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On set colorings of complete bipartite graphs

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Abstract

In *European J. Combin.* **30** (2009), 986–995, S. M. Hegde recently introduced *set colorings* of a graph G as an assignment (function) of distinct subsets of a finite set X of *colors* to the vertices of G, where the colors of the edges are obtained as the symmetric difference of the sets assigned to their end vertices (which are also distinct). A set coloring is called a *proper set coloring* if all the nonempty subsets of X are obtained on the edges. A graph is called *properly set colorable* if it admits a proper set coloring.

In this paper we give a proof for Hegde's conjecture that the complete bipartite graph $K_{a,b}$ is properly set colorable if and only if one of the partition sets is of cardinality 1, and the other one of cardinality $2^n - 1$ for some positive integer n.

1 Terminology and introduction

In this paper we consider finite and simple graphs only. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let X be a nonempty

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set of colors and 2^X denote the power set of X. For any two subsets $Y, Z \subseteq X$ let $Y \oplus Z = (Y \cup Z) \setminus (Y \cap Z)$ denote the symmetric difference of Y and Z.

Given a function $f : V(G) \to 2^X$ we define $f^{\oplus} : E(G) \to 2^X$ by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for all edges $uv \in E(G)$. We call f a *set coloring* of G if both f and $f^{\oplus}$ are injective functions. A graph is called *set colorable* if it admits a set coloring. A set coloring f is called a *proper set coloring* if $f^{\oplus}(E(G)) = \{f^{\oplus}(e) | e \in E(G)\} = 2^X \setminus \{\emptyset\}$. If a graph admits such a set coloring where all subsets of X except the empty set are obtained on the edges, then G is called *properly set colorable.*

$K_{a,b}$ denotes a complete bipartite graph with a and b being the cardinality of the partition sets $A, B \subset V(G)$, respectively.

An earlier approach to distinguish the edges of a graph by the colors of their adjacent vertices is due to Frank, Harary and Plantholt [2], who introduced the line distinguishing chromatic number of a graph in 1982. Another approach is due to Hopcroft and Krishnamoorthy [4], who established the common notion of harmonious colorings in 1983. Among others Zhang, Liu and Wang [5], and Balister, Riordan and Schelp [1] studied edge colorings of graphs, where the vertices are distinguishable by the set of colors on their incident edges.

The concept of set colorings was recently introduced by S. M. Hegde in [3], and describes a way to distinguish edges of a graph by subsets from the color set X. In this paper we will prove his conjecture on proper set colorability of complete bipartite graphs. Therefore, a graph $K_{a,b}$ without loss of generality with $a \leq b$ is supposably properly set colorable if and only if it satisfies $a = 1$ and $b = 2^n - 1$ for some positive integer n.

Before we prove our result we need some properties and terminology of set colorings of complete bipartite graphs. Therefore, from now on we will consider G to be a complete bipartite graph $K_{a,b}$ with a proper set coloring f. A necessary condition for f to be a proper set coloring of G with the color set $X = \{c_0, \ldots, c_{n-1}\}$ of order $n = |X|$ is

$$a \cdot b = |2^X| - 1 = 2^n - 1. \tag{1}$$

We define a binary representation $binary(i)$ of a non-negative integer i by the bijective function

$$binary : \mathbb{N}_0 \to \{(d_m)_{m \in \mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0} \mid \exists m_0 \in \mathbb{N}\ \forall m > m_0 : d_m = 0\},$$

$$i = \sum_{m=0}^{\infty} 2^m \cdot \underbrace{d_m[i]}_{\in \{0,1\}} \mapsto (d_m[i])_{m \in \mathbb{N}_0}.$$

Using this definition, we introduce the symmetric difference between non-negative integers as a bitwise exclusive or (XOR) of their binary representations

$$\oplus : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, i \oplus j = k \iff d_m[k] = d_m[i] + d_m[j]\ (modulo\ 2)\ \forall m \in \mathbb{N}_0.$$

