AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 221–233

The triangle intersection problem for nested Steiner triple systems*

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Abstract

We give a solution for the triangle intersection problem for nested Steiner triple systems, with three possible exceptions.

1 Introduction

A *Steiner triple system* $(V, \mathcal{B})$, of order v, is a collection $\mathcal{B}$ of unordered triples of elements chosen from a set V of size v in such a way that each unordered pair of elements of V occurs in precisely one triple in $\mathcal{B}$. The triples in $\mathcal{B}$ are called *blocks* or *triangles*, and the notation STS(v) is also used for a Steiner triple system of order v. It is well-known that the spectrum for STS(v), that is, the set of admissible orders v, is all positive integers $v \equiv 1$ or 3 (mod 6); this was first shown in [3]. We remark that $|\mathcal{B}| = v(v-1)/6$, and we usually set $|\mathcal{B}| = b$.

* Project supported by the Australian Research Council, Project Numbers LX0453416 and DP0664030

An STS(v), $(V, \mathcal{B})$, is said to be *nested* if it is possible to adjoin one element to each block in $\mathcal{B}$ to form a set of blocks

$$\mathcal{B}^* = \{B \cup \{w\} \mid B \in \mathcal{B},\ w \in V \setminus B\}$$

with the property that every pair $x, y \in V$ occurs in precisely *two* blocks of $\mathcal{B}^*$. Note that since $\mathcal{B}$ is the set of blocks of a Steiner triple system, and every pair $x, y \in V$ occurs in one block of $\mathcal{B}$, for each unordered pair $\{x, y\} \subseteq V$, there either exists a block $B \cup \{y\} \in \mathcal{B}^*$ such that $x \in B$, or else a block $B \cup \{x\} \in \mathcal{B}^*$ such that $y \in B$. Clearly, the blocks of $\mathcal{B}^*$ are all of size four. It will be useful to have shorthand notation for representing the nesting of a block and so given a block $B = \{x_1, x_2, \ldots, x_k\}$, the set $\{x_1, x_2, \ldots, x_k \mid y\}$, or more concisely $x_1\ x_2\ \ldots\ x_k \mid y$, will represent the nested block. We refer to a nested Steiner triple system of order v as an NSTS(v).

Counting arguments can be used to verify that a necessary condition for the existence of an NSTS(v) is $v \equiv 1 \pmod 6$, and Stinson [7] showed that for all such v there exists an STS(v) which can be nested. Consider two Steiner triple systems, $S_1 = (V, \mathcal{B}_1)$ and $S_2 = (V, \mathcal{B}_2)$. The *intersection number* for S_1 and S_2 is $|\mathcal{B}_1 \cap \mathcal{B}_2|$. Lindner and Rosa [4] have shown that the set of possible intersection numbers for STS(v), $v \equiv 1 \text{ or } 3 \pmod 6$ and $v \neq 9$, is $\{0, 1, 2, 3, \ldots, b-6, b-4, b\}$, where $b = v(v-1)/6$. If the Steiner triple systems S_1 and S_2 can be extended to nested designs with block sets $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$, then the *triangle intersection number* for the nested systems is also $|\mathcal{B}_1 \cap \mathcal{B}_2|$. We stress that it is the underlying triples or triangles from the STS which we consider in the intersection of two nestings, rather than the blocks of size 4.

In this paper we show that for all values of $v \equiv 1 \pmod 6$, except possibly for $v = 19$, 31 and 37, the set of possible triangle intersection numbers for *nested* Steiner triple systems is equal to the set of possible intersection numbers for Steiner triple systems. For systems of order v, we use the notation ISTS(v) to denote the set of achievable intersection numbers for Steiner triple systems, and INSTS(v) to denote the set of achievable triangle intersection numbers for nested Steiner triple systems. Clearly, for $v \equiv 1 \pmod 6$, INSTS(v) $\subseteq$ ISTS(v).

The constructions presented in this paper rely on establishing the set of the triangle intersection numbers for certain nested group divisible designs. A *group divisible design*, GDD($v; G; K$) consists of a set V, of size v, which is partitioned into *groups* of sizes $g \in G$ and a collection of subsets (*blocks*), of sizes $k \in K$, chosen from V in such a way that each pair of elements from different groups occurs in precisely one block and each pair of elements from the same group never occurs together in a block. The *type* of a GDD is the collection of group sizes, and it is usual to let g^h represent the occurrence of h groups of size $g \in G$ and g^* to represent the occurrence of a single group of size $g \in G$. A similar asterisk convention is applied to the block sizes if there is a single block of a particular size. If all the groups have size g and all the blocks have size k then we write GDD($v; g; k$).

