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# Feedback numbers of Kautz undirected graphs*

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**Abstract**

The feedback number $f(d,n)$ of the Kautz undirected graph $\mathrm{UK}(d,n)$ is the minimum number of vertices whose removal results in an acyclic graph. This paper shows $\lceil (d^{n+1}-d^{n-1}-\frac{1}{2}d(d+1)+1)/(2d-1)\rceil \leq f(d,n) \leq d^n-(\lfloor \frac{d^2}{4}\rfloor+1)d^{n-2}$, which implies that $f(2,n)=2^{n-1}$, as obtained by Královič and Ružička [*Information Processing Letters* 86 (4) (2003), 191–196].

## 1 Introduction

Let $G=(V,E)$ be a simple graph, i.e., loopless and without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. It is well-known that the cycle rank of a graph $G$ is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if $G$ has $\upsilon$ vertices, $\varepsilon$ edges, and $\omega$ connected components, then the minimum number of edges whose deletion from $G$ leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G)=\varepsilon-\upsilon+\omega$ (see, for example Xu [21]). A corresponding problem is the removal of vertices. A subset $F\subset V(G)$ is called a *feedback vertex set* if the subgraph $G-F$ is acyclic, that is, if $G-F$ is a forest. The minimum cardinality of all feedback vertex sets is called the *feedback number* (or *decycling number* proposed first by Beineke and Vandell [5]) of $G$. A feedback vertex set of this cardinality is called a *minimum feedback vertex set.*

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Determining the feedback number of a graph $G$ is equivalent to finding the greatest order of an induced forest of $G$, proposed first by Erdős, Saks and Sós [8], since the sum of the two numbers equals the order of $G$. Thus, it has attracted the interest of many researchers, both for general graphs and for specific graphs. A review of recent results and open problems on the decycling number is provided by Bau and Beineke [4].

Apart from its graph-theoretical interest, the minimum feedback vertex set problem has important application to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [16], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters' placement problem (see [7, 9]).

In fact, the problem of finding a feedback number is NP-hard for general graphs [11]. The best known approximation algorithm for this problem has approximation ratio 2 [1]. There are also polynomial time algorithms for a number of topologies, such as reducible graphs [18], cocomparability graphs [11], convex bipartite graphs [11], cyclically reducible graphs [19], and interval graphs [14]. Recently, the lower and the upper bounds of the feedback numbers have been established and improved on some graphs, such as hypercubic graphs, meshes, toroids, butterflies, cube-connected cycles, hypercubes, directed split-stars, de Bruijn digraphs, generalized de Bruijn digraphs and de Bruijn undirected graphs (see [1, 2, 3, 7, 9, 10, 14, 15, 17, 18, 19, 22, 23, 24]).

In this paper, we consider an important class of interconnection networks, namely, Kautz networks first proposed by Kautz in 1969 [12]. Structurally, Kautz networks are very similar to de Bruijn networks, and thus have as many desirable properties as de Bruijn networks (see, for example, Section 3.3 in [20]). Moreover, Kautz networks are an improvement over de Bruijn networks, and have also been thought of as good candidates for the next generation of parallel system architectures, after the hypercube networks [6].

For two given integers $d \geq 2$ and $n \geq 1$, the Kautz digraph $K(d, n)$ is defined as follows. The vertex set of $K(d, n)$ is

$$V(d,n) = \{x_1x_2\ldots x_n \mid x_i \in \{0,1,2,\ldots,d\},\ x_i \neq x_{i+1}, i = 1,2,\ldots,n-1\}$$

and the edge set $E(d, n)$ consists of all edges from one vertex $x_1x_2\ldots x_n$ to $d$ other vertices $x_2x_3\ldots x_n\alpha$, where $\alpha \in \{0, 1, \ldots, d\}$ and $\alpha \neq x_n$. A pair of edges in $K(d, n)$ is said to be symmetric if they have the same end-vertices but different orientations.

