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Minimizing the weight of the union-closure of families of two-sets

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Abstract

It is proved that, for any positive integer m, the weight of the union-closure of any m distinct 2-sets is at least as large as the weight of the union-closure of the first m 2-sets in squashed (antilexicographic) order, where all i-sets have the same non-negative weight w_i with $w_i \leq w_{i+1}$ for all i, and the weight of a family of sets is the sum of the weights of its members. As special cases, solutions are obtained for the problems of minimising size and volume of the union-closure of a given number of distinct 2-sets.

1 Introduction

In recent years there has been a number of papers published on Frankl's Union-Closed Sets Conjecture, which is that for any non-empty union-closed collection of sets there is an element appearing in at least half the sets (see [1] and its list of references, see also [3]). There has, however, been little investigation of such collections independent of this conjecture.

This paper is a step towards solving the following problem: Given positive integers m, i, find a family of m i-sets such that its union-closure is of smallest possible size. The only known results on this problem of minimising union-closure appear in [2], where it is shown that the family of the first m 2-sets in squashed (antilexicographic) order (see definition below) solves the above problem for $i = 2$, and under the assumption of a ground set of smallest possible size. This paper removes the assumption of a minimum size ground set, and a more general weighted version of the result is given.

Let $[n] := \{1, 2, \ldots, n\}$, and let $2^{[n]}$ denote the power set of $[n]$. All sets and families of sets in this paper are subsets of $[n]$ and $2^{[n]}$, respectively. If $|F| = i$, then F is called an i-set. The family of i-subsets of $[n]$ is denoted by $\binom{[n]}{i}$. Given a family $\mathcal{F}$, the following notations are used:

$$\begin{aligned} \mathcal{F}_x &:= \{F \in \mathcal{F} : x \in F\}, \\ \mathcal{F}_{\overline{x}} &:= \{F \in \mathcal{F} : x \notin F\}. \end{aligned}$$

$\mathbb{R}^+$ denotes the set of non-negative real numbers. A function $w : 2^{[n]} \mapsto \mathbb{R}^+$ is called a *weight function* on $2^{[n]}$, and the weight of a family $\mathcal{F}$ of sets is defined as $w(\mathcal{F}) := \sum_{F\in\mathcal{F}} w(F)$ with $w(\mathcal{F}) = 0$ if $\mathcal{F}$ is empty.

Property 1. *A weight function w on $2^{[n]}$ is said to have Property 1 if there are $w_0, w_1, \ldots, w_n \in \mathbb{R}^+$ such that $w_0 \le w_1 \le \cdots \le w_n$ and $w(F) = w_i$ whenever $|F| = i$.*

Two particular weight functions with Property 1 are of special interest. If $w(F) = 1$ for all $F \subseteq [n]$, then the weight $w(\mathcal{F})$ of a family $\mathcal{F}$ is equal to its size $|\mathcal{F}|$. If $w_i = i$ for all i, then $w(\mathcal{F})$ is equal to the *volume*

$$V(\mathcal{F}) := \Sigma_{F\in\mathcal{F}} |F|$$

of $\mathcal{F}$.

The *union-closure* of a family $\mathcal{F}$ is

$$\mathrm{UC}(\mathcal{F}) := \left\{\bigcup_{G\in\mathcal{G}} G : \emptyset \ne \mathcal{G} \subseteq \mathcal{F}\right\} \cup \left(\mathcal{F} \cap \{\emptyset\}\right).$$

$\mathcal{F}$ is said to be *union-closed* if $\mathrm{UC}(\mathcal{F}) = \mathcal{F}$. The *generating set* of a union-closed family $\mathcal{F}$ is the family $\mathcal{B}_\mathcal{F}$ of all members of $\mathcal{F}$ that are not the union of two distinct sets in $\mathcal{F}$, and $\mathcal{F}$ is said to be *generated by* $\mathcal{B}_\mathcal{F}$.

