THE CLASSIFICATION OF COMBINATORIAL SURFACES USING 3-GRAPHS

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ABSTRACT

A 3-graph is a cubic graph endowed with a proper edge colouring in three colours. Surfaces can be modelled by means of 3-graphs. We show how 3-graphs can be used to establish the standard classification of surfaces by orientability and Euler characteristic.

1. INTRODUCTION

In [8], Tutte approaches topological graph theory from a combinatorial viewpoint. In particular, an entirely combinatorial approach to the classification of surfaces is given. He uses the idea of a premap, which is expressed in [6] as a special kind of 3-graph. In the next section, we define 3-graphs as cubic graphs endowed with a proper edge colouring in three colours. They are also studied in [3, 9] in the more general setting of n-graphs, a variation of the traditional simplicial complex approach to algebraic topology. In fact, the classification of surfaces in terms of 3-graphs is a direct consequence of the main theorem in [3], and is explicitly stated in [10]. Our purpose here is to show how the classification of surfaces by means of 3-graphs follows from Tutte's approach in [8] and the relationship between 3-graphs and premaps. We shall find that this approach provides a possible tool for proving theorems about cubic graphs with a proper edge colouring in three colours. Since most of this paper consists of reproving a known result, it may be classed as expository.

Throughout this paper, the sum of sets is defined as their symmetric difference.

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The graphs we consider lack loops, unless we indicate otherwise, but may have multiple edges. This paper is concerned only with *finite* graphs, those graphs G in which the vertex set VG and edge set EG are both finite. Two distinct edges of a graph are said to be *adjacent* if they are incident on a common vertex. If $T \subseteq EG$, then we write $G[T]$ for the subgraph of G whose edge set is T and whose vertex set is the set of all vertices of G incident with at least one edge of T. We sometimes write $VT = VG[T]$ when no ambiguity results.

A *path* P joining two vertices, a and b, in the same component of G is the edge set of a minimal connected subgraph of G containing a and b. If x and y are vertices or edges of a path P, then we denote by $P[x, y]$ the edge set of the unique minimal connected subgraph of $G[P]$ containing x and y. A *circuit* in G is the edge set of a non-empty connected subgraph in which each vertex has degree 2. If C is a circuit, the elements of VC are sometimes referred to as vertices of C. If v is a vertex in VC incident on edges e_1 and e_2 of C, then C_v denotes the path $C - \{e_1, e_2\}$. Thus if $x, y \in VC - \{v\}$, then $C_v[x, y]$ denoted the path in C which joins x and y and does not pass through v. The *length* of a path or circuit is its cardinality.

2. 3-GRAPHS AND PREMAPS

Let K be a cubic graph. A *proper edge colouring* of K is a colouring of the edges so that adjacent edges receive distinct colours. A 3-*graph* is defined as an ordered triple $(K, \boldsymbol{P}, \boldsymbol{O})$ where K is a non-empty cubic graph endowed with a proper edge colouring $\boldsymbol{P}$ in three colours and $\boldsymbol{O}$ is an ordering of the three colours. We shall assume throughout that the three colours are red, yellow and blue. We write $K = (K, \boldsymbol{P}, \boldsymbol{O})$ when no ambiguity results. The set obtained from EK by deletion of the edges of a specified colour is the union of a set of disjoint circuits, called *bigons*. Thus bigons are of three types: red-yellow, red-blue and blue-yellow. Following Lins[4, 5], we define a *gem* to be a 3-graph in which the red-blue bigons are quadrilaterals (circuits of length 4), called *bisquares*. A 2-cell embedding of a graph G, which may have loops, in a closed surface can be modelled by means of a gem (see [1, 4, 5, 7]).

The relationship between gems and premaps is established in [6]. Let X be a set such that $|X|$ is divisible by 4. Let θ and ϕ be permutations on X satisfying the conditions $\theta^2 = \phi^2 = I$ and $\theta\phi = \phi\theta$, and suppose x, θx, ϕx, $\theta\phi x$ are distinct for each $x \in X$. Let P be another permutation on X such that $P\theta = \theta P^{-1}$, and for each x let the orbits of P through x and θx be distinct. Then (X, θ, ϕ, P) defines a *premap*, M. Tutte also defines Ψ_L as the permutation group generated by a non-empty set L of permutations of X. Then the premap M is a *map* if for each $x \in X$ and $y \in X$ there is a permutation $\pi \in \Psi_{\{\theta, \phi, P\}}$ such that $\pi x = y$.