Of course, this symmetric difference between two non-negative integers i and j will in general differ from their sum and/or difference ($i+j \neq i \oplus j \neq i-j$), but we have the properties:

$$\begin{array}{l} \oplus \text{ is commutative,} \\ i \oplus j \in \{0,\ldots,2^n-1\} \;\forall i,j \in \{0,\ldots,2^n-1\}, \end{array}$$

and

$$i \oplus 2^{n-1} = \begin{cases} i+2^{n-1}, & \text{if } 0 \leq i \leq 2^{n-1}-1 \\ i-2^{n-1}, & \text{if } 2^{n-1} \leq i \leq 2^n-1 \end{cases}. \tag{2}$$

Furthermore, the binary representation induces a natural correspondence between the non-negative integers less than 2^n and the subsets of $X = \{c_0,\ldots,c_{n-1}\}$ according to the bijective function

$$\mathit{subset} : \{0,\ldots,2^n-1\} \to 2^X, i \mapsto \{c_m \mid 0 \leq m \leq n-1, d_m[i] = 1\}.$$

Then

$$i \oplus j = k \quad \Leftrightarrow \quad \mathit{subset}(i) \oplus \mathit{subset}(j) = \mathit{subset}(k) \;\forall i,j \in \{0,\ldots,2^n-1\},$$

justifying our definition of the symmetric difference between non-negative integers. Let

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^n-1} \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2^n-1} \end{pmatrix}$$

be the two vectors of order 2^n representing the color sets assigned to the partition sets A and B by the set coloring f in the sense that $a_j = 1$ if $\mathit{subset}(j) \in f(A)$ and $a_j = 0$ otherwise. Since a set coloring is injective, we have $a_j + b_j \leq 1$ for all $0 \leq j \leq 2^n - 1$,

$$\sum_{j=0}^{2^n-1} a_j = a \quad\text{and}\quad \sum_{j=0}^{2^n-1} b_j = b. \tag{3}$$

An important part of the proof of our main result Theorem 2.1 involves the set of vectors U_n defined by

$$\begin{array}{rcl} U_1 &=& \{(1,-1)\} \text{ and} \\ U_n &=& \{(u_0,\ldots,u_{2^{n-1}-1},u_0,\ldots,u_{2^{n-1}-1}) \mid \vec{u} = (u_0,\ldots,u_{2^{n-1}-1}) \in U_{n-1}\} \\ &\cup& \{(u_0,\ldots,u_{2^{n-1}-1},-u_0,\ldots,-u_{2^{n-1}-1}) \mid \vec{u} = (u_0,\ldots,u_{2^{n-1}-1}) \in U_{n-1}\} \\ &\cup& \{(\underbrace{1,\ldots,1}_{2^{n-1}},\underbrace{-1,\ldots,-1}_{2^{n-1}})\} \text{ for } n \geq 2. \end{array}$$

It is clear that $|U_n| = 2^n - 1$ and each vector in U_n is of length 2^n. By definition the vectors $\vec{u} \in U_n$ have the following useful properties:

$$\vec{u} \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = -1 \tag{4}$$

and

$$\sum_{\vec{u} \in U_n} \vec{u} = (2^n - 1, \underbrace{-1, \dots, -1}_{2^n - 1}). \tag{5}$$

Since each vector $\vec{u} \in U_n$ has an equal number of $+1$ and -1 entries, but always with a $+1$ in the first component, it is trivial to see (4). To see (5), use induction on n using the recursive definition of U_n.

However, the most important and non-trivial property of these vectors is

$$(\vec{u} \cdot \vec{a}) + (\vec{u} \cdot \vec{b}) = 0, \tag{6}$$

which holds for all vectors $\vec{u} \in U_n$ and $\vec{a}, \vec{b}$ being the vectors representing the proper set coloring of the complete bipartite graph $K_{a,b}$ from above. With these properties it is straightforward to prove Theorem 2.1 in the next section, and we will show the more technical proof of (6) in Proposition 2.2 right after the proof of our main result.