For two distinct $F, G \subseteq [n]$, F occurs before G in *squashed order*, denoted by $F <_S G$, whenever

$$\max(F \Delta G) \in G,$$

where $F \Delta G$ denotes the symmetric difference of F and G.

The *compressed* family $\mathcal{C}(m, i)$ is the family of the first m i-sets in squashed order. Throughout this paper, $\mathcal{C}(m, 2)$ is abbreviated by $\mathcal{C}(m)$, and $\mathcal{U}(m)$ denotes the union-closure of $\mathcal{C}(m)$.

Furthermore, integers m with $0 \le m \le \binom{n}{2}$ will be represented in the form

$$m = \binom{a}{2} + b \quad \text{with} \quad 0 \le b \le a, \tag{1}$$

where a and b are integers. Note that, for given m, the integers a, b are uniquely determined, unless $m = \binom{c}{2}$ for some integer $c \ge 1$. In the latter case, both, $a = c$, $b = 0$ or $a = b = c - 1$, are allowed. In particular, if $m = 0$, then $a \in \{0, 1\}$ and $b = 0$. This convention will come in handy in the proof of the main result (Theorem 3 below).

Proposition 2. *Let m and $\mathcal{U}(m)$ be as above, and let $w : 2^{[n]} \mapsto \mathbb{R}^+$ be a weight function having Property 1, then*

$$w(\mathcal{U}(m)) \;=\; \sum_{i=2}^{a+1} \left(\binom{a+1}{i} - \binom{a-b}{i-1} \right) w_i.$$

In particular, $w(\mathcal{U}(0)) = 0$.

Proof. By the definition of the squashed order,

$$\mathcal{C}(m) \;=\; \binom{[a]}{2} \cup \big\{ \{h, a+1\} : h \in [b] \big\} \;=\; \binom{[a+1]}{2} \setminus \big\{ \{j, a+1\} : j \in [a] \setminus [b] \big\}.$$

This implies that

$$\mathcal{U}(m) \cap \binom{[n]}{i} \;=\; \binom{[a+1]}{i} \setminus \big\{ S \cup \{a+1\} : S \subseteq [a] \setminus [b],\ |S| = i - 1 \big\}$$

for $i \ge 2$. Hence, the number of i-sets in $\mathcal{U}(m)$ is $\binom{a+1}{i} - \binom{a-b}{i-1}$ for every $i \ge 2$, which implies the claim. ∎

In the next section we prove a lemma which will be used to prove our main result, namely Theorem 3. Its proof is given in Section 3 where it is followed by a discussion of the theorem's consequences and by two conjectures.

Theorem 3. *Let $m \le \binom{n}{2}$ be a non-negative integer. If $\mathcal{G}$ is a family of m 2-sets and $w : 2^n \mapsto \mathbb{R}^+$ is a weight function having Property 1, then*

$$w(\mathrm{UC}(\mathcal{G})) \;\ge\; w(\mathcal{U}(m)). \tag{2}$$

2 Preparations

Lemma 4. *Let $\mathcal{G}$ be a non-empty subset of $\binom{[n]}{2}$ and $\mathcal{F} := \mathrm{UC}(\mathcal{G})$. Furthermore, let w and w' be weight functions on $2^{[n]}$ such that w has Property 1 and $w'(F) = w_{i+1}$ whenever $|F| = i < n$. Then there exists an $x \in [n]$ with $|\mathcal{G}_x| \ge 1$ such that*

$$w(\mathcal{F}_x) \;\ge\; w'(\mathcal{F}_{\overline{x}}) + |\mathcal{G}_x| \cdot w_2. \tag{3}$$

Proof. The proof is based on the simple argument that was used by Sarvate and Renaud to show that the Union Closed Sets Conjecture is true for families containing a 2-set (see [4]).