We now show how to construct a gem $K(M)$ that *represents* a premap M. Each element of X is represented by a vertex. For each $x \in X$, let us draw a red edge joining x to θx, a blue edge joining x to ϕx and a yellow edge joining x to $P\theta x$. (See Figure 1.) It is shown in [6] that this construction yields a gem.

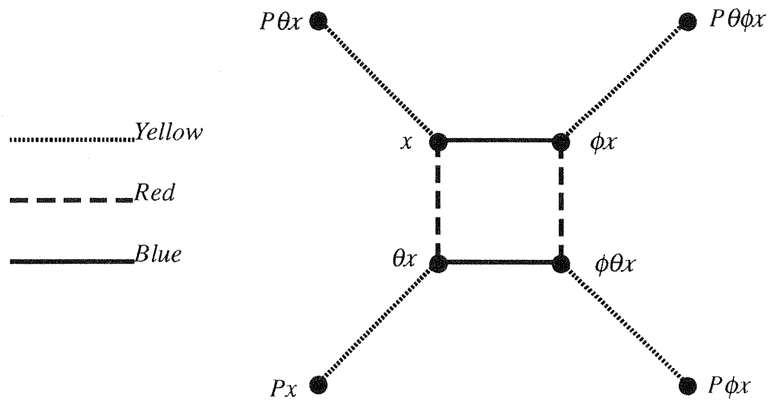


FIGURE 1

Conversely for any gem K it is easy to construct a premap M for which $K = K(M)$. X is the vertex set of K, and the red and blue edges determine the involutions θ and ϕ respectively. Moreover, for any vertex x, Px is the unique vertex joined to θx by a yellow edge.

Evidently the premap M is a map if and only if $K(M)$ is connected.

Let M be a map. It is shown in [8, p.257] that the number of equivalence classes determined by $\Psi_{\{\theta\phi, P\}}$ is either 1 or 2, where two objects are regarded as equivalent if some permutation in $\Psi_{\{\theta\phi, P\}}$ maps one onto the other. We call these equivalence classes the *orientation classes* of M. The premap M is *orientable* if the number of orientation classes is 2, and *non-orientable* otherwise. The following lemma is proved in [5, 6] and a generalisation of it appears in [9].

LEMMA 1. *A map M is orientable if and only if $K(M)$ is bipartite.*

In general, we say a 3-graph is *orientable* if it is bipartite, and *non-orientable* otherwise.

We let $\boldsymbol{B}(K)$, $\boldsymbol{Y}(K)$, $\boldsymbol{R}(K)$ denote the sets of red-yellow, red-blue, and blue-yellow bigons respectively in a 3-graph K. Let $r(K)$ be the total number of bigons in K. We define the *Euler characteristic* of K to be

$$\chi(K) = r(K) - \frac{|VK|}{2}.$$

(See [1, 5, 6].) The *Euler characteristic* of a map M is $\chi(K(M))$.

Tutte [8] defines a surface as the class of all maps with a given Euler characteristic and given orientability character, provided that the class is non-empty. In our setting, such a class corresponds to a class of connected gems. However, we will work in the more general setting of 3-graphs and define a *surface* as the class of all connected 3-graphs with a given Euler characteristic and a given orientability character, provided that the class is non-empty. The main theorem of this paper classifies all such surfaces. It states that one 3-graph can be obtained from another by a finite number of "moves" if and only if they belong to the same surface, where the moves are the crystallisation moves of [3] and

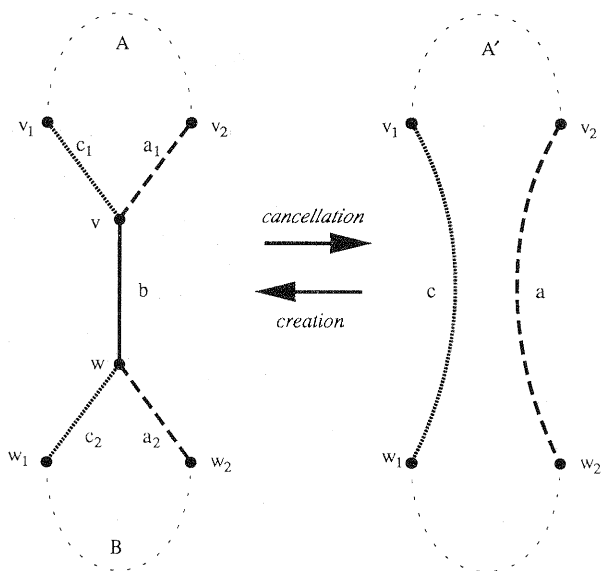


FIGURE 2

are defined in the next section.