It is clear that $K(d, n)$ is $d$-regular, $|V(d, n)| = d^n + d^{n-1}$ and $|E(d, n)| = d^{n+1} + d^n$. Moreover, $K(d, n)$ has $\frac{1}{2}\,d(d + 1)$ pairs of symmetric edges.

The Kautz undirected graph, denoted by $\mathrm{UK}(d, n)$, is an undirected graph obtained from $K(d, n)$ by deleting the orientation of all edges and omitting multiple edges.

It is clear that $\mathrm{UK}(d, n)$ has $d^{n+1} + d^n - \frac{1}{2}d(d + 1)$ edges, the maximum degree is $2d$ for $n \geq 3$ and the minimum degree is $2d - 1$ for $n \geq 2$.

Use $f(d, n)$ to denote the feedback number of $\mathrm{UK}(d, n)$. Královič and Ružička [13] determined that $f(2, n) = 2^{n-1}$. In this paper, by refining their technique, we obtain the following result. For any $d \geq 2$ and $n \geq 3$, the following holds:

$$\left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d-1} \right\rceil \leq f(d,n) \leq d^n - \left( \left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}.$$

This result implies that $f(2,n) = 2^{n-1}$, due to Královič and Ružička [13].

The main result of the theorem is the upper bound of $f(d,n)$. To obtain this upper bound, we first construct a cycle-free set in Section 2; its complementary set is a feedback vertex set. Thus the upper bound can be established in Section 3.

## 2 Constructing a Cycle-free Set

Throughout this paper, we follow Xu [21] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a graph and $S \subset V(G)$. The symbol $N_G(S)$ denotes the set of neighbors of $S$, namely, $N_G(S) = \{x \in V(G - S) : xy \in E(G), y \in S\}$. The subgraph induced by $S$ is denoted by $G[S]$. The set $S$ is independent if no two of vertices in $S$ are adjacent in $G$, and is cycle-free if $G[S]$ has no cycles.

Throughout this paper, we assume that $G = \mathrm{UK}(d,n)$ with $n \geq 3$. For $d \geq 2$, let $I_d = \{0, 1, 2, \ldots, d\}$, and $J_d$ be the set of integers in the interval $[\lfloor \frac{d}{2} \rfloor + 1, d]$. For $x = x_1 x_2 \ldots x_n \in V(G)$, let

$$N_G^{(L)}(x) = \{\alpha x_1 x_2 \ldots x_{n-1} \mid \alpha \in I_d, \alpha \neq x_1\},$$
$$N_G^{(R)}(x) = \{x_2 x_3 \ldots x_n \beta \mid \beta \in I_d, \beta \neq x_n\}.$$

Then $N_G(x) = N_G^{(L)}(x) \cup N_G^{(R)}(x)$. Define $\lfloor \frac{d}{2} \rfloor + 1$ subsets of $V(G)$ as follows.

$$\begin{aligned}
F_0 &= \{x_1 x_2 x_3 \ldots x_n \in V(G) \mid x_1 = 0\}, \\
F_1 &= \{x_1 x_2 x_3 \ldots x_n \in V(G) \mid x_1 = 1, x_2 \in \{0\} \cup J_d\}, \\
F_2 &= \{x_1 x_2 x_3 \ldots x_n \in V(G) \mid x_1 = 2, x_2 \in J_d\}, \\
&\vdots \\
F_{\lfloor \frac{d}{2} \rfloor} &= \{x_1 x_2 x_3 \ldots x_n \in V(G) \mid x_1 = \lfloor \frac{d}{2} \rfloor, x_2 \in J_d\}, \\
F &= F_0 \cup F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_{\lfloor \frac{d}{2} \rfloor}.
\end{aligned} \tag{2.1}$$