A GDD($v; G; K$) with collection of blocks $\mathcal{B}$ is said to be *nested* if it is possible to extend $\mathcal{B}$ to form a new set of blocks

$$\mathcal{B}^* = \{B \cup \{w\} \mid B \in \mathcal{B},\ w \in V \setminus B\}$$

with the property that every pair $\{x, y\} \subseteq V$ from distinct groups occurs in precisely two blocks of $\mathcal{B}^*$ and that every pair $\{x, y\} \subseteq V$ from the same group does not occur in a block of $\mathcal{B}^*$. As previously noted, we will represent the nesting of a block $\{x_1, x_2, \ldots, x_k\}$ with a point y by $x_1\ x_2\ \ldots\ x_k \mid y$. If GDD$(v; G; K)$ is a group divisible design which can be nested, we denote the nested design by NGDD$(v; G; K)$.

We shall also make use of the following notation. If A and B are sets of nonnegative integers, we let $A + B = \{a + b \mid a \in A,\ b \in B\}$. For any positive integer x and any subset A of nonnegative integers, we denote a sum of x not necessarily distinct elements of A by $x \star A$.

The following three calculations are made frequently throughout this paper. Let b be some positive integer. Then we have:

$$\begin{aligned}
b \star \{0, 2, 8\} &= \{0, 2, 4, 6, 8, \ldots, 8(b-3) + 3.0,\ 8(b-3) + 1.2 + 2.0,\\
&\quad 8(b-3) + 2.2 + 1.0,\ 8(b-3) + 3.2,\ 8(b-2) + 2.0,\\
&\quad 8(b-2) + 1.2 + 1.0,\ 8(b-2) + 2.2,\ 8(b-1) + 0,\\
&\quad 8(b-1) + 2,\ 8b\}\\
&= \{0, 2, 4, 6, 8, \ldots, 8b-24,\ 8b-22,\ 8b-20,\ 8b-18,\\
&\quad 8b-16,\ 8b-14,\ 8b-12,\ 8b-8,\ 8b-6,\ 8b\}\\
&= \{2i \mid 0 \le i \le 4b-6\} \cup \{8b-8,\ 8b-6,\ 8b\};
\end{aligned}$$

$$\begin{aligned}
b \star \{0, 1, 3, 7\} &= \{0, 1, 2, 3, 4, \ldots, 7(b-3) + 1.3 + 1.1 + 1.0,\\
&\quad 7(b-3) + 1.3 + 2.1, 7(b-3) + 2.3 + 1.0,\ 7(b-2) + 2.0,\\
&\quad 7(b-2) + 1.1 + 1.0,\ 7(b-2) + 2.1,\ 7(b-2) + 1.3 + 1.0,\\
&\quad 7(b-2) + 1.3 + 1.1,\ 7(b-2) + 2.3,\\
&\quad 7(b-1) + 1.0,\ 7(b-1) + 1.1, 7(b-1) + 1.3, 7b\}\\
&= \{0, 1, 2, 3, 4, \ldots, 7b-17, 7b-16, 7b-14, 7b-13, 7b-12,\\
&\quad 7b-11,\ 7b-10,\ 7b-8,\ 7b-7,\ 7b-6,\ 7b-4,\ 7b\}\\
&= \{i \mid 0 \le i \le 7b-10\} \cup \{7b-8, 7b-7, 7b-6, 7b-4, 7b\};
\end{aligned}$$

$$\begin{aligned}
&b \star \{0, 1, 2, \ldots, 20, 22, 26\}\\
&= \{0, 1, 2, 3, 4, \ldots, 26b-9,\ 26b-8,\ 26b-7,\ 26b-6,\ 26b-4,\ 26b\}\\
&= \{i \mid 0 \le i \le 26b-6\} \cup \{26b-4, 26b\}.
\end{aligned}$$

2 General method of construction

The proofs presented in this paper will be based on the following recursive constructions. These constructions are a special case of Wilson's more general construction (see [8]).

Construction 1

Assume there exists a GDD$(v; G; k)$ and an NGDD$(mk; m; 3)$. Let the number of blocks in the GDD$(v; G; k)$ be t. We construct a nested group divisible design NGDD$(mv; \overline{G}; 3)$, where $\overline{G} = \{mg \mid g \in G\}$, as follows. Let X represent the point set and $\mathcal{B}$ represent the block set of the GDD$(v; G; k)$. The NGDD$(mv; \overline{G}; 3)$ has point set $\overline{X} = \{(x, j) \mid x \in X,\ 0 \leq j \leq m-1\}$ and block set $\overline{\mathcal{B}} = \bigcup_{i=1}^{t} \mathcal{B}_i$ where

- for $i = 1 \ldots, t$ the block set $\mathcal{B}_i$ is obtained by taking the ith block, B_i, of the GDD$(v; G; k)$ and constructing an NGDD$(mk; m; 3)$ on the point set $\{(x, j) \mid x \in B_i,\ 1 \leq j \leq m\}$, where for each $x \in B_i$ the set $\{(x, j) \mid 1 \leq j \leq m\}$ forms a group.