2 Main Result

Theorem 2.1. *A graph $K_{a,b}$ without loss of generality with $a \leq b$ is properly set colorable if and only if it is $a = 1$ and $b = 2^n - 1$ for some positive integer n.*

Proof. Let $\vec{a}$ and $\vec{b}$ be the vectors representing the proper set coloring f of the partite sets A and B in the sense that $a_j = 1$ if $subset(j) \in f(A)$, $b_j = 1$ if $subset(j) \in f(B)$ and $a_j, b_j = 0$ otherwise, and let n be the cardinality of the color set $n = |X|$. We deduce

$$\begin{aligned} 0 &= \sum_{\vec{u} \in U_n} (\vec{u} \cdot \vec{a} + \vec{u} \cdot \vec{b}) && \text{by (6)} \\ &= \Big(\sum_{\vec{u} \in U_n} \vec{u}\Big) \cdot (\vec{a} + \vec{b}) \\ &= (2^n - 1, \underbrace{-1, \dots, -1}_{2^n - 1}) \cdot (\vec{a} + \vec{b}) && \text{by (5)} \end{aligned}$$

$$= 2^n \cdot (a_0 + b_0) - \sum_{i=0}^{2^n-1} (a_i + b_i)$$
$$= 2^n \cdot (a_0 + b_0) - (a + b). \qquad \text{by } (3)$$

Thus, we have $a + b = 2^n \cdot (a_0 + b_0)$, which implies $(a_0 + b_0) = 1$ due to $a + b \neq 0$ and $a_0 + b_0 \leq 1$. By using (1) we have

$$a + b = 2^n = a \cdot b + 1,$$

leading us directly to

$$0 = a \cdot b - a - b + 1 = (a - 1) \cdot (b - 1).$$

Conversely, we obtain a proper set coloring f for the complete bipartite graph $K_{a,b}$ with $a = 1$ and $b = 2^n - 1$ for a positive integer n, if we assign the empty color set to the vertex in A, and each of the $2^n - 1$ non-empty subsets of the color set $\{c_0, \ldots, c_{n-1}\}$ to one of the vertices in B. □

We will now complete this proof by showing property (6) for the vectors in U_n. Because all values in $\vec{a}, \vec{b}$ and $\vec{u}$ are integers and, therefore, $(\vec{u} \cdot \vec{a})$ and $(\vec{u} \cdot \vec{b})$ are integers as well, the condition $(\vec{u} \cdot \vec{a}) + (\vec{u} \cdot \vec{b}) = 0$ is a direct consequence of the equation $(\vec{u} \cdot \vec{a}) \cdot (\vec{u} \cdot \vec{b}) = -1$, which is the property we now prove.

Proposition 2.2. *Let f be a proper set coloring of the complete bipartite graph $K_{a,b}$ with the color set X and the partition sets A and B with $a = |A|$ and $b = |B|$, and let the vectors $\vec{a}, \vec{b}$ represent the subsets of X in $f(A)$ and $f(B)$, respectively, such that $a_i = 1$ if $subset(i) \in f(A)$, $b_i = 1$ if $subset(i) \in f(B)$ and $a_i, b_i = 0$ otherwise for $0 \leq i \leq 2^n - 1$ with $n = |X|$. Then for all $\vec{u} \in U_n$ we have*

$$(\vec{u} \cdot \vec{a}) \cdot (\vec{u} \cdot \vec{b}) = -1.$$

Proof. We define the $(2^n \times 2^n)$-matrix B_n by

$$B_n = \begin{pmatrix} b_{0\oplus 0} & \cdots & b_{0\oplus 2^n-1} \\ \vdots & \ddots & \vdots \\ b_{2^n-1\oplus 0} & \cdots & b_{2^n-1\oplus 2^n-1} \end{pmatrix}$$

Each row and each column of the matrix B_n contains each coefficient $b_0, \ldots, b_{2^n-1}$ exactly once, and according to this construction we can express all possible ways to obtain a certain subset of X on an edge - as the symmetric difference of two sets $subset(i) \in f(A)$ and $subset(j) \in f(B)$ (i.e. $a_i = 1 \wedge b_j = 1$) - as the product of the matrix B_n and the vector $\vec{a}$:

$$B_n \cdot \vec{a} = \begin{pmatrix} b_{0\oplus 0} & \cdots & b_{0\oplus 2^n-1} \\ \vdots & \ddots & \vdots \\ b_{2^n-1\oplus 0} & \cdots & b_{2^n-1\oplus 2^n-1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_{2^n-1} \end{pmatrix}$$
$$= \begin{pmatrix} \sum\limits_{0\leq i\leq 2^n-1} a_i b_{0\oplus i} \\ \vdots \\ \sum\limits_{0\leq i\leq 2^n-1} a_i b_{2^n-1\oplus i} \end{pmatrix} \tag{7}$$

Here the sum in the first row contains all combinations of sets in A and B that could produce the empty set on an edge, because if a_i and $b_{0\oplus i}$ both have the value one, then there is an edge with the color set $subset(i) \oplus subset(0 \oplus i) = subset(i \oplus 0 \oplus i) = subset(0) = \emptyset$. Analogously, the second row describes the possibilities to produce the set $subset(1) = \{c_0\} \subseteq X$, and so on.