**Lemma 2.1.** $|F| = d^{n-1} + (\lfloor \frac{d^2}{4} \rfloor + 1)\, d^{n-2}$.

*Proof.* By the definition of $F$ in (2.1), it is easy to verify that $F_i \cap F_j = \emptyset$ for $0 \leq i \neq j \leq \lfloor \frac{d}{2} \rfloor$, and

$$|F_0| = d^{n-1}, \;\; |F_1| = (d - \lfloor \tfrac{d}{2} \rfloor + 1)\, d^{n-2} = (\lceil \tfrac{d}{2} \rceil + 1)\, d^{n-2},$$
$$|F_i| = (d - \lfloor \tfrac{d}{2} \rfloor)\, d^{n-2} = \lceil \tfrac{d}{2} \rceil d^{n-2} \text{ for each } i = 2, 3, \ldots, \lfloor \tfrac{d}{2} \rfloor.$$

It follows that

$$\begin{aligned}
|F| &= |F_0| + |F_1| + |F_2| + \cdots + |F_{\lfloor \frac{d}{2} \rfloor}| \\
&= d^{n-1} + (\lceil \tfrac{d}{2} \rceil + 1)\, d^{n-2} + (\lfloor \tfrac{d}{2} \rfloor - 1) \lceil \tfrac{d}{2} \rceil d^{n-2} \\
&= d^{n-1} + \lfloor \tfrac{d}{2} \rfloor \lceil \tfrac{d}{2} \rceil d^{n-2} + d^{n-2} \\
&= d^{n-1} + (\lfloor \tfrac{d^2}{4} \rfloor + 1)\, d^{n-2},
\end{aligned}$$

as required. □

**Lemma 2.2.** *$F_i$ is an independent set of* UK$(d,n)$ *for each* $i = 0, 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$.

*Proof.* Vertices in $F_i$ are all starting with the same entry; it is clear that no edge connects any two of them in UK$(d,n)$. Thus $F_i$ is an independent set for each $i = 0, 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$. □

**Lemma 2.3.** *$F_0 \cup F_1$ is a cycle-free set of $UK(d,n)$.*

*Proof.* Let $G$ = UK$(d,n)$. For the sake of contradiction, we assume that there exists a cycle in $G[F_1 \cup F_2]$. Then the length of $C$ is even since both $F_0$ and $F_1$ are independent sets by Lemma 2.2. Let $C = (v_1, v_2, \dots, v_{2m+1})$, where $v_{2m+1} = v_1$. Without loss of generality, we may suppose $v_1 \in F_0$. Then $v_{2i-1} \in F_0$ and $v_{2i} \in F_1$ for each $i = 1, 2, \dots, m+1$.

If $C$ is a directed cycle in the digraph $K(d,n)$, then $v_1 = 0101\dots01$ when $n$ is even and $v_1 = 0101\dots010$ when $n$ is odd. Thus $C$ is a digraph cycle of length 2, a contradiction.

If $C$ is not a directed cycle, then there must be a vertex $u$ with in-degree 2. Suppose $x, y \in N_G^{(L)}(u)$ and $x, y$ are vertices in $C$. Without loss of generality, we may suppose $u \in F_0$ and $u = 0u_2 \dots u_n$. Then $N_G^{(L)}(u) = \{\alpha 0 u_2 \dots u_{n-1} \mid \alpha \in I_d, \alpha \neq 0\}$. But $x, y$ have the same first entry 1, which means $x = y$, a contradiction.

The lemma follows. □

**Lemma 2.4.** *$F$ is a cycle-free set of* UK$(d,n)$.

*Proof.* Let $G$ = UK$(d,n)$, and for each $i = 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$, let

$$U_i \;=\; F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_i \;\text{ and }\; G_i = G[U_i].$$

To prove that $F$ is a cycle-free set of UK$(d,n)$, we only need to prove that $G_{\lfloor \frac{d}{2} \rfloor}$ is acyclic. The proof proceeds by induction on $i \geq 1$.

By Lemma 2.3, $G_1$ is acyclic. Assume the induction hypothesis for $i-1$ with $i \geq 2$. Let $v$ be any vertex in $F_i$. We only need to prove that only one of the neighbors of $v$ is in $G_{i-1}$ since $F_i$ is an independent set by Lemma 2.2.