Let $\{x,y\} \in \mathcal{G}$, and define

$$\begin{aligned} \mathcal{X} &:= \{F \in \mathcal{F} : \{x,y\} \subseteq F,\ F\setminus\{x\} \notin \mathcal{F},\ F\setminus\{y\} \in \mathcal{F},\ F\setminus\{x,y\} \in \mathcal{F}\}, \\ \mathcal{Y} &:= \{F \in \mathcal{F} : \{x,y\} \subseteq F,\ F\setminus\{y\} \notin \mathcal{F},\ F\setminus\{x\} \in \mathcal{F},\ F\setminus\{x,y\} \in \mathcal{F}\}. \end{aligned}$$

Without loss of generality, we can assume

$$w(\mathcal{X}) \ \geq\ w(\mathcal{Y}). \tag{4}$$

We will show that we then have

$$w(\mathcal{F}_x \setminus \mathcal{G}_x) \ \geq\ w'(\mathcal{F}_{\overline{x}}), \tag{5}$$

which clearly implies (3).

For $F \in \mathcal{Y}$, we define $f(F) := F \setminus \{x,y\}$ and $\mathcal{Y}' := \{f(F) : F \in \mathcal{Y}\}$. By the definitions of $\mathcal{X}$ and $\mathcal{Y}$, we have $\mathcal{X} \subseteq \mathcal{F}_x \setminus \mathcal{G}_x$ and $\mathcal{Y}' \subseteq \mathcal{F}_{\overline{x}}$. Furthermore, $f : \mathcal{Y} \mapsto \mathcal{Y}'$ is a bijection and $|F| = |f(F)| + 2$ for all $F \in \mathcal{Y}$. This implies that $w(\mathcal{Y}) \geq w'(\mathcal{Y}')$, and by (4) we obtain

$$w(\mathcal{X}) \ \geq\ w'(\mathcal{Y}'). \tag{6}$$

For $F \in \mathcal{F}_{\overline{x}} \setminus \mathcal{Y}'$, we define

$$g(F) \ :=\ \begin{cases} F \cup \{x\} & \text{if } F \cup \{x\} \in \mathcal{F}, \\ F \cup \{x,y\} & \text{otherwise.} \end{cases}$$

Let $F \in \mathcal{F}_{\overline{x}} \setminus \mathcal{Y}'$. We will show that $g(F) \in (\mathcal{F}_x \setminus \mathcal{G}_x) \setminus \mathcal{X}$. As $\{x,y\} \in \mathcal{G}$, we have $F \cup \{x,y\} \in \mathcal{F}_x$, which implies $g(F) \in \mathcal{F}_x$. As $|g(F)| \geq |F| + 1 \geq 3$, it follows that $g(F) \notin \mathcal{G}_x$. Finally, assume for a contradiction that $g(F) \in \mathcal{X}$. Then $\{x,y\} \subseteq g(F)$ and $g(F) \setminus \{x\} \notin \mathcal{F}$, which implies that $F = g(F) \setminus \{x,y\}$. As $g(F) \in \mathcal{X}$, we have $g(F) \setminus \{y\} \in \mathcal{F}$. The definition of g and $g(F) \setminus \{y\} = F \cup \{x\}$ imply that $g(F) = F \cup \{x\} \neq g(F)$, which is a contradiction.

We next show that $g : \mathcal{F}_{\overline{x}} \setminus \mathcal{Y}' \mapsto (\mathcal{F}_x \setminus \mathcal{G}_x) \setminus \mathcal{X}$ is injective. Assume for a contradiction that there are sets $F, F' \in \mathcal{F}_{\overline{x}} \setminus \mathcal{Y}'$, $F \neq F'$, and H, such that $g(F) = g(F') = H$. By the definition of g, we have $x \in H$ and

$$F, F' \in \big\{H \setminus \{x\},\ H \setminus \{x,y\}\big\}.$$

As $F \neq F'$, without loss of generality we can assume that $F = H \setminus \{x\}$ and $F' = H \setminus \{x,y\}$, where $y \in H$. As $F' \notin \mathcal{Y}'$, we have $H \notin \mathcal{Y}$. As $F = H \setminus \{x\}$ and $F' = H \setminus \{x,y\}$ are in $\mathcal{F}$, $H \setminus \{y\} \in \mathcal{F}$. On the other hand, $H \setminus \{y\} = F' \cup \{x\}$, and by the definition of g, we obtain $g(F') = H \setminus \{y\} \neq H$, which is a contradiction.