Construction 2

Assume there exists an NGDD$(v; G; 3)$ and, for each $g \in G$, there exists an NSTS$(g+1)$. Let $V = G_1 \cup \cdots \cup G_r$ represent the point set of the NGDD$(v; G; 3)$ with groups G_i, $1 \leq i \leq r$. Let $\mathcal{B}$ represent the block set of the NGDD$(v; G; 3)$. For $i = 1, \ldots, r$, we construct an NSTS$(g+1)$ with point set $V_i = G_i \cup \{\infty\}$, and with block set denoted $\mathcal{B}_i$. When all block sets are combined we obtain an NSTS$(v+1)$ on the point set $V \cup \{\infty\}$, with block set

$$\overline{\mathcal{B}} = \left(\bigcup_{i=1}^{r} \mathcal{B}_i \right) \ \cup\ \mathcal{B}.$$

3 Construction of some useful GDD

The constructions set out in the previous section rely on the existence of a number of essential group divisible designs. In this section we note the following existence results established by Stinson [7].

Lemma 3.1. *If* $n \equiv 0$ *or* 3 $(mod\ 12)$, $n \geq 12$, *then there exist group divisible designs GDD*$(n; 3; 4)$. *For* $n \equiv 6$ *or* 9 $(mod\ 12)$ *and* $n \geq 21$, *there exist group divisible designs GDD*$(n; 3; \{4, 7^*\})$, *and GDD*$(n; \{3, 6^*\}; 4)$.

We also establish the following existence results.

Lemma 3.2. *There exists an NGDD*$(8; 2; 3)$.

Proof. The following design is an NGDD$(8; 2; 3)$, with the groups $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ and $\{7, 8\}$.

$$\begin{array}{llll} 1\,3\,5 \mid 7, & 1\,4\,6 \mid 8, & 1\,6\,7 \mid 4, & 2\,5\,8 \mid 3, \\ 4\,5\,7 \mid 2, & 3\,6\,8 \mid 1, & 1\,4\,8 \mid 5, & 2\,3\,7 \mid 6. \end{array}$$

□

Lemma 3.3. *There exists an NGDD*$(48; \{6^4, 12^2\}; 3)$.

Proof. Since there exist a GDD$(24; \{3^4, 6^2\}; 4)$ (see [1], Section IV.1) and a NGDD$(8; 2; 3)$ (see Lemma 3.2) we may apply Construction 1 to obtain a NGDD$(48; \{6^4, 12^2\}; 3)$. The GDD$(24; \{3^4, 6^2\}; 4)$ has 39 blocks and each NGDD$(8; 2; 3)$ has eight blocks; we combine these to construct the 312 blocks of the NGDD$(48; \{6^4, 12^2\}; 3)$. □

Lemma 3.4. *There exists an NGDD*$(54; \{6^1, 12^4\}; 3)$.

Proof. Since there exists a GDD$(27; \{3^1, 6^4\}; 4)$ (see [1], Section IV.1) and a NGDD$(8; 2; 3)$ (see Lemma 3.2) we may apply Construction 1 to obtain a NGDD$(54; \{6^1, 12^4\}; 3)$. The GDD$(27; \{3^1, 6^4\}; 4)$ has 48 blocks and each NGDD$(8; 2; 3)$ has eight blocks; we combine these to construct the 384 blocks of the NGDD$(54; \{6^1, 12^4\}; 3)$. □

Lemma 3.5. *There exists an NGDD*$(72; 12; 3)$.

Proof. Since there exists a GDD$(36; 6; 4)$ (see [1], Section IV.1) and a NGDD$(8; 2; 3)$ (see Lemma 3.2) we may apply Construction 1 to obtain an NGDD$(72; 12; 3)$. The GDD$(36; 6; 4)$ has 90 blocks, each NGDD$(8; 2; 3)$ has eight blocks and we combine these to construct the 720 blocks of the NGDD$(72; 12; 3)$. □

4 Triangle intersection numbers: NSTS of small order

The constructions presented in this paper are recursive in nature. So in order to establish the recursion we require the triangle intersection numbers for some nested Steiner triple systems of small orders. We present these now.

Lemma 4.1. *The set INSTS*(7) *of triangle intersection numbers of NSTS*(7)*, is equal to the set ISTS*$(7) = \{0, 1, 3, 7\}$.

Proof. Consider the following four NSTS(7), denoted $S_i(7)$, $1 \leqslant i \leqslant 4$.