3. DIPOLES

Let v and w be a pair of adjacent vertices in a 3-graph K. Suppose that v and w are linked by one edge b, which is blue. Following Ferri and Gagliardi [3], we say that b is a *blue* 1-*dipole* if the red-yellow bigons A and B passing through v and w respectively are distinct. Let c_1 and c_2 be the yellow edges incident on v and w respectively. Let a_1 and a_2 be the red edges incident on v and w respectively. Let v_1, v_2, w_1, w_2 be the vertices other than v and w incident on c_1, a_1, c_2, a_2 respectively. The *cancellation* of this blue 1-dipole b is the operation of deletion of the vertices v and w followed by the insertion of edges c and a linking v_1 to w_1 and v_2 to w_2 respectively. (See Figure 2.) We denote the resulting 3-graph by $K - [b]$. We observe that A and B have coalesced into one red-yellow bigon A'. The *creation* of a blue 1-dipole is the inverse operation. Similar definitions can be made for red and yellow 1-dipoles.

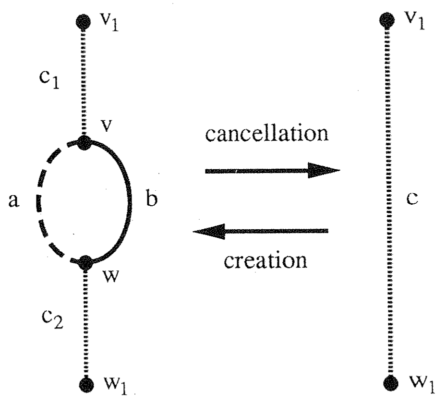


FIGURE 3

Now suppose that v and w are linked by two edges a and b coloured red and blue respectively. Following Ferri and Gagliardi [3], we say that $\{a, b\}$ is a *red-blue* 2-*dipole* if the yellow edges c_1 and c_2 incident on v and w respectively are distinct. Let c_1 link v and v_1 and let c_2 link w and w_1. The *cancellation* of this red-blue 2-dipole is the operation of deletion of the vertices v and w followed by the insertion of an edge c linking v_1 to w_1. (See Figure 3.) We denote the resulting 3-graph by $K - [a, b]$. We observe that c_1 and c_2

have coalesced into one yellow edge c. The *creation* of a red-blue 2-dipole is the inverse operation. Similar definitions can be made for red-yellow and blue-yellow 2-dipoles.

We note that the yellow edge c_1 is a yellow 1-dipole in K and that the 3-graph $K - [c_1]$ is isomorphic to $K - [a, b]$. Hence cancellation or creation of a 2-dipole is in fact a special case of a 1-dipole cancellation or creation.

A *μ-move* is a cancellation or creation of a 1-dipole. Two 3-graphs are *μ-equivalent* if one can be obtained from the other by a finite sequence of μ-moves. It is shown in [3] that two 3-graphs are equivalent if and only if the corresponding surfaces are homeomorphic. Thus the following theorem is equivalent to the classification of surfaces, due to Dehn and Heegaard [2]. Our proof is essentially theirs translated into the setting of coloured graphs. A similar proof in terms of premaps appears in [8]. The proof of the necessity appears as Lemma 2, and the proof of the sufficiency appears in Sections 4 and 5.

THEOREM 1. *Two connected 3-graphs K and J are μ-equivalent if and only if they have the same Euler characteristic and orientability character.*

LEMMA 2. *If two connected 3-graphs K and J are μ-equivalent, then they belong to the same surface.*

Proof. We may assume that J is obtained from K by cancellation of a 1-dipole. We will show that $\chi(K) = \chi(J)$ and that K is bipartite if and only if J is bipartite. Indeed, in a 1-dipole cancellation the number of bigons drops by one and the number of vertices by two. Therefore

$$\begin{aligned}\chi(J) &= r(J) - \frac{|VJ|}{2} \\ &= r(K) - 1 - \frac{|VK|}{2} + 1 \\ &= \chi(K).\end{aligned}$$

Now assume that K is bipartite. Therefore, one may colour the vertices of K black or white so that adjacent vertices receive distinct colours. Evidently v and w receive distinct colours, as do v_1 and w_1, and v_2 and w_2. Hence we conclude that J is bipartite. Similarly K is bipartite if J is bipartite. ❑

We conclude this section by describing another operation on 3-graphs called cancellation of a red-blue bigon. This operation is in fact a pair of dipole cancellations.

