1,2,3,5,7\}$	$\{3,5,6\}$	$\{1,2,5,6,7\}$	$\{1,2,3,7\}$	$\{2,3,5,6,7\}$	$\{1,2,6,7\}$
	$\{3,5,7\}$		$\{4,5,6,7\}$	$\{3,4,7\}$	$\{3,4,5,6,7\}$	$\{4,6,7\}$
			$\{1,5,6,7\}$	$\{1,7\}$	$\{5,6,7\}$	$\{1,6,7\}$

Figure 2: The bad set in the partial latin square of Figure 1, with lists

6	8	4	5	2	3	7	1
5	6	8	3	7	4	1	2
7	3	1	2	8	5	6	4
4	7	2	1	3	8	5	6
2	5	6	8				
8	4	7	6				
3	1	5	7				
1	2	3	4				

1	2	3	4	5	6
3	6	1	2	4	5
5	4	2	6	3	1
2	5				
4	1				
6	3				

Figure 3: Two incomplete partial latin squares satisfying Hall's conditiion, due to J. L. Goldwasser; neither can be completed.

author, that S is the only "bad set" of cells in $V(K_7 \square K_7)\backslash V_0$, with respect to the partial proper 7-coloring given in Figure 1. We leave this claim for the reader's contemplation.

Despite this example, the answer to Cropper's question is no. The first known examples demonstrating this, both discovered by the second author, are given in Figure 3.

In both cases it is easy to check that the partial latin square cannot be completed: start at any unfilled cell with a 2-symbol list, fill it with one of the two symbols on its list, and follow the resulting spreading chain of forced colorings until reaching impasse. Then do the same starting with the other symbol.

The verification of Hall's condition for the second partial latin square in Figure 3 is relatively easy. The list assignment L_φ to $V(K_6 \square K_6)\backslash V_0$ for this p.l.s. is given in Figure 4. Let $G \simeq K_3 \square K_4$ be the graph underlying Figure 4. By Lemma 1, we need only verify that G and the depicted list assignment satisfy Hall's condition. The key to seeing this is to observe that $G_i = G(i, L_\varphi) \simeq C_4$ for each $i \in \{1, \dots, 6\}$. It follows that for every induced subgraph H of G, and $1 \leq i \leq 6$, if i appears on the lists of t_i vertices of H, then $\alpha(H_i) \geq \lceil \frac{t_i}{2} \rceil$. Therefore, $\sum_{i=1}^{6} \alpha(H_i) \geq \sum_{i=1}^{6} t_i/2 = |V(H)|$, by appeal to the fact that every vertex of H has two symbols on its list.

Now let $G \simeq K_4 \square K_4$ be the graph underlying the unfilled array in the first p.l.s. in Figure 3, and let L denote the list assignment to it induced by the entries in the filled cells. Cells $(3,4)$ and $(4,3)$ in the 4×4 array have list $\{8\}$, and cells $(3,3)$

4, 6	1, 3	1, 6	3, 4
5, 6	3, 5	2, 6	2, 3
4, 5	1, 5	1, 2	2, 4

Figure 4: List assignment to the unfilled cells in the second partial latin square in Figure 3.

and $(4,4)$ have lists $\{2,4,8\}$ and $\{5,7,8\}$, respectively. Let G' denote the graph obtained by deleting vertices (cells) $(3,4)$ and $(4,3)$ from G, and let L' be the list assignment to G' obtained by removing 8 from the L-lists on cells $(3,3)$ and $(4,4)$ and letting $L = L'$ on the other cells of G'. It is easy to see that G and L satisfy Hall's condition if G' and L' satisfy Hall's condition. But $|L'(v)| = 2$ for every $v \in V(G')$, and $G'(i, L') \simeq C_4$ for each $i \in \{1, \ldots, 7\}$, so G', L' satisfy Hall's condition, by the argument above.

If we were interested only in Hall chromatic completability the demise of Cropper's question would largely kill our interest in the graphs $K_n \Box K_n$. But completing partial latin squares is a subject of special importance in combinatorics (whether inevitably or by historical accident we leave to debate), and there is evidence, in the results of [6], [13] and [16], that there may be interesting answers to the following question: For which sets $V_0 \subseteq V(K_n \Box K_n)$ is it the case that whenever V_0 is the set of prescribed cells in a partial latin square of order n—i.e., whenever V_0 is the domain of a partial proper n-coloring φ of $K_n \Box K_n$—such that $K_n \Box K_n$ and L_φ satisfy Hall's condition, there is necessarily a completion of the partial latin square? The results in [6] and [13] are that subrectangles and subrectangles minus one cell have this property, but even more: in the case of a subrectangle, a single instance of $(*)$, i.e. satisfaction of the inequality for a single choice of H, suffices to guarantee the existence of a completion, and in the case of a subrectangle minus one cell, satisfaction of $(*)$ for 3 choices of H implies completability.

In Section 3 we revisit the case where V_0 is a subrectangle, giving the result a bit differently than in [6] and [13], and then deal with 4 other cases, in each of which a theorem about completing partial latin squares can be reconstructed to the form: if a partial latin square has prescribed cells forming such-and-such a shape (after permuting rows and columns), then Hall's condition is sufficient (as well as necessary, of course) for the existence of a completion—and a relatively small number of instances of $(*)$ suffice for completion.

The 6 "shape-of-the-domain" theorems are preceded by two classical results, the precursor of Ryser's theorem due to M. Hall [11] and the affirmation of the Evans conjecture [4], [20]. Theorems of Andersen and Hilton sharpening the Evans conjecture yield corollaries concerning Hall's condition. We suspect that there is a general theorem awaiting discovery to which these results are clues. More such results and clues will appear in [16].

In the last section we find that a theorem due to Andersen and Hoffman on completing partial commutative latin squares can be reconstructed in a way similar to the other reconstructions. A main difference is that the underlying graph is not $K_n \Box K_n$.

This paper is not about Hall m-completability; that definition was given only for the purpose of placing our work on completing partial latin squares in a more general context that might be suitable for future exploration. However, having given the definition, there is no harm in ending this section with a compendium of elementary results on the subject. Proofs will be given only when they are not obvious. None of these results will bear on the work in Section 3.