$S_1(7)$		$S_2(7)$		$S_3(7)$		$S_4(7)$	
1 2 4	7	1 2 4	7	1 2 4	7	1 5 7	2
2 3 5	1	3 4 5	1	4 3 5	1	1 2 6	3
3 4 6	2	2 3 6	4	2 5 6	4	2 3 7	4
4 5 7	3	2 5 7	3	2 3 7	5	3 4 1	5
5 6 1	4	5 6 1	2	3 6 1	2	4 5 2	6
6 7 2	5	6 7 4	5	6 7 4	3	5 6 3	7
7 1 3	6	7 1 3	6	1 5 7	6	7 6 4	1

Note that the nested Steiner triple systems $S_1(7)$ and $S_4(7)$ intersect in zero triangles, $S_1(7)$ and $S_3(7)$ intersect in one triangle, $S_1(7)$ and $S_2(7)$ intersect in three triangles, while $S_1(7)$ intersects itself in seven triangles. We list these explicitly for future reference although the result is immediate from the fact that up to isomorphism there is only one Steiner triple system of order seven (see [1], Section II.1). Since ISTS(7) $= \{0, 1, 3, 7\}$ (see [4]) and since there exists an NSTS(7) (see [7]), the required result is achieved. □

Lemma 4.2. *The set INSTS*(13) *of triangle intersection numbers for STS*(13) *is equal to the set ISTS*(13) $= \{0, 1, 2, \dots, 20, 22, 26\}$.

Proof. It is noted that the set of intersection numbers for Steiner triple systems of order 13 is ISTS(13) $= \{0, 1, 2, \dots, 20, 22, 26\}$ (see [4]). There are precisely two non-isomorphic STS(13) (see [1], Section II.1) and we shall show that an STS(13) from each isomorphism class can be nested; hence for each $s \in$ ISTS(13) there exists a pair of nested Steiner triple systems with triangle intersection number s. The following two nested Steiner triple systems achieve this; they are on the vertex set $\{0, 1, \dots, 9, a, b, c\}$.

0 1 2	3	2 6 a	8
0 3 4	8	2 7 c	b
0 5 6	4	2 8 b	1
0 7 8	9	3 6 b	9
0 9 a	2	3 7 a	6
0 b c	6	3 8 c	7
1 3 5	c	4 6 c	2
1 4 7	a	4 8 9	c
1 6 8	5	4 a b	3
1 9 b	4	5 7 b	0
1 a c	0	5 8 a	b
2 3 9	5	5 9 c	a
2 4 5	7	6 7 9	1

$S_1(13)$

0 1 2	6	2 6 a	3
0 3 4	5	2 7 b	4
0 5 6	8	2 8 c	7
0 7 8	9	3 6 b	c
0 9 a	2	3 7 c	0
0 b c	1	3 8 a	b
1 3 5	7	4 6 c	9
1 4 7	a	4 8 9	3
1 6 8	4	4 a b	0
1 9 b	5	5 7 a	6
1 a c	8	5 8 b	2
2 3 9	1	5 9 c	a
2 4 5	c	6 7 9	b

$S_2(13)$

□

5 Triangle intersection numbers: NGDD of small orders

The Steiner triple systems presented in the previous section are not quite enough to establish intersection numbers for general systems. We also require *ad hoc* constructions for some NGDDs. In this section we establish some triangle intersection numbers for the nested group divisible designs NGDD$(8; 2; 3)$, NGDD$(24; 6; 3)$, NGDD$(48; \{6^4, 12^2\}; 3)$, NGDD$(54; 6^1, \{12^4\}; 3)$ and NGDD$(72; 12; 3)$.

Lemma 5.1. *The set of values* $\{0, 2, 8\}$ *is a subset of the set INGDD*$(8; 2; 3)$ *of triangle intersection numbers for NGDD*$(8; 2; 3)$.

Proof. Consider the following three NGDD$(8; 2; 3)$, denoted $S_1(8)$, $S_2(8)$, $S_3(8)$.

1	3	5	7
2	4	6	8
1	6	7	4
2	5	8	3
4	5	7	2
3	6	8	1
1	4	8	5
2	3	7	6

$S_1(8)$

1	3	7	5
2	4	8	6
1	4	5	8
2	3	6	7
3	5	8	2
4	6	7	1
1	6	8	3
2	5	7	4

$S_2(8)$

1	3	5	7
2	4	6	8
1	6	8	3
2	5	7	4
4	5	8	1
3	6	7	2
1	4	7	6
3	2	8	5

$S_3(8)$

The nested group divisible designs $S_1(8)$ and $S_2(8)$ intersect in zero triangles, $S_1(8)$ and $S_3(8)$ intersect in two triangles, as do $S_2(8)$ and $S_3(8)$, and $S_1(8)$ intersects itself in all eight triangles. ☐

Lemma 5.2. *The set of values* $\{2i \mid 0 \leq i \leq 30\} \cup \{63, 64, 66, 72\}$ *is a subset of the set INGDD*$(24; 6; 3)$.

Proof. Since there exists a GDD$(12; 3; 4)$ (see Lemma 3.1) and an NGDD$(8; 2; 3)$ (see Lemma 3.2) we may apply Construction 1 to obtain an NGDD$(24; 6; 3)$; there are nine blocks in a GDD$(12; 3; 4)$, so we use nine NGDD$(8; 2; 3)$ in the construction.