We will now consider the product $\vec{u} \cdot B_n$ with $\vec{u} = (u_0, \ldots, u_{2^n-1}) \in U_n$. By induction on n we show for $0 \leq k \leq 2^n - 1$ and $j \in \mathbb{N}$

$$\vec{u} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^n-1\oplus j\oplus k} \end{pmatrix} = u_k \cdot \sum_{i=0}^{2^n-1} u_i x_{i\oplus j} \text{ for any vector } \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^n-1\oplus j\oplus k} \end{pmatrix}. \tag{8}$$

For $n = 1$ we have:

Case 1. Let $k = 0$.

$$(1, -1) \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ x_{1\oplus j\oplus k} \end{pmatrix} = x_{0\oplus j\oplus 0} - x_{1\oplus j\oplus 0} = x_{0\oplus j} - x_{1\oplus j}$$

Case 2. Let $k = 1$.

$$(1, -1) \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ x_{1\oplus j\oplus k} \end{pmatrix} = x_{0\oplus j\oplus 1} - x_{1\oplus j\oplus 1} = -1 \cdot (x_{0\oplus j} - x_{1\oplus j})$$

So now let (8) be true for some $n - 1 \geq 1$.

Case 1.1. Let $\vec{u} = (v_0, \ldots, v_{2^{n-1}-1}, v_0, \ldots, v_{2^{n-1}-1})$ for $\vec{v} = (v_0, \ldots, v_{2^{n-1}-1}) \in U_{n-1}$ and $0 \leq k \leq 2^{n-1} - 1$.

$$\vec{u} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^n-1\oplus j\oplus k} \end{pmatrix} = \vec{v} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^{n-1}-1\oplus j\oplus k} \end{pmatrix} + \vec{v} \cdot \begin{pmatrix} x_{(0\oplus 2^{n-1})\oplus j\oplus k} \\ \vdots \\ x_{(2^{n-1}-1\oplus 2^{n-1})\oplus j\oplus k} \end{pmatrix} \text{ by (2)}$$

$$= \vec{v} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^{n-1}-1\oplus j\oplus k} \end{pmatrix} + \vec{v} \cdot \begin{pmatrix} x_{0\oplus(2^{n-1}\oplus j)\oplus k} \\ \vdots \\ x_{2^{n-1}-1\oplus(2^{n-1}\oplus j)\oplus k} \end{pmatrix}$$

$$= v_k \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus j} + v_k \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus(2^{n-1}\oplus j)}$$

$$= v_k \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus j} + v_k \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{(i\oplus 2^{n-1})\oplus j}$$

$$= u_k \cdot \sum_{i=0}^{2^n-1} u_i x_{i\oplus j} \qquad \text{by } (2)$$

Case 1.2. Let $\vec{u} = (v_0, \ldots, v_{2^{n-1}-1}, v_0, \ldots, v_{2^{n-1}-1})$ for $\vec{v} = (v_0, \ldots, v_{2^{n-1}-1}) \in U_{n-1}$ and $2^{n-1} \leq k \leq 2^n - 1$. Note, that because of (2) we have $0 \leq 2^{n-1} \oplus k = k - 2^{n-1} \leq 2^{n-1} - 1$ $(*)$.