Elementary results on m-completions and Hall m-completability

1. *If $m \geq \Delta(G) + 1$ then every partial proper m-coloring of G has a completion.*
2. *G is Hall m-completable (assuming, $m \geq \chi(G)$) if and only if every component of G is Hall m-completable.*
3. *If G is one of the following then G is Hall m-completable for all $m \geq \chi(G)$:*
   (a) *an odd cycle;*
   (b) *complete multipartite;*
   (c) *a graph in which every block is a clique.*

Proof of 3: (a) follows from 1, above, and (c) follows from Theorem HJW.
(b): Suppose that $G = K_{n_1,\ldots,n_r}$, $m \geq r$, and $\varphi : V_0 \to \{1, \ldots, m\}$ is a partial proper m-coloring of G such that G and L_φ satisfy Hall's condition. Any two vertices from the same part of G that are not in V_0 have the same list assigned by L_φ. Therefore, if K is a clique in G induced by a collection of single representatives not in V_0 of the parts of G that have vertices not in V_0, then the proper L_φ-colorability of K would imply the proper L_φ-colorability of G, and that would finish the proof. Because G and L_φ satisfy Hall's condition, K and the restriction of L_φ to $V(K)$ satisfy Hall's condition, so there is a proper L_φ-coloring of K, by Theorem HJW (or by Hall's Theorem, of which Theorem HJW is a generalization). □

4. *Every bipartite graph is Hall chromatic completable, but for every $m \geq 3$ there is a bipartite graph which is not Hall m-completable.*

Proof: To show that every bipartite graph is Hall chromatic completable, by 2 and common sense about isolated vertices it suffices to consider G bipartite, connected, of order > 1. Suppose $\varphi : V_0 \to \{1, 2\}$ is a partial proper 2-coloring of G such that L_φ and G satisfy Hall's condition. Let X, Y be a bipartition of $V(G)$. There can fail to be a proper 2-completion of φ only if either the same color is prescribed by φ to vertices in X and in Y, say $\varphi(x) = \varphi(y) = 1$ for some $x \in X \cap V_0$, $y \in Y \cap V_0$,

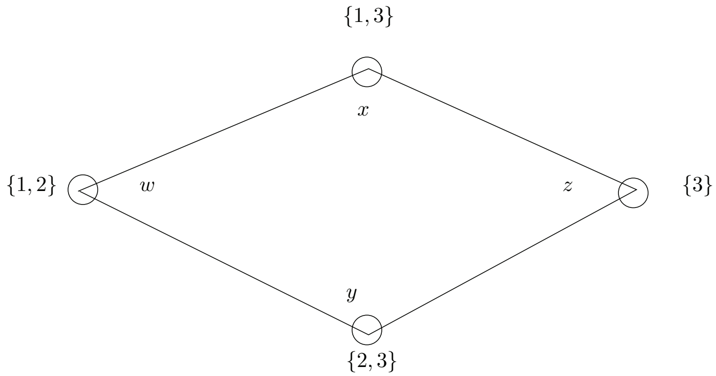

Figure 5: The smallest graph with a list assignment satisfying Hall's condition from which there is no proper coloring

or different colors are prescribed by φ to vertices in the same part, say $\varphi(x) = 1$, $\varphi(z) = 2$ for some x, $z \in X \cap V_0$. In the first case, because G is connected, there is a path P in G with ends x and y. Because G and L_φ satisfy Hall's condition, P and L_φ must satisfy Hall's condition, so P is properly L_φ-colorable, by Theorem HJW. But since P is a path from a vertex in X to a vertex in Y, P is of odd length; since there are only 2 colors available, the ends of P cannot have the same color, in a proper coloring. So $\varphi(x) = \varphi(y)$ for $x \in X \cap V_0$, $y \in Y \cap V_0$ cannot happen. Similarly, $\varphi(x) \neq \varphi(z)$ for x, $z \in X \cap V_0$ cannot happen. Thus G is Hall 2-completable.

Regarding the other assertion in 4, consider the 4-cycle with list assignment in Figure 5. Clearly there is no proper coloring of $G = C_4$ from these lists, and it is not hard to see that Hall's condition is satisfied. [Verify that $G - v$ is properly colorable from the lists, for each $v \in V(G)$.] For each $m \geq 3$ attach pendant vertices to the vertices of the graph and prescribe colors to these to obtain a partial proper m-coloring φ of the augmented graph such that the lists L_φ on w, x, y, and z are as indicated. For instance, if $m = 4$, 2 leafs will be attached to w with prescribed colors $3, 4$; 2 to x with prescribed colors $2, 4$; 2 to y with prescribed colors, $1, 4$; and 3 to z with prescribed colors, $1, 2, 4$.

The resulting graph is bipartite, and by Lemma 1 it satisfies Hall's condition with the list assignment generated by the prescription, but this prescription admits no completion. □

Item 4 affirms the ghastly possibility that a graph may be Hall m-completable but not Hall $(m + 1)$-completable. At present $m = 2$ is the only value for which this is known to occur, among graphs of chromatic number $\leq m$.

In the graphs in the proof of item 4 which are not Hall m-completable, and in all other examples that we know of, the maximum degree of the constructed graph is $m + 1$. Item 1 implies that every graph G is Hall $(\Delta(G) + 1)$-completable. Is it true that every connected graph G which is not a complete graph or an odd cycle (the only connected graphs for which $\chi > \Delta$, by Brooks' theorem [7]) is Hall $\Delta(G)$-completable?

Item 1 and the existence of graphs G which are not Hall m-completable for some $m \geq \chi(G)$ implies that for some G and $m \geq \chi(G)$, G is not Hall m-completable but is Hall $(m+1)$-completable. It is possible, in such a case, that G could be not Hall $(m+k)$-completable for some $k \geq 2$?

3 Six theorems on completing partial latin squares, restated

Suppose that $m \geq \chi(G)$ and $V_0 \subseteq V(G)$. We will say that V_0 is *Hall*(m, G) *easy* if every partial proper m-coloring $\varphi : V_0 \to \{1, \ldots, m\}$ such that L_φ and G satisfy Hall's condition has an m-completion. So, for any G and $m \geq \chi(G)$, $\emptyset$ and $V(G)$ are Hall(m, G) easy; and G is Hall m-completable if and only if every subset of $V(G)$ is Hall(m, G) easy.

If V_0 is Hall(m, G) easy and π is an automorphism of G, then $\pi(V_0)$ is Hall(m, G) easy. Therefore, if $G = K_n \square K_n$ and $V_0 \subseteq V(G)$ is Hall(n, G) easy, then so is any set of cells obtained from V_0 by permuting rows and columns of the $n \times n$ array constituting the standard representation of $K_n \square K_n$, or by reflecting about the main diagonal. This should be kept in mind in considering the six theorems below, each of which shows that particular forms of cell sets in $V(K_n \square K_n)$ are Hall(n, G) easy.

There are two famous theorems that belong with the six. The first of these appeared before any of the others; it is the first application of Hall's theorem to the problem of completing partial latin squares.