Note that by Lemma 5.1 there exist NGDD$(8; 2; 3)$ which intersect in 0, 2 or 8 blocks. We shall construct several NGDD$(24; 6; 3)$. In each construction we use the same GDD$(12; 3; 4)$ whilst varying the NGDD$(8; 2; 3)$ used; hence, the intersection numbers we can achieve occur from combinations of the intersection numbers for the NGDD$(8; 2; 3)$. We use nine NGDD$(8; 2; 3)$, so the possible intersection numbers are $9 \star \{0, 2, 8\} = \{2i \mid 0 \leq i \leq 30\} \cup \{64, 66, 72\}$.

Finally, to construct NGDD$(24; 6; 3)$ with intersection number 63, we take a variant of Construction 1.

We begin with an NGDD$(8; 2; 3)$ with point set V and block set $\mathcal{B}$ and construct an NGDD$(24; 6; 3)$ with point set $\overline{V} = \{(x, i) \mid x \in V,\ 1 \leq i \leq 3\}$ and block set obtained by taking each block $B = \{a\,b\,c \mid d\} \in \mathcal{B}$ and constructing either the nine nested blocks $S_1(9)$ or nine nested blocks $S_2(9)$ given below.

$$
\begin{array}{ccc|c}
(a,1) & (b,4) & (c,7) & (d,1) \\
(a,2) & (b,6) & (c,8) & (d,1) \\
(a,3) & (b,5) & (c,9) & (d,1) \\
(a,1) & (b,5) & (c,8) & (d,2) \\
(a,2) & (b,4) & (c,9) & (d,2) \\
(a,3) & (b,6) & (c,7) & (d,2) \\
(a,1) & (b,6) & (c,9) & (d,3) \\
(a,2) & (b,5) & (c,7) & (d,3) \\
(a,3) & (b,4) & (c,8) & (d,3)
\end{array}
\qquad\qquad
\begin{array}{ccc|c}
(a,1) & (b,4) & (c,8) & (d,1) \\
(a,2) & (b,6) & (c,9) & (d,1) \\
(a,3) & (b,5) & (c,7) & (d,1) \\
(a,1) & (b,5) & (c,9) & (d,2) \\
(a,2) & (b,4) & (c,7) & (d,2) \\
(a,3) & (b,6) & (c,8) & (d,2) \\
(a,1) & (b,6) & (c,7) & (d,3) \\
(a,2) & (b,5) & (c,8) & (d,3) \\
(a,3) & (b,4) & (c,9) & (d,3)
\end{array}
$$

$S_1(9)$ $\qquad\qquad$ $S_2(9)$

The process is repeated for each block of the NGDD$(8;3;2)$ and we obtain an NGDD $(24;6;3)$. Since NGDD$(8;3;2)$ contains eight blocks, NGDD$(24;6;3)$ will contain $9 \times 8 = 72$ blocks.

We construct two different NGDD$(24;6;3)$ labelled $S_1(24)$ and $S_2(24)$, having triangle intersection number 63, by varying the number of copies of $S_1(9)$ and $S_2(9)$: Let $S_1(24)$ be formed using 8 copies of $S_1(9)$ (and no copies of $S_2(9)$), while $S_2(24)$ is formed from 7 copies of $S_1(9)$ and one copy of $S_2(9)$. $\square$

Lemma 5.3. *The set of values* $\{2i \mid 0 \le i \le 150\} \cup \{304, 306, 312\}$ *is a subset of the set INGDD*$(48; \{6^4, 12^2\}; 3)$.

Proof. Consider the construction of the NGDD$(48; \{6^4, 12^2\}; 3)$ in Lemma 3.3. We shall construct several NGDD$(48; \{6^4, 12^2\}; 3)$. Note that by Lemma 5.1 there exist pairs of NGDD$(8;2;3)$ which intersect in 0, 2, or 8 blocks. In each construction we use the same GDD$(24; \{3^4, 12^2\}; 4)$ whilst varying the NGDD$(8;2;3)$ used; hence, the intersection numbers we achieve occur from combinations of the intersection numbers for the NGDD$(8;2;3)$. There are thirty-nine blocks in the GDD$(24; \{3^4, 12^2\}; 4)$; hence, we use thirty-nine NGDD$(8;2;3)$, so the possible intersection numbers are clearly $39 \star \{0,2,8\} = \{2i \mid 0 \le i \le 150\} \cup \{304, 306, 312\}$. $\square$

Lemma 5.4. *The set of values* $\{2i \mid 0 \le i \le 186\} \cup \{376, 378, 384\}$ *is a subset of the set INGDD*$(54; \{6^1, 12^4\}; 3)$.