Suppose Y to be a red-blue bigon of length 4 in a 3-graph K. Label the edges and vertices incident on Y as in Figure 4a. If b is a blue 1-dipole then let $K' = K - [b]$. (See Figure 4b.) Let a' denote the red edge of K' that links v' and w'. If $[a', b']$ is a red-blue 2-dipole then let $K'' = K' - [a', b']$. (See Figure 4c.) We say that K'' is obtained from K by *cancellation* of the red-blue bigon Y. Let c and c' denote the yellow edges that link v_1 and w_1, and v_2 and w_2, respectively. The inverse operation is described as *splitting* c and c' to *create* the red-blue bigon Y. By definition K and K'' are μ-equivalent.

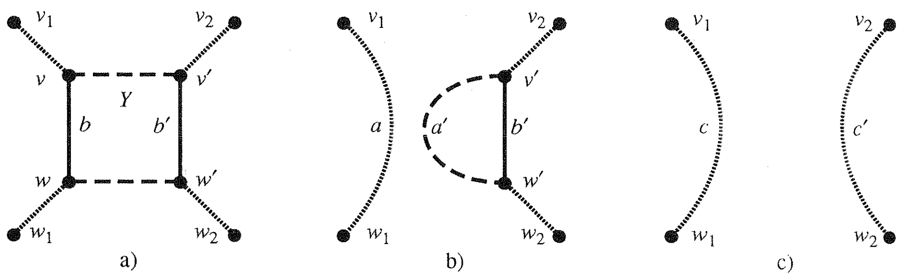


FIGURE 4

4. REDUCED AND UNITARY 3-GRAPHS

The 3-graph with two vertices is *trivial*.

We assume given a connected 3-graph K. Suppose K has at least two red-yellow bigons. Since K is connected there must exist a blue 1-dipole in K. Cancelling this dipole reduces the number of red-yellow bigons by one. Proceeding inductively, we obtain a connected 3-graph which has exactly one red-yellow bigon. Similarly, we reduce to 1 the numbers of blue-yellow and red-blue bigons by red 1-dipole and yellow 1-dipole cancellations. The resulting 3-graph is a *reduced 3-graph* of K, and has just 3 bigons, one of each type. Note that a reduced 3-graph of K is μ-equivalent to K.

LEMMA 3. *If K is a connected 3-graph then $\chi(K) \leq 2$. Moreover if $\chi(K) = 2$ then K is bipartite and the only reduced 3-graph of K is trivial.*

Proof. Let K' denote a reduced 3-graph of K. Since K and K' are μ-equivalent, we have $\chi(K) = \chi(K')$. Furthermore

$$\chi(K') = r(K') - \frac{|VK'|}{2}$$

$$\leq 2$$

since all 3-graphs have at least 2 vertices. If $\chi(K) = \chi(K') = 2$ then $|VK'| = 2$, and K' is the trivial 3-graph. Since the trivial 3-graph is bipartite, we have that K is bipartite by Lemma 2. ❑

The *combinatorial sphere* is the class of all connected 3-graphs with Euler characteristic 2, and all 3-graphs in it are called *planar*. Thus the trivial 3-graph is planar.

A connected 3-graph K is *unitary* if $|\boldsymbol{B}(K)| = |\boldsymbol{R}(K)| = 1$. Hence reduced 3-graphs are unitary.

LEMMA 4. *A connected 3-graph K is μ-equivalent to the trivial 3-graph or a unitary gem.*

Proof. Let K' denote a reduced 3-graph of K. If K is planar then K' is the trivial 3-graph by Lemma 3 and we are done. Hence we assume otherwise. Let B denote the red-yellow bigon in K' and let Y be the red-blue bigon in K'. If Y is a digon (a circuit of length 2), then each yellow edge incident on Y is a yellow 1-dipole, a contradiction. If Y is a bisquare, then K' is a unitary gem and we are done. In the remaining case, let a_1 and a_2 be red edges of Y both adjacent to a common blue edge b. Let P_1 and P_2 be the two paths of $B - \{a_1, a_2\}$. Clearly there exist yellow edges $c_1 \in P_1$ and $c_2 \in P_2$, and therefore we split c_1 and c_2 to create a red-blue bisquare Y_1. Let K'' denote the resulting graph. Evidently b is a blue 1-dipole in K'', and therefore we cancel it to obtain a unitary 3-graph U. The red-blue bigon in U corresponding to Y has one blue edge less than Y. Proceeding inductively we obtain a unitary 3-graph U' where the red-blue bigon corresponding to Y is a bisquare. However U' is a unitary gem since all new red-blue bigons created were bisquares. ❑

Let us now study the case in which the surface is not the combinatorial sphere. Starting with an arbitrary 3-graph, we change it by μ-moves, as in Lemma 4, into a unitary gem U.