As $g : \mathcal{F}_{\overline{x}} \setminus \mathcal{Y}' \mapsto (\mathcal{F}_x \setminus \mathcal{G}_x) \setminus \mathcal{X}$ is injective, we have

$$w((\mathcal{F}_x \setminus \mathcal{G}_x) \setminus \mathcal{X}) \ \geq\ w'(\mathcal{F}_{\overline{x}} \setminus \mathcal{Y}'). \tag{7}$$

Adding (6) and (7), we obtain (5) which completes the proof. ∎

3 Proof of the main result

Proof of Theorem 3. Let $\mathcal{G}$ be a family of m sets in $\binom{[n]}{2}$, and let $w : 2^n \mapsto \mathbb{R}^+$ have Property 1.

To show (2), we proceed by induction on m. If $m = 0$, then trivially $w(\mathrm{UC}(\mathcal{G})) = w(\mathcal{U}(0)) = 0$. Assume that $m \geq 1$ is represented as in (1), and that the assertion is true for any integer m' with $0 \leq m' < m$.

Let $\mathcal{F}$, w' and x be as in Lemma 4, and define $g_x := |\mathcal{G}_x|$. Without loss of generality, we can assume that $w'(\emptyset) = 0$. By the assumption in Lemma 4, the weight function w' also has Property 1.

Obviously, we have

$$w(\mathcal{F}) \;=\; w(\mathcal{F}_{\overline{x}}) + w(\mathcal{F}_x). \tag{8}$$

Case 1. Assume that $g_x \leq a - 1$.
Note that if $a = b$, then with $a' = a + 1$ we have $m = \binom{a'}{2}$ and $g_x < a' - 1$. Hence, without loss of generality, we can assume $b \leq a - 1$.

By (3), (8), and the induction hypothesis, we obtain

$$w(\mathcal{F}) \;\geq\; w(\mathcal{U}(m - g_x)) + w'(\mathcal{U}(m - g_x)) + g_x w_2.$$

It is clear that the size of $\mathcal{U}(m)$ is strictly increasing with m, so the right hand side decreases as g_x increases, and thus it attains its minimum when $g_x = a - 1$. Hence,

$$w(\mathcal{F}) \;\geq\; w\left(\mathcal{U}\left(\binom{a-1}{2} + b\right)\right) + w'\left(\mathcal{U}\left(\binom{a-1}{2} + b\right)\right) + (a-1)w_2$$

and, by Proposition 2,

$$\begin{aligned} w(\mathcal{F}) \;&\geq\; \sum_{i=2}^{a} \left(\binom{a}{i} - \binom{a-b-1}{i-1}\right)(w_i + w_{i+1}) + (a-1)w_2 \\ &=\; \sum_{i=2}^{a+1} \left(\binom{a+1}{i} - \binom{a-b}{i-1}\right) w_i \\ &=\; w(\mathcal{U}(m)). \end{aligned}$$

Case 2. Assume that $g_x = a \geq b \geq 1$.
In this case, (8) and the induction hypothesis imply

$$\begin{aligned} w(\mathcal{F}) \;&\geq\; w\left(\mathcal{U}\left(\binom{a-1}{2} + (b-1)\right)\right) + w(\mathcal{F}_x) \\ &\geq\; \sum_{i=2}^{a} \left(\binom{a}{i} - \binom{a-b}{i-1}\right) w_i + \sum_{i=2}^{a+1} \binom{a}{i-1} w_i \\ &=\; w(\mathcal{U}(m)). \end{aligned}$$

Case 3. Assume that $g_x = a$, $b = 0$ or $g_x \geq a + 1$.
Note that if $g_x = a$ and $b = 0$, then with $a' = a - 1$ we have $m = \binom{a'}{2} + a'$ and $g_x = a' + 1$. Hence, without loss of generality, we can assume that $g_x \geq a + 1$ and obtain

$$w(\mathcal{F}) \;\geq\; w(\mathcal{F}_x) \;\geq\; \sum_{i=2}^{a+2} \binom{a+1}{i-1} w_i \;\geq\; w(\mathcal{U}(m)).$$