Theorem MH [11] *If the set of prescribed cells in a partial latin square of order n is the union of some rows of the array, then the partial latin square is completable.*

Corollary 1 *Any union of rows of $K_n \square K_n$ is Hall$(n, K_n \square K_n)$ easy.*

The second famous theorem is the confirmation of Evans' conjecture.

Theorem E ([4] and [20]) *Any partial square of order n with $n-1$ or fewer prescribed cells can be completed to a latin square of order n.*

We have not given these results the same status as the other 6 because there is nothing we can add to the original result. Given an $r \times n$ latin rectangle on n symbols, as a partial latin square of order n it is completable and therefore satisfies Hall's condition; the set of inequalities $(*)$ to be checked is empty. The same holds for any partial latin square of order n in which the number of prescribed cells is no greater than $n - 1$.

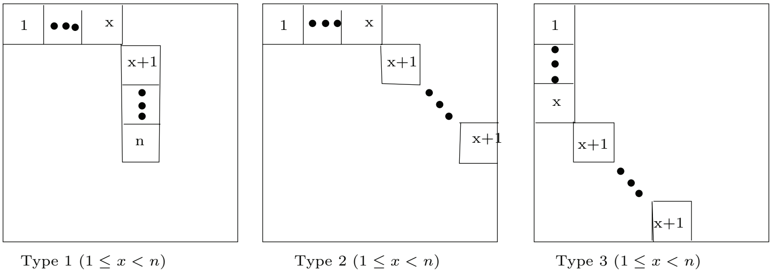

Figure 6: The noncompletable partial latin squares of side n with n nonempty cells

3.1 A theorem of Andersen combined with one of Andersen and Hilton

In [4], Andersen and Hilton not only proved Evans' Conjecture, they also gave a complete characterization of the partial latin squares of order n with exactly n prescribed cells which are not completable to latin squares of order n. These are given in Figure 6, as they appeared in [5]. It is to be understood that these represent, also, any partial latin squares that can be obtained from them by permuting rows and/or columns, permuting the names of the symbols, and/or interchanging the roles of the rows and columns—i.e., reflecting the array about the main diagonal. Therefore Types 2 and 3 represent the same batch of bad partial latin squares.

Andersen went on, in [3], to characterize the incompletable partial latin squares of order n in which exactly $n+1$ cells are prescribed. These are given in Figure 7, as they appeared in [5]. Again, there is some redundancy: Types 4 and 5 represent the same lot, as do Types 7 and 8, and Types 10 and 11. The compound theorem alluded to in the title of this section is:

Theorem AH *For all integers $n > 1$ the partial latin squares of order n, with either exactly n or $n+1$ prescribed cells, which are not completable to latin squares of order n, are represented by the 13 Types in Figures 6 and 7.*

Corollary 2 *Any set of no more than $n+1$ vertices in $K_n \Box K_n$ is Hall $(n, K_n \Box K_n)$ easy.*

Proof: Equivalently, the claim is that any partial latin square of order n in which no more than $n+1$ cells are prescribed, such that the induced list assignment to $K_n \Box K_n$ satisfies Hall's condition, is completable to a proper n-coloring of $K_n \Box K_n$ (i.e., a latin square of order n).

Since the only such partial latin squares which are not completable are equivalent to one of Types 1–13 in Figures 6 and 7, to prove the corollary it suffices to give, for each type, a set of vertices $V(H)$ of an induced subgraph H of $K_n \Box K_n$ such that H and the induced list assignment in question do not satisfy Hall's condition. We

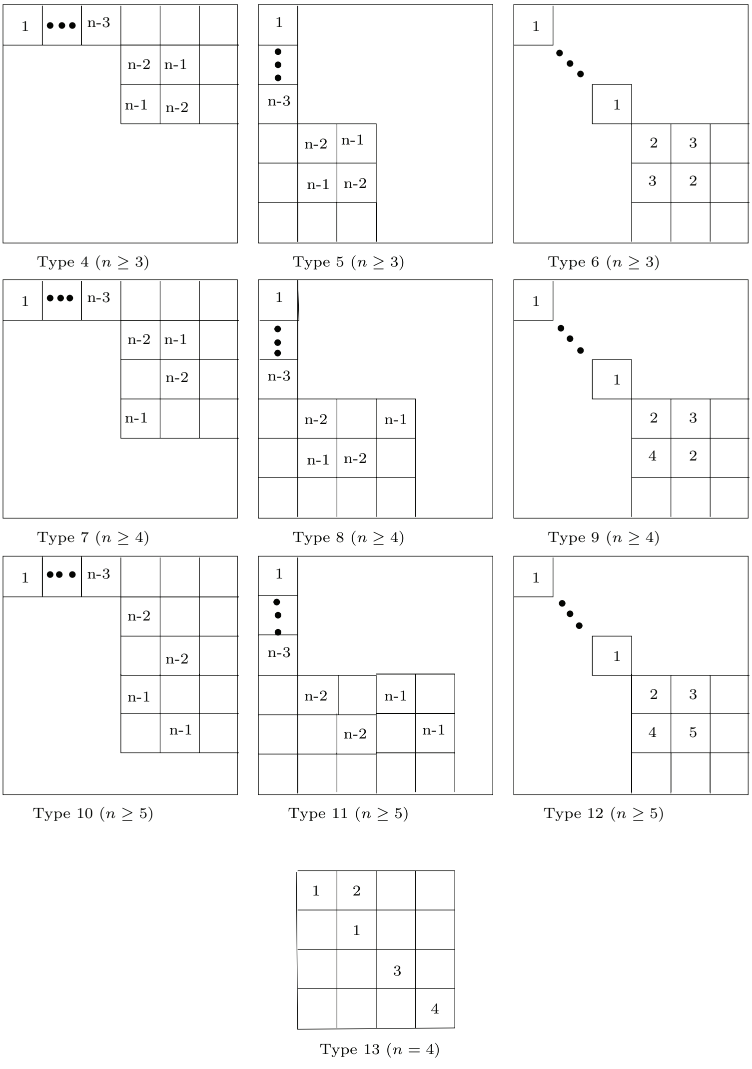

Figure 7: Noncompletable partial latin squares of side n with $n+1$ nonempty cells, other than those obtained by filling in one more cell in the instances in Figure 6.

give $V(H)$ as follows, skipping redundant types, and leave the verification that Hall's condition is violated to the reader.

Type 1: $V(H) = \{v(1, x+1)\}$.

Type 2: $V(H) = \{v(1, j) \mid x+1 \le j \le n\}$.

Type 4: $V(H) = \{v(1, n-2), v(1, n-1)\}$.