Proof. Consider the construction of the NGDD$(54; \{6^1; 12^4\}; 3)$ in Lemma 3.4. We will consruct several NGDD$(54; \{6^1; 12^4\}; 3)$. Note that by Lemma 5.1 there exist NGDD$(8;2;3)$ that intersect in 0, 2 or 8 blocks. In each construction we use the same GDD$(27; \{3^1, 6^4\}; 4\})$ whilst varying the NGDD $(8;2;3)$ used; hence, the intersection numbers we achieve occur from combinations of the intersection numbers for the NGDD$(8;2;3)$. There are forty-eight blocks in the GDD$(27; \{3^1, 6^4\}; 4\})$; hence, we use forty-eight NGDD$(8;2;3)$, so, the possible intersection numbers are clearly $48 \star \{0,2,8\} = \{2i \mid 0 \le i \le 186\} \cup \{376, 378, 384\}$. $\square$

Lemma 5.5. *The set of values* $\{2i \mid 0 \leq i \leq 354\} \cup \{712, 714, 720\}$ *is a subset of the set INGDD*(72; 12; 3) *of triangle intersection numbers for NGDD*(72; 12; 3).

Proof. Consider the construction of the NGDD(72; 12; 3) in Lemma 3.5. We will construct several NGDD(72; 12; 3). Note that by Lemma 5.1 there exist NGDD(8; 2; 3) that intersect in 0, 2 or 8 blocks. In each construction we use the same GDD(36; 6; 4) whilst varying the NGDD(8; 2; 3) used; hence, the intersection numbers we achieve occur from combinations of the intersection numbers for the NGDD(8; 2; 3). There are ninety blocks in the GDD(36; 6; 4); hence, we use ninety NGDD(8; 2; 3), so, the possible intersection numbers are clearly $90 \star \{0, 2, 8\} = \{2i \mid 0 \leq i \leq 354\} \cup \{712, 714, 720\}$. □

6 Triangle intersection numbers: NSTS of orders 25, 49, 55 and 73

In this section we apply Construction 2 to the small designs constructed in the previous three sections to obtain the set of triangle intersection numbers for nested Steiner triple systems of order $v = 25$, 49, 55 and 73.

Lemma 6.1. *The set of the triangle intersection numbers for NSTS*(25) *is*

$$INSTS(25) = \{0, 1, 2, \ldots, 94, 96, 100\}.$$

Proof. By Lemma 5.2 there exists NGDD(24; 6; 3) with triangle intersection numbers including $\{2i \mid 0 \leq i \leq 30\} \cup \{63, 64, 66, 72\}$ and by Lemma 4.1 there exists NSTS(7) with triangle intersection numbers $\{0, 1, 3, 7\}$. Using Construction 2 we take the seventy-two blocks of the NGDD(24; 6; 3) and on each of the four groups together with the point ∞ we take an NSTS(7) to form an NSTS(25).

There are four NSTS(7) and one NGDD(24; 6; 3) used in the construction. By varying the triangle intersection numbers for these essential designs we obtain the following triangle intersection numbers $4 \star \{0, 1, 3, 7\} + \{2i \mid 0 \leq i \leq 30\} \cup \{63, 64, 66, 72\} = \{i \mid 0 \leq i \leq 18\} \cup \{20, 21, 24, 28\} + \{2i \mid 0 \leq i \leq 30\} \cup \{63, 64, 66, 72\} = \{0, 1, 2, \ldots, 94, 96, 100\}$. □

Lemma 6.2. *The set of triangle intersection numbers for NSTS*(49) *is INSTS*(49) = $\{0, 1, 2, \ldots, 386, 388, 392\}$.

Proof. By Lemma 5.3 there exists NGDD(48; $\{6^4, 12^2\}$; 3) with triangle intersection numbers including $\{0, 2, 4, \ldots, 300, 304, 306, 312\}$, by Lemma 4.1 we have INSTS(7) $= \{0, 1, 3, 7\}$, and by Lemma 4.2 we have INSTS(13) $= \{0, 1, 2, \ldots, 20, 22, 26\}$. Using Construction 2 we start with an NGDD(48; $\{6^4, 12^2\}$; 3); on each of the four groups of size six together with the point ∞ we take an NSTS(7), and on each of the two groups of size twelve together with the point ∞ we take an NSTS(13). This yields an NSTS(49).

There are four NSTS(7), two NSTS(13)s and one NGDD(48; $\{6^4, 12^2\}$; 3) used in the construction. By varying the triangle intersection numbers for these essential designs we obtain the triangle intersection numbers

$$\begin{aligned} & 4\star\{0,1,3,7\}+2\star\{0,1,2,\dots,20,22,26\}+\{0,2,4,\dots,\\ & \qquad\qquad 300,304,306,312\}\\ = & \{i \mid 0\le i\le 18\}\cup\{20,21,24,28\}+\{i\mid 0\le i\le 45\}\cup\{48,52\}\\ & +\{0,2,4,\dots,300,304,306,312\}\\ = & \{0,1,2,\dots,386,388,392\}. \end{aligned}$$

□

Lemma 6.3. *The set of triangle intersection numbers for NSTS*(55) *is INSTS*(55) $=$ $\{0, 1, 2, \dots, 489, 491, 495\}$.