Let B be the one red-yellow bigon in U. Let Y be an arbitrary red-blue bigon of U. Label the edges and vertices incident on Y as in Figure 4a. If $w' \in V(B_{v'}[v, w])$ then we say that Y is a *cross-cap* of B. (See Figure 5.) It is an *assembled* cross-cap of B if there exists a subpath of B with just two red edges, both in Y. If Y is not a cross-cap then it is a *cap* of B.

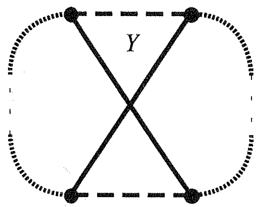


FIGURE 5

Suppose there exist two caps X and Y of B. Again we label the edges and vertices incident on Y as in Figure 4a. If $|X \cap B_{w'}[v, w]| = 1$ then we say that X and Y are *bound* in B, and $\{X, Y\}$ is a *handle* of B. (See Figure 6.) Such a handle is *assembled* if there exists a subpath of B with just four red edges, all in $X \cup Y$.

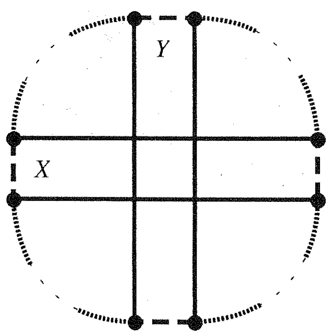


FIGURE 6

LEMMA 5. *A given unitary gem U is μ-equivalent to another unitary gem J for which all the cross-caps are assembled.*

Proof. Let B be the one red-yellow bigon in U and let Y be an unassembled cross-cap. Label the edges of B incident on vertices of VY as in Figure 7a. Split d and d' to create a red-blue bigon X and let U' denote the resulting gem. (Figure 7b.) In U', a and a' belong to distinct red-yellow bigons. Hence we may let U'' denote the gem obtained by cancelling Y in U'. (Figure 7c.) Clearly $|\boldsymbol{B}(U'')| = |\boldsymbol{B}(U)| = 1$ and $|\boldsymbol{R}(U'')| = |\boldsymbol{R}(U)| = 1$. It is also clear that X is an assembled cross-cap in U''. Moreover, no assembled cross-cap in U has been lost; the red edges of such a cross-cap are in $B - \{c, a, d, c', a', d'\}$.

By this procedure we can reduce to the case in which every cross-cap of U is assembled. ❑

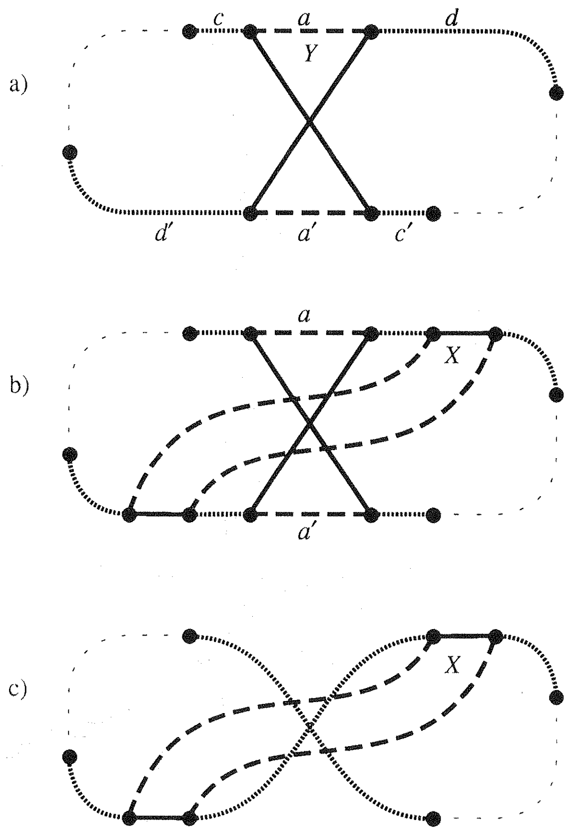


FIGURE 7

LEMMA 6. *Let U be a unitary gem with all cross-caps assembled. Then all caps in U are bound.*