Type 6: $V(H) = \{v(i, j) \mid n-2 \le i \le n-1, 1 \le j \le n-3\} \cup \{v(n-2, n), v(n-1, n)\}$

Type 7: Same as Type 4.

Type 9: Same as Type 6.

Type 10: Same as Type 4.

Type 12: Same as Type 6.

Type 13: $V(H) = \{v(1, 3), v(1, 4), v(2, 3), v(2, 4)\}$. □

Can $n+2$ replace $n+1$ in Corollary 2? What is the largest integer $f(n)$ such that $n + f(n)$ can replace $n+1$ in Corollary 2?

3.2 Ryser's theorem

Suppose that $1 \le r,\ s \le n$ and that $R = \{v(i, j) \mid 1 \le i \le r, 1 \le j \le s\}$, the $r \times s$ rectangle in the upper left of $G = K_n \square K_n$. Suppose that R is filled in with symbols from $\{1, \ldots, n\}$ so that it is a latin rectangle —and so G becomes a partial latin square of order n, with R as the set of prescribed cells. For each $i \in \{1, \ldots, n\}$, let $N_R(i)$ be the number of occurrences of i in R.

Ryser's Theorem [19] *A partial latin square of order n with prescribed cell set R can be completed if and only if, for each $i \in \{1, \ldots, n\}$, $N_R(i) \ge r + s - n$.*

Restatement *A partial latin square with prescribed cell set R can be completed if, and only if, the Hall inequality $(*)$ holds when $V(H) = [v(i, j) \mid 1 \le i \le r, s+1 \le j \le n\}$.*

[In this restatement and in all to follow, it is to be understood that the L in inequality $(*)$ is the list assignment L_φ associated with the partial proper n-coloring φ constituting the partial latin square, and C is any set containing $\{1, \ldots, n\}$.]

Proof: The necessity of $(*)$ for completability follows from the necessity of Hall's condition for a list coloring. Now suppose that $(*)$ holds, for this choice of H. For each $i \in \{1, \ldots, n\}$, $V(H_i)$ is an $(r - N_R(i)) \times (n-s)$ subrectangle of $V(H)$, because i is available on the list of every cell of H not in a row where i appears in R, and only on the lists of such cells. Therefore, $\alpha(H_i) = \min(r - N_R(i), n-s) = \alpha(K_{r-N_R(i)} \square K_{n-s})$. Applying $(*)$,

$$\begin{aligned}|V(H)| \;=\; (n-s)r \;&\leq \sum_{i=1}^{n} \alpha(H_i)\\ &\leq \sum_{i=1}^{n}(r - N_R(i))\\ &= nr - \sum_{i=1}^{n} N_R(i)\\ &= nr - |R| = nr - sr.\end{aligned}$$

Equality implies termwise equality in the second inequality in the chain; that is, for each $i \in \{1, \ldots, n\}$, $\min(r - N_R(i), n-s) = \alpha(H_i) = r - N_R(i)$, so $r - N_R(i) \leq n - s$. Then a completion exists, by Ryser's theorem. $\square$

In [6] and in [13] it was shown that the conditions for completability in Ryser's theorem could be replaced by a single inequality of type $(*)$: in [6], $H = K_r \square K_n$, the first r rows of the array, and in [13], $H = K_n \square K_n$. (In [13], the reformulation of Ryser's theorem preceded the formulation of Hall's condition, and it was not realized that the inequality in the reformulation was an instance of one of the inequalities constituting Hall's condition.) Although the reformulation here is not much different from that in [6], we like it as a warm-up—it is a lot easier than what is to come, but bears a great family resemblance to 3 of the remaining 4 restatements.

As mentioned previously, in [6] it is shown that a subrectangle minus one cell is Hall$(n, K_n \square K_n)$ easy. If the set of prescribed cells is $R \backslash \{v(1,s)\}$, R as above, then satisfaction of $(*)$ for 3 choices of H implies completability. These 3 can be: (i) $H_1 =$ the copy of $K_r \square K_{n-s}$ in the upper right of $K_n \square K_n$, as in the reformulation of Ryser's theorem; (ii) the single vertex $v(1,s)$; and (iii) $H_1 \cup v(1,s)$.

In [16] will be found much more on Hall$(n, K_n \square K_n)$ easy subsets of subrectangles of $K_n \square K_n$.

3.3 A theorem of Buchanan and Ferencak

In the theorem referred to (in [8]), the prescribed cells of a p.l.s. of order n occupy the first r rows of the $n \times n$ array and the first d columns of row $r+1$. In [8] it is proven that such a p.l.s. is completable if and only if there do not exist sets X of columns and $\sum \subseteq \{1, \ldots, n\}$ such that

(i) the d prescribed symbols in the d filled cells of row $r+1$ are in $\sum$;

(ii) X is among the columns numbered $d+1, \ldots, n$;

(iii) every $\sigma \in \{1, \ldots, n\} \backslash \sum$ appears in each column of X, in the first r rows; and

(iv) $|X| > |\sum| - d$.

Observe that, in view of Theorem MH, for partial latin squares of the sort featured in this theorem, there is a completion if and only if the last $n-d$ cells in row $r+1$ can be properly colored from the list assignment induced by the specified cells. Since these cells, as vertices in $K_n\Box K_n$, induce a clique, K_{n-d}, it follows, from P. Hall's theorem, or Theorem HJW, that there is a completion if and only if the K_{n-d} and its list assignment satisfy Hall's condition. That is, it is necessary and sufficient for completion that the $2^{n-d}-1$ instances of the Hall inequality $(*)$ corresponding to the non-empty subsets of $\{v(r+1,d+1),\dots,v(r+1,n)\}$ hold.

It is beside our point, which is that a set of rows together with part of another row in $K_n\Box K_n$ is necessarily Hall$(n,K_n\Box K_n)$ easy, with a relatively small collection of instances of the Hall inequality $(*)$ sufficient to imply completability, but it bears mention that the necessary and sufficient conditions for completability given by Buchanan and Ferencak boil down to the $2^{n-d}-1$ instances of $(*)$ referred to above. To see this it suffices to observe that any set X of columns among the last $n-d$ columns corresponds to a set $S\subseteq\{v(r+1,d+1),\dots,v(r+1,n)\}$, namely the set of intersections of the columns of X with row $r+1$, and if X and $\sum$ satisfy (i) - (iii) then $\bigcup_{u\in S}L_\varphi(u)\subseteq\sum\setminus D$, where D is the set of symbols appearing in cells $v(r+1,1),\dots,v(r+1,d)$. Conversely, given S, let X be the set of columns of the cells in S and let $\sum=(\bigcup_{u\in S}L_\varphi(u))\cup D$. Then $\sum$ and X satisfy (i) – (iii).