Let $v = i\alpha x_3 x_4 \dots x_n$, where $\alpha \in J_d$. Then

$$\begin{aligned} N_G^{(L)}(v) &= \{\gamma i \alpha x_3 x_4 \dots x_{n-1} \mid \gamma \in I_d, \gamma \neq i\}, \\ N_G^{(R)}(v) &= \{\alpha x_3 x_4 \dots x_n \beta \mid \beta \in I_d, \beta \neq x_n\}. \end{aligned}$$

Since $\alpha > i$, $N_G^{(R)}(v) \cap U_{i-1} = \emptyset$. Moreover, $N_G^{(L)}(v) \cap U_{i-1} \neq \emptyset$ implies $\gamma = 0$ in $N_G^{(L)}(v)$, i.e., $N_G^{(L)}(v) \cap U_{i-1} = \{0 i \alpha x_3 x_4 \dots x_{n-1}\}$, a single vertex in $F_0$. This is due to, in $F_k$ $(k \geq 1)$, any vertex $u = k\alpha x_3 x_4 \dots x_n$ with $\alpha \in J_d$ but clearly satisfying $i \notin J_d$. Thus $u = 0 i \alpha x_3 x_4 \dots x_{n-1}$ is a unique neighbor of $v$ in $G_{i-1}$.

So we have completed the proof of the lemma. □

## 3 Establishing Bounds

In this section, we prove the main result of our paper. We first restate a lower bound on the size of the feedback vertex set in any graph, due to Beineke and Vandell [5].

**Lemma 3.1.** *For feedback vertex set $F$ in a graph $G = (V, E)$ with maximum degree $\triangle$, the following holds:*

$$|F| \geq \left\lceil \frac{|E| - |V| + 1}{\triangle - 1} \right\rceil.$$

**Theorem 3.1.** *For any $d \geq 2$ and $n \geq 3$, the following holds:*

$$\left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d - 1} \right\rceil \;\leq\; f(d,n) \;\leq\; d^n - \left( \left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}.$$

**Proof.** Noting that $\mathrm{UK}(d,n)$ has $d^n + d^{n-1}$ vertices and $d^{n+1} + d^n - \frac{1}{2}d(d+1)$ edges, and the maximum degree is $2d$ for $n \geq 3$, by Lemma 3.1, we immediately obtain the lower bound:

$$f(d,n) \;\geq\; \left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d - 1} \right\rceil. \tag{3.2}$$

We now establish the upper bound

$$f(d,n) \leq d^n - \left( \left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}. \tag{3.3}$$

Let $\overline{F} = V(\mathrm{UK}(d,n)) \backslash F$, where $F$ is defined in (2.1). Then $\overline{F}$ is a feedback vertex-set of $\mathrm{UK}(d,n)$ since $F$ is a cycle-free set of $\mathrm{UK}(d,n)$ by Lemma 2.4. It follows from Lemma 2.1 that

$$\begin{aligned} f(d,n) &\leq |\overline{F}| = |V(d,n)| - |F| \\ &= (d^n + d^{n-1}) - (d^{n-1} + (\lfloor \frac{d^2}{4} \rfloor + 1)\, d^{n-2}) \\ &= d^n - (\lfloor \frac{d^2}{4} \rfloor + 1)\, d^{n-2}. \end{aligned}$$

as required. □

**Corollary 3.1.** (Královič and Ružička [13]) *If $n \geq 3$, then $f(2,n) = 2^{n-1}$.*

*Proof.* On the one hand, by (3.3) we have

$$f(2,n) \;\leq\; 2^n - 2 \cdot 2^{n-2} \;=\; 2^{n-1}.$$

On the other hand, by (3.2) we have

$$f(2,n) \;\geq\; \left\lceil \frac{2^{n+1} - 2^{n-1} - 2}{3} \right\rceil \;=\; \left\lceil 2^{n-1} - \frac{2}{3} \right\rceil \;=\; 2^{n-1}.$$

Thus, $f(2,n) = 2^{n-1}$. □

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