Proof. By Lemma 5.4 there exists NGDD(54; $\{6^1, 12^4\}$; 3) with triangle intersection numbers including $\{0, 2, 4\dots, 372, 376, 378, 384\}$, by Lemma 4.1 there exists NSTS(7) with triangle intersection numbers $\{0, 1, 3, 7\}$, and by Lemma 4.2 there exist pairs of NSTS(13) with triangle intersection numbers $\{0, 1, 2, \dots, 20, 22, 26\}$. Using Construction 2 we start with an NGDD(54; $\{6^1, 12^4\}$; 3); on the group of size six together with the point ∞ we take an NSTS(7), and on each of the four groups of size twelve together with the point ∞ we take an NSTS(13). This yields an NSTS(55).

There are: one NSTS(7), four NSTS(13) and one NGDD(54; $\{6^1, 12^4\}$; 3) used in the construction. By varying the triangle intersection numbers for these essential designs we obtain triangle intersection numbers

$$\begin{aligned} & \{0,1,3,7\}+4\star\{0,1,2,\dots,20,22,26\}+\{0,2,4\dots,372,376,378,384\}\\ = & \{0,1,3,7\}+\{i\mid\ 0\le i\le 97\}\cup\{100,104\}+\{0,2,4\dots,372,376,\\ & \qquad\qquad 378,384\}\\ = & \{0,1,2,\dots,489,491,495\}. \end{aligned}$$

□

Lemma 6.4. *The set of triangle intersection numbers for NSTS*(73) *is INSTS*(73) $=$ $\{0, 1, 2, \dots, 870, 872, 876\}$.

Proof. By Lemma 5.5 there exists NGDD(72; 12; 3) with triangle intersection numbers including $\{0, 2, 4, \dots, 708, 712, 714, 720\}$, and by Lemma 4.2 there exists NSTS(13) with triangle intersection numbers $\{0, 1, 2, \dots, 20, 22, 26\}$. Using Construction 2 we start with an NGDD(72 : 12; 3), and on each of the six groups of size twelve together with the point ∞ we take an NSTS(13). This yields an NSTS(73).

There are six NSTS(13) and one NGDD(72; 12; 3) used in the construction. By varying the triangle intersection numbers for these essential designs we obtain triangle intersection numbers

$$\begin{aligned} & 6\star\{0,1,2,\ldots,20,22,26\}+\{0,2,4,\ldots,708,712,714,720\} \\ = \;& \{i \mid 0\le i\le 149\}\cup\{152,156\}+\{0,2,4,\ldots,708,712,714,720\} \\ = \;& \{0,1,2,\ldots,870,872,876\}. \end{aligned}$$

□

7 The main result

Theorem 7.1. *For all $n\equiv 1\,(mod\ 6)$, the set INSTS(n) is equal to the set ISTS(n), with the possible exceptions $n=19$, 31 and 37.*

Proof. We begin by considering $n=2m+1$, where $m\equiv 0$ or $3\,(\text{mod } 12)$ and $m\ge 39$; we then consider the case $m\equiv 6$ or $9\,(\text{mod } 12)$ and $m\ge 21$. In both cases we apply Construction 1 to obtain the required nested group divisible design and then Construction 2 to extend this group divisible design to a Steiner triple system.

For $m\equiv 0$ or $3\,(\text{mod } 12)$ and $m\ge 39$, there exists a GDD$(m;\{3^\alpha,12^*\};4)$, where $\alpha\ge 9$, (see [1], Section IV.1) with point set V and block set $\mathcal{B}$ consising of $|\mathcal{B}|=b$ $=3\alpha(\alpha{+}7)/4$ blocks of size 4. Denote the α groups of size 3 by G_i, $1\le i\le\alpha$, and the one group of size 12 by G_0. By Lemma 3.2 there exists an NGDD$(8;2;3)$; so applying Construction 1 to the GDD$(m;\{3^\alpha,12^*\};4)$ we obtain an NGDD$(2m;\{6^\alpha,24^*\};3)$. This NGDD$(2m;\{6^\alpha,24^*\};3)$ has a point set of size $2m$, partitioned into α groups of size 6 and one group of size 24. The blocks are of size 3.

Next we note that by Lemmas 4.1 and 6.1, respectively, there exists an NSTS(7) and an NSTS(25). Thus we may adjoin the point ∞ to each group of the NGDD$(2m;$ $\{6^\alpha,24^*\};3)$ and apply Construction 2 to obtain an NSTS$(2m+1)$.

By Lemmas 4.1 and 6.1, the triangle intersection numbers for NSTS(7) and NSTS(25) are, respectively, $\{0,1,3,7\}$ and $\{0,1,2,\ldots,94,96,100\}$. By Lemma 5.1 the triangle intersection numbers for NGDD$(8;2;3)$ are $\{0,2,8\}$.