3.4 Another theorem of Buchanan and Ferencak

In this theorem, also in [8], the prescribed set of cells is the upper right $r\times(n-d)$ rectangle R together with the set Y of the first d cells in row $r+1$. The situation is depicted in Figure 8.

For $i\in\{1\dots,n\}$, let $N_R(i)$ denote the number of occurrences of i in R. It is shown in [8] that the conjunction of the following three conditions is necessary and sufficient for completability, in these circumstances.

1. There do not exist X and $\sum$ as described in the preceding section, satisfying (i) – (iv).
2. For each $i\in\{1,\dots,n\}, N_R(i)\geq r-d$.
3. If $N_R(i)=r-d$, then i does not appear in Y.

Restatement *If the prescribed part of a partial latin square of order n is $R\cup Y$, as depicted in Figure 8, then the satisfying of the Hall inequalities $(*)$ when H is each of the $2^{n-d}-1$ cliques induced by non-empty subsets of $\{v(r+1,d+1),\dots v(r+1,n)\}$, and when H is the copy of $K_r\Box K_d$ depicted in Figure 8, is necessary and sufficient for completability.*

Proof: Necessity follows from the necessity of Hall's condition for a proper coloring from lists.

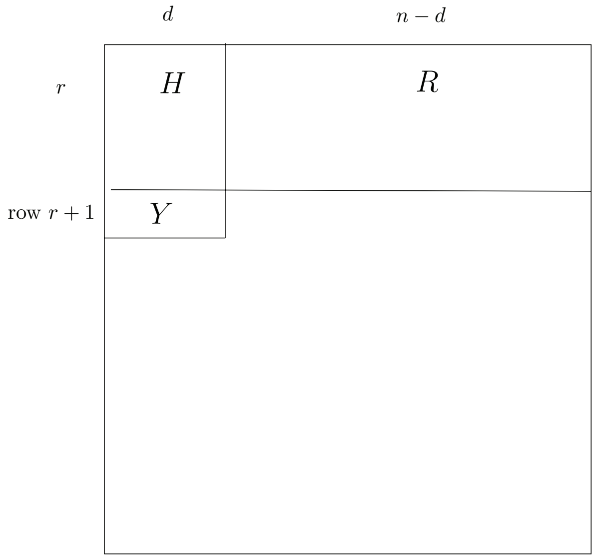

Figure 8: $\{\text{prescribed cells}\} = R \cup Y$

As indicated in Section 3.3 (even though the situation here is slightly different from the situation there, the explanation there still works), condition 1 of Buchanan and Ferencak here is equivalent to $(*)$ holding for each of the cliques induced by subsets of $\{v(r+1, d+1), \ldots, v(r+1, n)\}$. So to complete the proof of sufficiency in the restatement, it suffices to show that the instance of the Hall inequality $(*)$ for the H in Figure 8 implies Buchanan and Ferencak's conditions 2 and 3. So, suppose that $(*)$ holds for that H.

If $i \in \{1, \ldots, n\}$ then H_i is a subrectangle of H of dimensions either $(r - N_R(i)) \times d$, if i does not appear in Y, or $(r - N_R(i)) \times (d-1)$, if $i \in D$, the set of symbols appearing in Y. Therefore, applying the Hall inequality $(*)$, $rd = |V(H)| \leq \sum_{i=1}^{n} \alpha(H_i) \leq \sum_{i=1}^{n} (r - N_R(i)) = rn - \sum_{i=1}^{n} N_R(i) = rn - r(n-d) = rd$. Equality implies termwise equality in the second inequality in the string; that is, $\alpha(H_i) = r - N_R(i)$ for each $i = 1, \ldots, n$. Therefore,

$$\begin{aligned} r - N_R(i) &\leq d, \quad \text{if } i \neq D, \text{ and} \\ r - N_R(i) &\leq d-1, \text{ if } i \in D. \end{aligned}$$

These inequalities, together, restate Buchanan and Ferencak's conditions 2 and 3. □

Corollary 3 *Suppose that the prescribed cells in a partial latin square of order n consist of an $r \times (n-d)$ rectangle in the upper right part of the square, together with part of the $r + 1^{st}$ row, including the first d cells. The p.l.s. can be completed if and only if the Hall inequality $(*)$ holds (with $L = L_\varphi$, as usual) with H ranging* *over all of the cliques induced by sets of unprescribed cells in row $r+1$, and with H being the $K_r \square K_d$ in th upper left part of the square.*

Proof: By Hall's theorem, or Theorem HJW, the hypothesis implies that row $r+1$ can be properly completed. Now consider the partial latin square obtained by blanking out all entries in $v(r+1, d+1), \ldots, v(r+1, n)$; that is, these cells are unprescribed, in the new p.l.s., which fits the form treated in Buchanan and Ferencak's second theorem, the subject of this subsection. Then the fact that row $r+1$ could be properly completed, in this situation, and the fact that the lists on H, the upper left $K_r \square K_d$, are the same in this new p.l.s. as in the one we started with, give that the new p.l.s. satisfies the hypothesis of the restatement of Buchanan and Ferencak's theorem, and so is completable. Therefore, the new p.l.s. is also completable to an $(r+1) \times n$ latin rectangle on n symbols, and therefore, since the filling in of H places no restraints on the filling in of the last $n-d$ entries of row $r+1$, it follows that the original p.l.s. is completable to an $(r+1) \times n$ latin rectangle on n symbols. Therefore, by Theorem MH, the original p.l.s. is completable. □

3.5 Another theorem of Andersen and Hilton

In this theorem the prescribed cells of the p.l.s. are, for some r, $s \in \{1, \ldots, n-1\}$, say with $r \leq s$, and $t \in \{1, \ldots, n-s\}$,

$$\{v(i,j) \mid 1 \leq i \leq r, 1 \leq j \leq s\} \cup \{v(r+i, s+i) \mid 1 \leq i \leq t\},$$

with the additional proviso that if $r = s$ then $t \leq n-s-1$. See Figure 9.

Let R be the $r \times s$ rectangle shown in Figure 9, and let H be the copy of $K_r \square K_{n-s}$, depicted in Figure 9, with vertices in the unprescribed area to the right of R. For $i \in \{1, \ldots, n\}$, let $N_R(i)$ denote the number of occurrences of i in R, and let $f(i)$ denote the number of occurrences of i in the cells $v(r+j, s+j)$, $1 \leq j \leq t$. Clearly $\sum_{i=1}^{n} f(i) = t$.