$$\vec{u} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^n-1\oplus j\oplus k} \end{pmatrix}$$

$$= \vec{v} \cdot \begin{pmatrix} x_{0\oplus(2^{n-1}\oplus j)\oplus(2^{n-1}\oplus k)} \\ \vdots \\ x_{2^{n-1}-1\oplus(2^{n-1}\oplus j)\oplus(2^{n-1}\oplus k)} \end{pmatrix} + \vec{v} \cdot \begin{pmatrix} x_{0\oplus j\oplus(2^{n-1}\oplus k)} \\ \vdots \\ x_{2^{n-1}-1\oplus j\oplus(2^{n-1}\oplus k)} \end{pmatrix} \qquad \text{by } (2)$$

$$= v_{(2^{n-1}\oplus k)} \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus(2^{n-1}\oplus j)} + v_{(2^{n-1}\oplus k)} \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus j} \qquad \text{by } (*)$$

$$= u_k \cdot \sum_{i=0}^{2^n-1} u_i x_{i\oplus j} \qquad \text{by } (2)$$

Case 2.1. Let $\vec{u} = (v_0, \ldots, v_{2^{n-1}-1}, -v_0, \ldots, -v_{2^{n-1}-1})$ for $\vec{v} = (v_0, \ldots, v_{2^{n-1}-1}) \in U_{n-1}$ and $0 \leq k \leq 2^{n-1} - 1$.

$$\vec{u} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^n-1\oplus j\oplus k} \end{pmatrix} = \vec{v} \cdot \begin{pmatrix} x_{0\oplus j\oplus k} \\ \vdots \\ x_{2^{n-1}-1\oplus j\oplus k} \end{pmatrix} - \vec{v} \cdot \begin{pmatrix} x_{0\oplus(2^{n-1}\oplus j)\oplus k} \\ \vdots \\ x_{2^{n-1}-1\oplus(2^{n-1}\oplus j)\oplus k} \end{pmatrix} \qquad \text{by } (2)$$

$$= v_k \cdot \sum_{i=0}^{2^{n-1}-1} v_i x_{i\oplus j} - v_k \cdot \sum_{i=0}^{2^{n-1}-1} \underbrace{v_i}_{=-u_{i\oplus 2^{n-1}}} x_{i\oplus(2^{n-1}\oplus j)}$$

$$= u_k \cdot \sum_{i=0}^{2^n-1} u_i x_{i\oplus j} \qquad \text{by } (2)$$

Case 2.2. Let $\vec{u} = (v_0, \dots, v_{2^{n-1}-1}, -v_0, \dots, -v_{2^{n-1}-1})$ for $\vec{v} = (v_0, \dots, v_{2^{n-1}-1}) \in U_{n-1}$ and $2^{n-1} \leq k \leq 2^n - 1$.

$$\begin{aligned}
&\vec{u} \cdot \begin{pmatrix} x_{0 \oplus j \oplus k} \\ \vdots \\ x_{2^n - 1 \oplus j \oplus k} \end{pmatrix} \\
&= \vec{v} \cdot \begin{pmatrix} x_{0 \oplus (2^{n-1} \oplus j) \oplus (2^{n-1} \oplus k)} \\ \vdots \\ x_{2^{n-1}-1 \oplus (2^{n-1} \oplus j) \oplus (2^{n-1} \oplus k)} \end{pmatrix} - \vec{v} \cdot \begin{pmatrix} x_{0 \oplus j \oplus (2^{n-1} \oplus k)} \\ \vdots \\ x_{2^{n-1}-1 \oplus j \oplus (2^{n-1} \oplus k)} \end{pmatrix} && \text{by } (2) \\
&= \underbrace{v_{(2^{n-1} \oplus k)}}_{=-u_k} \cdot \sum_{i=0}^{2^{n-1}-1} \underbrace{v_i}_{=-u_{i \oplus 2^{n-1}}} x_{i \oplus (2^{n-1} \oplus j)} - \underbrace{v_{(2^{n-1} \oplus k)}}_{=-u_k} \cdot \sum_{i=0}^{2^{n-1}-1} \underbrace{v_i}_{=u_i} x_{i \oplus j} && \text{by } (*) \\
&= u_k \cdot \sum_{i=0}^{2^n - 1} u_i x_{i \oplus j} && \text{by } (2)
\end{aligned}$$

Case 3.1. Let $\vec{u} = (\underbrace{1, \dots, 1}_{2^{n-1}}, \underbrace{-1, \dots, -1}_{2^{n-1}})$ and $0 \leq k \leq 2^{n-1} - 1$.