Proof. Suppose there exists an unbound cap Y in U. Label the vertices and edges adjacent to Y as in Figure 4a. Since there is no blue edge in U that has terminal vertices in both $VB_v[v', w']$ and $VB_{v'}[v, w]$, b and b' must belong to distinct blue-yellow bigons. This contradicts the fact that U is unitary. ❑

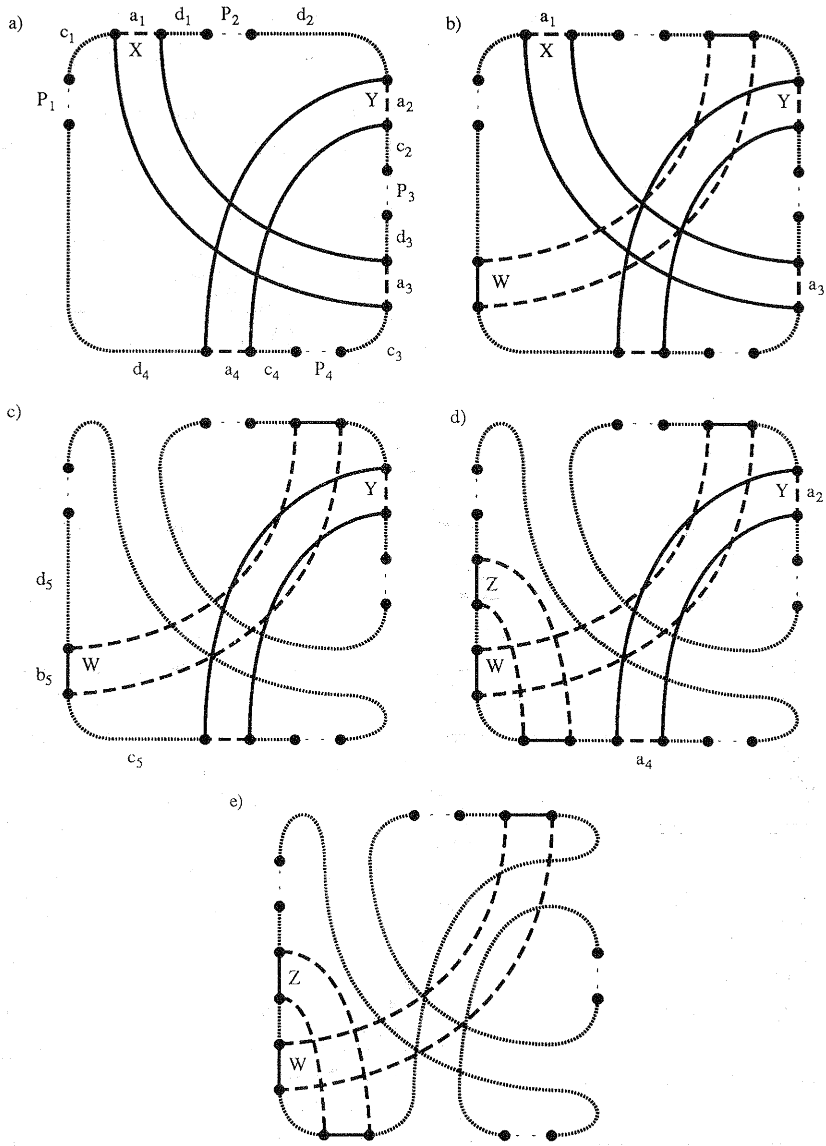
Consider our unitary gem in which all cross-caps are assembled. Then Lemma 6 tells us that either the red-blue bigons are all assembled cross-caps or there are two red-blue bigons that constitute a handle. The next lemma deals with the assembly of handles.

LEMMA 7. *By a finite sequence of μ-moves, we can convert a given unitary gem into one in which each red-blue bigon is an assembled cross-cap or a member of an assembled handle.*

Proof. Suppose there exists a handle $\{X, Y\}$ in U. The following uses the notation in Figure 8a. Split d_2 and d_4 to create a red-blue bigon W and let U_1 denote the resulting gem. (See Figure 8b.) In U_1, a_1 and a_3 are in distinct red-yellow bigons, and so the operation of cancellation of a red-blue bigon is applicable to X. We apply it, and let U_2 denote the resulting graph. (See Figure 8c.) In U_2 let c_5 denote the yellow edge adjacent to a_4 other than c_4. Let b_5 denote the blue edge of W adjacent to c_5, and let d_5 denote the yellow edge other than c_5 adjacent to b_5. Split c_5 and d_5 to create a red-blue bigon Z and let U_3 denote the resulting gem. (See Figure 8d.) In U_3, a_2 and a_4 are in distinct red-yellow bigons, and so the operation of cancellation of a red-blue bigon is applicable to Y. We apply it and let U' denote the resulting gem. (See Figure 8e.)