It is proven in [4] that such a partial latin square is completable if and only if, for each $i \in \{1, \ldots, n\}$,

$$N_R(i) \geq r + s - n + f(i).$$

Restatement *In the circumstances described, the p.l.s. is completable if and only if the instance of the Hall inequality $(*)$ associated with H holds.*

Proof: Again, the necessity is automatic. For the sufficiency, suppose that $(*)$ holds for this particular H. For each $i \in \{1, \ldots, n\}$, H_i is an $(r - N_R(i)) \times (n - s - f(i))$ subrectangle of H. Therefore, invoking the Hall inequality $(*)$,

$$\begin{aligned} r(n-s) \; = \; |V(H)| \; &\leq \; \textstyle\sum_{i=1}^{n} \alpha(H_i) \\ &\leq \; \textstyle\sum_{i=1}^{n} (r - N_R(i)) \\ &= \; rn - \textstyle\sum_{i=1}^{n} N_R(i) \; = \; rn - rs. \end{aligned}$$

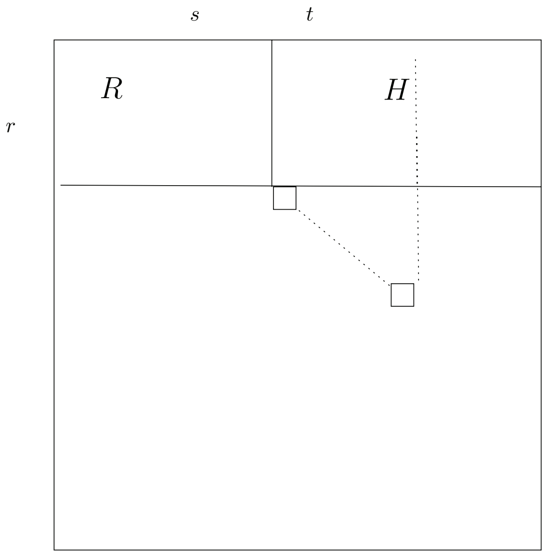

Figure 9: Prescribed cells in Andersen and Hilton's theorem: $R \cup \{$"diagonal" cells$\}$

Equality throughout implies termwise equality in the second inequality above;

$$r - N_R(i) = \alpha(H_i) = \min[r - N_R(i), n - s - f(i)]$$

implies $r - N_R(i) \leq n - s - f(i)$, for each i, which implies completability, by Andersen and Hilton's theorem. □

3.6 A theorem of Rodger

This theorem, in [18], deals with the missing case in Andersen and Hilton's theorem: $r = s$ and $t = n - s$. See Figure 10, and let D be as defined there.

Let N_R and f be defined as in the preceding section. Rodger [18] proved that a partial latin square with such a domain can be completed if and only if each of the following conditions hold:

- R1. For each $i \in \{1, \ldots, n\}$, $N_R(i) \geq 2r - n + f(i)$.
- R2. For each $i \in \{1, \ldots, n\}$, if $N_R(i) = r$ then $f(i) \neq n - r - 1$.
- R3. If R is a latin square on the symbols $1, \ldots, r$ and $n = 2r+1$, then $\sum_{i=1}^{r} f(i) \neq 1$.

(Actually, Rodger proved this for $n \geq 10$. Strangely, the cases $n < 10$ proved resistant, and were finally dispatched by Abueida [1].)

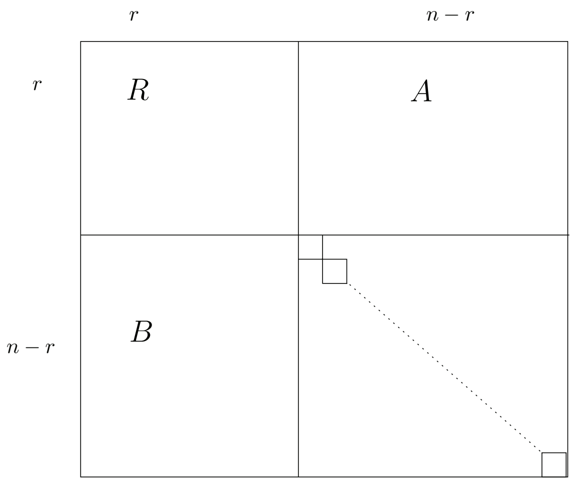

Figure 10: The domain of the p.l.s. in Rodger's theorem is $R \cup D$, $D = \{v(j,j) \mid r+1 \le j \le n\}$.

Restatement *If a partial latin square of order n has domain $R \cup D$, as depicted in Figure 10, then it can be completed to a latin square of order n if and only if inequality $(*)$ holds for each of the following choices of H:*

R(i) *The copy of $K_r \square K_{n-r}$ induced by the cells $A = \{v(i,j) | 1 \le i \le r, r+1 \le j \le n\}$.* [The rectangle B in Figure 8 could be substituted for A in R(i).]

R(ii) *The clique induced by the cells in column j*, for each $j = r+1, \dots, n$. [Rows could be substituted for columns, and, in each case, the j^{th} column or row minus $v(j,j)$ could be substituted for the full line.]

R(iii) *For each $k = r+1, \dots, n$, the subgraph $H^{(k)}$ induced by $A \cup B \cup (k^{th}$ row$) \cup (k^{th}$ column).* [$H^{(k)} - v(k,k)$ could be substituted for $H^{(k)}$.]

Proof: As usual, we need only prove sufficiency.

By the proof in the preceding section, the satisfying of $(*)$ when H is the subgraph induced by A implies R1, which is just the condition in Andersen and Hilton's theorem in the case $r = s$, $t = n - r$.

Suppose R2 does not hold. Then for some $i \in \{1, \dots, n\}$, $N_R(i) = r$ and $f(i) = n - r - 1$. Then i is the prescribed symbol in every cell of D except one; say $v(j,j)$ has prescribed symbol $\varphi(v(j,j)) = k \ne i$. Because i is in every row of R, and every cell of D except $v(j,j)$, i does not appear on the list of any cell in the j^{th} column of the array. Therefore, if H is the clique induced by the cells of that column,

$$\sum_{z=1}^{n} \alpha(H_z) \;\le\; n-1 \;<\; n \;=\; |V(H)|,$$

i.e., $(*)$ does not hold. Therefore, if the Hall inequality $(*)$ does hold for every clique induced by a column of the array among the last $n-r$ columns, then R2 holds.