There are $(m{-}12)/3$ STS(7), one STS(25) and $(m{-}12)(m{+}9)/12$ NGDD$(8;2;3)$ used in the construction. By varying the intersection numbers for these essential designs we obtain the triangle intersection numbers

$$\begin{aligned} & \tfrac{m-12}{3}\star\{0,1,3,7\}+\{0,1,2,\ldots,94,96,100\}+\tfrac{(m-12)(m+9)}{12}\star\{0,2,8\} \\ = \;& \{i\mid 0\le i\le 7(\tfrac{m-12}{3})-10\} \\ & \cup\{7(\tfrac{m-12}{3})-8,7(\tfrac{m-12}{3})-6,7(\tfrac{m-12}{3})-7,7(\tfrac{m-12}{3})-4,7(\tfrac{m-12}{3})\} \\ & \quad +\{0,1,2,\ldots,94,96,100\} \\ & \quad +\{2i\mid 0\leqslant i\leqslant \tfrac{2(m-12)(m+9)}{3}-12\} \\ & \qquad \cup\{\tfrac{2(m-12)(m+9)}{3}-8,\ \tfrac{2(m-12)(m+9)}{3}-6,\ \tfrac{2(m-12)(m+9)}{3}\} \\ = \;& \{0,1,2,\ldots,b-6,b-4,b\} \end{aligned}$$

where $b = \frac{(2m+1)(2m)}{6}$ for the NSTS$(2m+1)$ with $m \geq 39$.

For $m \equiv 6$ or $9 \pmod{12}$ and $m \geq 21$ there exists a GDD$(m; \{3^\alpha, 6^*\}; 4)$, where $\alpha \geq 5$, with point set V and block set $\mathcal{B}$ consisting of $|\mathcal{B}| = b = (9k^2 + 33k)/12$ blocks of size 4. Denote the α groups of size 3 by G_i, $1 \leq i \leq \alpha$ and the one group of size 6 by G_0. There exists an NGDD$(8; 2; 3)$ by Lemma 3.2 and so applying Construction 1 to the GDD$(m; \{3^\alpha, 6^*\}; 4)$ we obtain a NGDD$(2m; \{6^\alpha, 12^*\}; 3)$. This NGDD$(2m; \{6^\alpha, 12^*\}; 3)$ has a point set of size $2m$, partitioned into α groups of size 6 and one group of size 12. The blocks are of size 3.

Next we note that by Lemmas 4.1 and 4.2, respectively, there exists an NSTS(7) and an NSTS(13). Thus we may adjoin the point ∞ to each group of the NGDD$(2m; \{6^\alpha, 12^*\}; 3)$ and apply Construction 2 to obtain an NSTS$(2m+1)$.

By Lemmas 4.1 and 4.2, the triangle intersection numbers for NSTS(7) and NSTS(13) are, respectively, $\{0, 1, 3, 7\}$ and $\{0, 1, 2, \dots, 20, 22, 26\}$. By Lemma 5.1 the triangle intersection numbers for an NGDD$(8; 2; 3)$ are $\{0, 2, 8\}$.

There are $\frac{m-6}{3}$ STS(7), one STS(13) and $\frac{(m-6)(m+3)}{12}$ NGDD$(8; 2; 3)$ used in the construction. By varying the intersection numbers for these essential designs we obtain the triangle intersection numbers

$$\begin{aligned}
&\tfrac{m-6}{3} \star \{0,1,3,7\} + \{0,1,2,\dots,20,22,26\} + \tfrac{(m-6)(m+3)}{12} \star \{0,2,8\} \\
&= \{i \mid 0 \leq i \leq 7(\tfrac{m-6}{3}) - 10\} \cup \{7(\tfrac{m-6}{3}) - 8, 7(\tfrac{m-6}{3}) - 7, 7(\tfrac{m-6}{3}) - 4, \\
&\quad 7(\tfrac{m-6}{3})\} + \{0,1,2,\dots,20,22,26\} \\
&\quad + \{2i \mid 0 \leq i \leq 4(\tfrac{(m-6)(m+3)}{12}) - 6\} \\
&\qquad \cup \{8(\tfrac{(m-6)(m+3)}{12}) - 8, 8(\tfrac{(m-6)(m+3)}{12}) - 6, 8(\tfrac{(m-6)(m+3)}{12})\} \\
&= \{0,1,2,\dots,b-6,b-4,b\}
\end{aligned}$$

where $b = \frac{(2m+1)(2m)}{6}$ for the NSTS$(2m+1)$ with $m \geq 21$.

If $m \geq 39$ and $m \equiv 0$ or $3 \pmod{12}$, then the first construction proves that INSTS$(2m+1)$ = ISTS$(2m+1)$. If $m \geq 21$ and $m \equiv 6$ or $9 \pmod{12}$, the second construction proves that INSTS$(2m+1)$ = ISTS$(2m+1)$. The Lemmas 4.1, 4.2, 6.1, 6.2, 6.3 and 6.4 prove that, for $k \in \{7, 13, 25, 49, 55, 73\}$, INSTS$(k)$ = ISTS(k).

Hence, for all $n \equiv 1 \pmod 6$, the set INSTS(n) is equal to ISTS(n), with just the possible exceptions $n = 19$, 31 and 37 remaining. □

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(Received 24 Feb 2011)