The above process transforms U into another unitary gem U'. The handle $\{X, Y\}$ has been replaced by the assembled handle $\{W, Z\}$. Any assembled cross-cap or other assembled handle of B has edges in $B - \{a_1, c_1, d_1, a_2, c_2, d_2, a_3, c_3, d_3, a_4, c_4, d_4\}$, and is preserved.

By repetition of the operation just described, we can replace unassembled handles by assembled ones until we have a unitary gem of the kind required. (No red-blue bigons will be left over, by Lemma 6.) ❑


a)
a_1
d_1
P_2
d_2
c_1
X
P_1
Y
a_2
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d_4
a_4
c_4
P_4
b)
a_1
X
Y
W
a_3
c)
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FIGURE 8

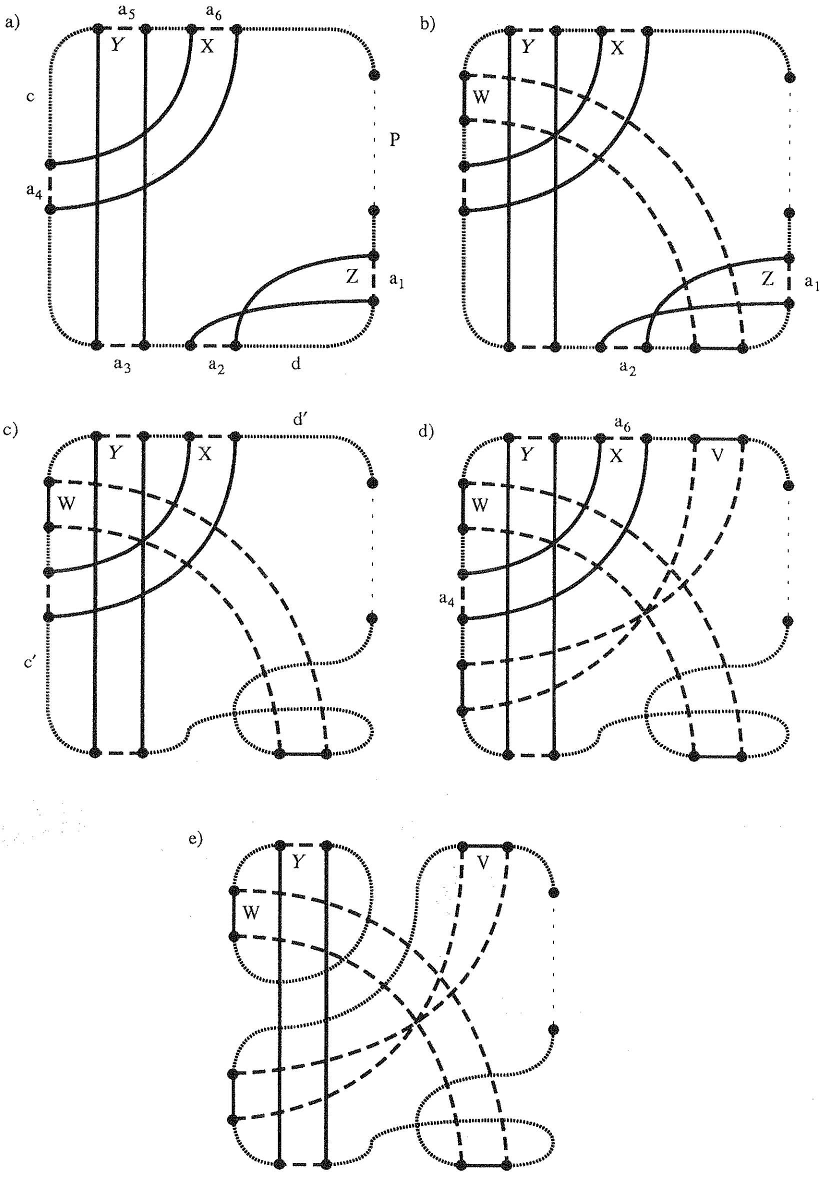

FIGURE 9

LEMMA 8. *By a finite sequence of μ-moves we can convert a given unitary gem into one in which all red-blue bigons are assembled cross-caps or all red-blue bigons are members of assembled handles.*