Now suppose that R3 is violated. Then R is a latin square on the symbols $1,\ldots,r$, $n=2r+1$, and $\sum_{i=1}^{r} f(i)=1$. Then exactly one of $1,\ldots,r$ appears on D, and it appears exactly once. Without loss of generality, we may suppose that $1=\varphi(v(k,k))$, i.e., 1 appears on $v(k,k)$, and neither 1 nor any other symbol in $\{1,\ldots,r\}$ appears anywhere else on D. We will show that the Hall inequality $(*)$ does not hold with $H=H^{(k)}$, which will finish the proof of the restatement.

$$\begin{aligned}|V(H^{(k)})| &= 2r(n-r)+2(n-r)-1\\ &= 2r(r+1)+2r+1,\end{aligned}$$

since $n=2r+1$.

To simplify notation, let $\alpha(H_i^{(k)})=\alpha(i)$, $i=1,\ldots,n$. Clearly $\alpha(1)=1$ and $\alpha(j)=2$, $j=2,\ldots,r$.

If $j\in\{r+1,\ldots,2r+1\}$, then j appears on the lists of the cells in an $r\times(n-r-f(j))$, i.e., $r\times(r+1-f(j))$, subrectangle of A, in an $(r+1-f(j))\times r$ subrectangle of B, and in $r-f(j)$ cells of row k outside B, and in $r-f(j)$ cells of column k outside A. It makes it easier to estimate $\alpha(j)$ in this case if we imagine that $k=r+1$. (In fact, we could arrange for $k=r+1$ by permuting rows and columns.) Then $H_j^{(k)}$ is contained in the union of two rectangles, one of dimensions $(r+1)\times(r+1-f(j))$ and the other of dimensions $(r+1-f(j))\times(r+1)$. Then $\alpha(j)\le 2(r+1-f(j))$. If $f(j)=0$, then $\alpha(j)\le 2r+1$, because we are considering a $(2r+1)\times(2r+1)$ array.

Let $S_1=\{j\in\{r+1,\ldots,2r+1\}\mid f(j)=0\}$ and $S_2=\{r+1,\ldots,2r+1\}\backslash S_1$. Then $\sum_{j\in S_2} f(j)=n-r-1=r$, so $|S_2|\le r$. We have

$$\begin{aligned}\sum_{j=1}^{n}\alpha(j) &= 1+2(r-1)+\sum_{j\in S_1}\alpha(j)+\sum_{j\in S_2}\alpha(j)\\ &\le 2r-1+(2r+1)|S_1|+2\sum_{j\in S_2}(r+1-f(j))\\ &= 2r-1+(2r+1)|S_1|+2(r+1)|S_2|-2\sum_{j\in S_2} f(j)\\ &= 2r-1+(2r+1)(|S_1|+|S_2|)+|S_2|-2r\\ &= -1+(2r+1)(r+1)+|S_2|\\ &\le 2r^2+3r+r=2r(r+1)+2r<2r(r+1)+2r+1=|V(H^{(k)})|.\end{aligned}$$

Regarding the alternatives proposed in brackets in R(i), R(ii), and R(iii): clearly rows can play the roles of columns, and B the role of A. Deleting any $v(j,j)$ from H reduces both sides of $(*)$ by 1. □

Corollary 4 (A corollary of the proof above) *Suppose, for a p.l.s. with domain $R\cup D$, as in Rodger's theorem, the hypothesis of R3 does not hold: that is, either $n\neq 2r+1$ or $n=2r+1$ but R, with its prescription, is not a latin square of order r. Then if the Hall inequality $(*)$ holds for the $n-r+1$ instances of H described in R(i) and R(ii), the p.l.s. is completable to a latin square of order n.*

4 A theorem of Andersen and Hoffman

A commutative latin square is a latin square which is symmetric, as a matrix. So, a partial commutative latin square of order n is an assignment to a symmetric set of cells, in an $n \times n$ array, of n symbols, say $1, \ldots, n$, so that symmetric cells are assigned the same symbol and no symbol appears more than once in any row or column.

A commutative latin square of order n can be regarded as a proper vertex coloring, with n colors, of a graph $G(n)$, to be defined shortly. Moreover a partial proper n-coloring of $G(n)$ corresponds to a partial commutative latin square. To define $G(n)$, $n = 1, 2, \ldots$, let $V_n = \{v(i,j) \mid 1 \le i \le j \le n\}$, the set of cells on and above the main diagonal in the $n \times n$ array that usually represents $K_n \square K_n$. V_n is the vertex set of $G(n)$. The edge set of $G(n)$ contains all the edges from $K_n \square K_n$ induced in that graph by V_n, and also contains other edges which reflect the symmetry of any commutative latin square that will arise from a proper n-coloring of $G(n)$. In a commutative latin square, if a symbol σ is placed in a cell $v(i,j)$ with $i < j$, then σ cannot appear elsewhere in row i or column j; but σ also appears in $v(j,i)$, so σ cannot appear in $v(j,t)$, $t \neq i$, nor in $v(t,i)$, $t \neq j$. If we declare an edge in $G(n)$ between vertex $v(i,j)$ and $v(j,k)$ whenever $i < j < k$, then for every commutative latin square of order n, the symbols on V_n will constitute a proper n-coloring of $G(n)$, and, conversely, every proper n-coloring of $G(n)$ determines a commutative latin square of order n.

The extra edges of $G(n)$ which are not in $K_n \square K_n$ are illustrated in Figure 11 in the case $n = 5$.

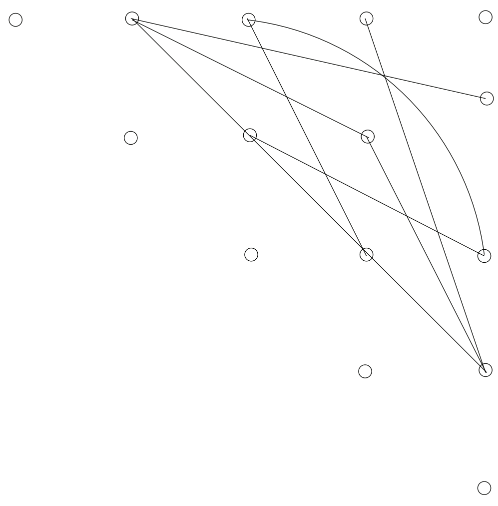

Figure 11: The edges of $G(5)$ which are not also in $K_5 \square K_5$

Suppose L is a list assignment to $G(n)$, and $S \subseteq V_n$. If the Hall inequality $(*)$

holds with H being the subgraph of $G(n)$ induced by S, then also $(*)$ holds with H being the subgraph of $K_n \Box K_n$ induced by S, because the independence numbers in the graph with more edges are no greater than those in the graph with fewer edges.