This completes the proof of (2). ∎

4 Cases of special interest

The next corollary provides formulas for size and volume of $\mathcal{U}(m)$. It follows immediately from Theorem 3, as size and volume correspond to weight functions that have Property 1 (see Section 1). The right hand sides of the inequalities in the corollary are equal to $|\mathcal{U}(m)|$ and $V(\mathcal{U}(m))$, respectively, which is easily derived from Proposition 2 and consistent with formulas given in [2].

Corollary 5. *Let $m \le \binom{n}{2}$ be a non-negative integer represented in the form* (1)*. If $\mathcal{G}$ is a family of m 2-subsets of $[n]$, then*

$$|\mathrm{UC}(\mathcal{G})| \;\ge\; 2^{a+1} - 2^{a-b} - a - 1$$

and

$$V(\mathrm{UC}(\mathcal{G})) \;\ge\; (a+1)\cdot 2^a - (a-b+2)\cdot 2^{a-b-1} - a.$$

These bounds are best possible.

As an example consider the case $m = 4$. Since $m = \binom{3}{1} + 1$ we have $a = 3$ and $b = 1$. The first four 2-sets in squashed order are 12, 13, 23, and 14. (For brevity, braces and commas are omitted.) The union-closure of these is

$$\{12, 13, 123, 23, 14, 124, 134, 1234\}.$$

So the size of the union-closure is 8 and its volume is 21, in agreement with the formulas above.

5 Concluding remarks

For two distinct sets $F, G \subseteq [n]$, F occurs before G in *order* U, denoted by $F <_U G$, whenever

$$\begin{aligned} &\max F < \max G, \text{ or} \\ &\max F = \max G \text{ and } \min(F \Delta G) \in F. \end{aligned}$$

Order U and squashed order coincide on $\binom{[n]}{i}$ for $i \le 2$ but not for larger i. For example, the family of the first seven 3-sets in squashed order is (again, using abbreviated notation without braces and commas)

$$\mathcal{F} = \{123, 124, 134, 234, 125, 135, 235\}$$

but the family of the first seven 3-sets in Order U is

$$\mathcal{G} = \{123, 124, 134, 234, 125, 135, 145\}.$$

It is easily checked that both the size and volume of the union-closed family generated by $\mathcal{G}$ are smaller than the corresponding values for the union-closed family generated by $\mathcal{F}$.

It is conjectured in [2] that choosing the first m i-sets in Order U simultaneously minimises both size and volume of the union-closure over all choices of generating families of m i-sets. That conjecture is generalised here.

Conjecture 6. *Let $i \leq n$ and $m \leq \binom{n}{i}$ be non-negative integers, and let $w : 2^{[n]} \mapsto \mathbb{R}^+$ be a weight function having Property 1. If $\mathcal{B}$ and $\mathcal{B}_U$ are a family of m i-subsets of $[n]$ and the family of the first m i-subsets of $[n]$ in order U, respectively, then*

$$w(\mathrm{UC}(\mathcal{B})) \geq w(\mathrm{UC}(\mathcal{B}_U)).$$

In principle, the proof of Theorem 3 can easily be generalised to prove Conjecture 6. However, it would rely upon a generalised version of Lemma 4, the formulation of which is straight-forward. In the case $w \equiv 1$, it becomes the following conjecture.

Conjecture 7. *If $\mathcal{B}$ is a collection of m i-sets and $\mathcal{U} = UC(\mathcal{B})$, then there is an $x \in [n]$ with $\mathcal{B}_x \neq \emptyset$ and such that x is contained in at least half of the sets in $\mathcal{U} \setminus \mathcal{B}_x$.*

In view of the Union Closed Sets Conjecture, a proof of this seems to be out of reach at this point.

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