Proof. We may suppose our unitary gem U already in the form specified in Lemma 7. Suppose it to have at least one assembled cross-cap and at least one assembled handle. Then the situation arising is depicted in Figure 9a. This displays a handle $\{X, Y\}$ immediately followed by a cross-cap Z. Split c and d to create a red-yellow bigon W and let U_1 denote the resulting graph. (See Figure 9b.) In U_1, a_1 and a_2 are in distinct red-yellow bigons, and so the operation of cancellation of a red-blue bigon is applicable to Z. We apply it, and let U_2 denote the resulting graph. (See Figure 9c.) In U_2 split c' and d' to create a red-blue bigon V, and let U_3 denote the resulting gem. (See Figure 9d.) In U_3, a_4 and a_6 are in distinct red-yellow bigons, and so the operation of cancellation of a red-blue bigon is applicable to X. We apply it, and let U_4 denote the resulting graph. (See Figure 9e.)

At this stage we still have a unitary gem U_4. We have an assembled cross-cap V, an unassembled cross-cap Y and a cap W. The original assembled handles and cross-caps which have edges in P are preserved.

Our next step is to replace the cross-cap Y by an assembled cross-cap, as in Lemma 5. The assembled cross-caps and handles of U_4 are clearly preserved. The red-blue bigon W is transformed into another cross-cap by Lemma 6. Finally, we assemble this cross-cap, too. We thus obtain a unitary gem, still of the form in Lemma 7, but with one handle fewer and two more cross-caps. Repetition of the above procedure leads us to a unitary gem in which all red-blue bigons are cross-caps.

Only if all red-blue bigons in U belong to assembled handles is the above procedure inapplicable. Hence the lemma follows. ❑

5. CANONICAL GEMS

Let us define a *canonical* 3-graph as either the trivial 3-graph or a unitary gem U in which $Y(U)$ consists entirely of assembled cross-caps or members of assembled handles.

The trivial 3-graph is orientable and so is a unitary canonical 3-graph whose red-blue bigons are members of handles. The *genus* of such a 3-graph is the number of handles that it contains. The *genus* of the trivial 3-graph is defined to be zero. On the other hand, a unitary canonical 3-graph whose red-blue bigons are cross-caps is non-orientable. The *cross-cap number* of such a 3-graph is the number of cross-caps that it contains.

We observe that an orientable canonical 3-graph can be constructed with an arbitrary nonnegative integer as genus, and that a non-orientable one can be constructed with an arbitrary positive integer as cross-cap number. There are no other possibilities. Hence we have the following lemma.

LEMMA 9. *There is at most one orientable canonical 3-graph with given genus and one non-orientable canonical 3-graph with given cross-cap number.* ❑

LEMMA 10. *An orientable canonical 3-graph of genus g has Euler characteristic $2-2g$. A non-orientable canonical 3-graph of cross-cap number k has Euler characteristic $2-k$.*

Proof. This follows from the fact that the number of red-blue bigons is $2g$ in the first case and k in the second. ❑

LEMMA 11. *There is exactly one canonical 3-graph on each surface.*

Proof. By Lemmas 4 and 8 there exists a canonical 3-graph on each surface. Its uniqueness follows from Lemmas 9 and 10. ❑

LEMMA 12. *Let K and J be 3-graphs on the same surface. Then K and J are μ-equivalent.*

Proof. Let U be the canonical 3-graph on the surface. Then each of K and J can be transformed into U by a sequence of μ-moves by Lemmas 4 and 8. The lemma follows. ❑

Theorem 1 now follows from Lemmas 2 and 12. The genus of an orientable canonical gem is also called the *genus* of its surface, and of any other 3-graph on that surface. Likewise, the cross-cap number of an non-orientable canonical gem is also called the *cross-cap number* of its surface, and of any other 3-graph on that surface. A surface is called *orientable* or *non-orientable* according as the 3-graphs on it are orientable or non-orientable. We can now say that there is just one orientable surface $\mathbb{S}_g$ whose genus is a given nonnegative integer g, and just one non-orientable surface $\mathbb{N}_k$ whose cross-cap number is a given positive integer k, and moreover there are no other surfaces.

6. CONCLUSION

We have established the classification of surfaces by means of simple operations (dipole cancellations and creations) on 3-graphs. These operations enabled us to reduce any given 3-graph to a simple canonical form. This observation provides us with a possible approach to proving results about cubic graphs with a proper edge colouring in three colours: first prove the required result for the canonical forms, and then prove that the result is preserved under the dipole cancellation and creation operations.

We would like to mention that Vince (private communication) has a similar method for obtaining the classification of surfaces. It is of interest to note that his canonical 3-graphs are different from ours.

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