Suppose that we have a partial commutative latin square which, were it to be considered a p.l.s. (with symmetric prescription), would have domain $R \cup D$, as in Figure 10—i.e., as in Rodger's theorem, of Subsection 3.6. For $i \in \{1, \ldots, n\}$, let $N_R(i)$ and $f(i)$ be as in that subsection, and let $g(i)$ be the total number of appearances of i on the main diagonal of R, so that $d(i) = f(i) + g(i)$ is the total number of appearances of i on the main diagonal of the $n \times n$ array. A theorem proven independently by Andersen [2] and Hoffman [17] asserts that this partial commutative latin square can be completed to a commutative latin square of order n if and only if:

For each $i \in \{1, \ldots, n\}$,
AH(1) $N_R(i) \geq 2r - n + f(i)$, and
AH(2) $d(i) \equiv n \mod 2$.

Proof that AH(1) and AH(2) are equivalent to $(*)$ holding for a small number of choices of H, with $G = G(n)$:

Obviously AH(1) and R1, in Rodger's theorem, Subsection 3.6, are the same, and it is shown in 3.6, by reference to 3.5, that that condition is implied by a single instance of $(*)$, the inequality in Hall's condition with reference to $K_n \Box K_n$, with H being the subgraph induced by the $r \times (n - r)$ rectangle A shown in Figure 10. By remarks above, if the Hall inequality $(*)$ holds with H being the subgraph of $G(n)$ induced by A, then it holds with respect to $K_n \Box K_n$, and so AH(1) holds.

Now we shall show that a single instance of the Hall inequality (*) implies AH(2), and thus that this theorem of Andersen and Hoffman can be restated as the other theorems have been restated. The only difference is that the underlying graph here is $G(n)$, not $K_n \Box K_n$.

Let the vertices (cells) $v(i, j)$, $1 \leq i \leq j \leq r$, and $v(i, i)$, $r + 1 \leq i \leq n$ have prescribed colors from $\{1, \ldots, n\}$, so that no cells adjacent in $G(n)$ have the same color, and let L be the list assignment to $G(n)$ induced by this partial proper coloring. Let $M(n) = M = \{v(i, i) \mid 1 \leq i \leq n\}$, the "main diagonal" of $G(n)$.

We shall show that if the Hall inequality (*) holds for L and $H = H(n) = G(n) - M(n)$, then condition AH(2) must hold. Suppose that $(*)$ holds for L and H while AH(2) does not hold.

Let L' denote the list assignment to $G(n)$ induced just by the diagonal cells, i.e. the cells of M. Since $L(v) \subseteq L'(v)$ for all $v \in V(G(n))$, $\sum_{i=1}^{n} \alpha(H(i, L)) \leq \sum_{i=1}^{n} \alpha(H(i, L'))$, so if the inequality (*) fails to hold for H and L', then it fails for H and L, and we have a contradiction. For each $i \in \{1, \ldots, n\}$, we will see that

$$\begin{aligned} \alpha(H(i, L')) &= \left\lfloor \tfrac{n-d(i)}{2} \right\rfloor \\ &= \begin{cases} \frac{n-d(i)}{2} & \text{if } n \equiv d(i) \bmod 2 \\ \frac{n-d(i)-1}{2} & \text{if } n \not\equiv d(i) \bmod 2 \end{cases} \end{aligned}$$

and this will imply that the inequality (*) fails for H and L'. For, letting $b > 0$ be the number of "bad" symbols, i.e. symbols i satisfying $d(i) \not\equiv n \bmod 2$, we would have

$$\begin{aligned} \sum_{i=1}^{n} \alpha(H(i,L')) &= \sum_{i=1}^{n} \tfrac{n-d(i)}{2} - \tfrac{b}{2} \\ &= \tfrac{n^2-n}{2} - \tfrac{b}{2} \\ &= |V(H)| - \tfrac{b}{2} < |V(H)|. \end{aligned}$$

Suppose $i \in \{1, \ldots, n\}$. Applying an automorphism of $G(n)$, we can assume that i occupies the last, i.e. the bottom right, $d(i)$ cells on M. Then $H(i, L')$ is induced by $\{v(i,j) \mid 1 \le i < j \le n - d(i)\}$. That is, $H(i, L') \simeq H(n - d(i)) = G(n - d(i)) - M(n - d(i))$. So, to finish the proof, it suffices to show that for each positive integer z, $\alpha(H(z)) = \lfloor \frac{z}{2} \rfloor$. (Actually, it suffices to show that $\alpha(H(z)) \le \lfloor \frac{z}{2} \rfloor$, but it does not hurt to prove the equality.)

Suppose that $v(i_1, j_1), \ldots, v(i_k, j_k)$ are distinct vertices in $V(H(z)) = \{v(i,j) \mid 1 \le i < j \le z\}$, no two adjacent. By the way adjacency is defined in $G(z)$, $i_1, \ldots, i_k$ must be distinct, $j_1, \ldots, j_k$ must be distinct, and no i_t is equal to any j_r. [For, if $i_t = j_r$ then $i_r < j_r = i_t < j_t$ and $v(i_t, j_t)$, $v(i_r, j_r)$ are adjacent.] Therefore, $i_1, \ldots, i_k$, $j_1, \ldots, j_k$ are $2k$ distinct integers in $\{1, \ldots, z\}$. Therefore $k \le \lfloor \frac{z}{2} \rfloor$; therefore, $\alpha(H(z)) \le \lfloor \frac{z}{2} \rfloor$.

To show equality, we exhibit an independent set of $\lfloor \frac{z}{2} \rfloor$ vertices in $H(z)$. Start in the upper right hand corner of the triangular array $V(H(z))$ and move along the back diagonal toward $M(z)$; the cells in the set are $v(1,z), v(2, z-1), \ldots, v(\alpha, z - \alpha + 1)$, where α satisfies either $\alpha + 1 = (z - \alpha + 1) - 1$ in which case $\alpha = \frac{z-1}{2} = \lfloor \frac{z}{2} \rfloor$, or $\alpha = (z - \alpha + 1) - 1$ in which case $\alpha = \frac{z}{2} = \lfloor \frac{z}{2} \rfloor$. □

Restatement of the theorem of Andersen and Hoffman

If a partial commutative latin square of order n has domain $R \cup D$, as depicted in Figure 10, then it can be completed to a commutative latin square of order n if and only if the Hall inequality $(*)$ holds for the following two choices of subgraphs H of $G = G(n)$:

(i) The subgraph induced by the cells $A = \{v(i,j) \mid 1 \le i \le r, r+1 \le j \le n\}$;

(ii) $H = G(n) - M(n)$.

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