$$\begin{aligned}
\vec{u} \cdot \begin{pmatrix} x_{0 \oplus j \oplus k} \\ \vdots \\ x_{2^n - 1 \oplus j \oplus k} \end{pmatrix} &= \sum_{i=0}^{2^{n-1}-1} \underbrace{x_{i \oplus k \oplus j}}_{i \oplus k \in \{0, \dots, 2^{n-1}-1\}} - \sum_{i=2^{n-1}}^{2^n - 1} \underbrace{x_{i \oplus k \oplus j}}_{i \oplus k \in \{2^{n-1}, \dots, 2^n - 1\}} \\
&= \sum_{i=0}^{2^{n-1}-1} x_{i \oplus j} - \sum_{i=2^{n-1}}^{2^n - 1} x_{i \oplus j} = u_k \cdot \sum_{i=0}^{2^n - 1} u_i x_{i \oplus j}
\end{aligned}$$

Case 3.2. Let $\vec{u} = (\underbrace{1, \dots, 1}_{2^{n-1}}, \underbrace{-1, \dots, -1}_{2^{n-1}})$ and $2^{n-1} \leq k \leq 2^n - 1$.

$$\begin{aligned}
\vec{u} \cdot \begin{pmatrix} x_{0 \oplus j \oplus k} \\ \vdots \\ x_{2^n - 1 \oplus j \oplus k} \end{pmatrix} &= \sum_{i=0}^{2^{n-1}-1} \underbrace{x_{i \oplus k \oplus j}}_{i \oplus k \in \{2^{n-1}, \dots, 2^n - 1\}} - \sum_{i=2^{n-1}}^{2^n - 1} \underbrace{x_{i \oplus k \oplus j}}_{i \oplus k \in \{0, \dots, 2^{n-1}-1\}} \\
&= \sum_{i=2^{n-1}}^{2^n - 1} x_{i \oplus j} - \sum_{i=0}^{2^{n-1}-1} x_{i \oplus j} = u_k \cdot \sum_{i=0}^{2^n - 1} u_i x_{i \oplus j}
\end{aligned}$$

Therefore, (8) holds for all $\vec{u} \in U_n$ and as a direct consequence we have

$$\vec{u} \cdot B_n = \left(\sum_{i=0}^{2^n - 1} u_i b_i, \dots, \sum_{i=0}^{2^n - 1} u_i b_i \right) \cdot \begin{pmatrix} u_0 & & 0 \\ & \ddots & \\ 0 & & u_{2^n - 1} \end{pmatrix}. \tag{9}$$

Since a proper set coloring has every subset of X exactly once on the edges of $K_{a,b}$ except for the empty set (10), we can now deduce

$$\begin{aligned}
-1 &= \vec{u} \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} && \text{by (4)} \\
&= \vec{u} \cdot \begin{pmatrix} \sum\limits_{0 \le i \le 2^n - 1} a_i b_{0 \oplus i} \\ \vdots \\ \sum\limits_{0 \le i \le 2^n - 1} a_i b_{2^n - 1 \oplus i} \end{pmatrix} && \text{by (10)} \\
&= \vec{u} \cdot B_n \cdot \vec{a} && \text{by (7)} \\
&= \left(\sum_{i=0}^{2^n-1} u_i b_i, \ldots, \sum_{i=0}^{2^n-1} u_i b_i \right) \cdot \begin{pmatrix} u_0 & & 0 \\ & \ddots & \\ 0 & & u_{2^n-1} \end{pmatrix} \cdot \vec{a} && \text{by (9)} \\
&= \left(\sum_{i=0}^{2^n-1} u_i b_i \right) \cdot \left(\sum_{i=0}^{2^n-1} u_i a_i \right) = (\vec{u} \cdot \vec{b}) \cdot (\vec{u} \cdot \vec{a}). \qquad \square
\end{aligned}$$

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References

[1] P. N. Balister, O. M. Riordan and R. S. Schelp, Vertex distinguishing edge colorings of graphs, *J. Graph Theory* **42** (2003), 95–109.

[2] O. Frank, F. Harary and M. Plantholt, The line distinguishing chromatic number of a graph, *Ars Combin.* **14** (1982), 241–252.

[3] S. M. Hegde, Set colorings of graphs, *European J. Combin.* **30** (2009), 986–995.

[4] J. Hopcroft and M. S. Krishnamoorthy, On the harmonious colouring of graphs, *SIAM J. Algebra Discrete Math.* **4** (1983), 306–311.

[5] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs, *Applied Math. Letters* **15** (2002), 623–626.

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