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Nonexistence of $(3, 2, 1)$-conjugate $(v + 7)$-orthogonal Latin squares*

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Abstract

Two Latin squares of order v are r-orthogonal if their superposition produces exactly r distinct ordered pairs. If the second square is the $(3, 2, 1)$-conjugate of the first one, we say that the first square is $(3, 2, 1)$-conjugate r-orthogonal, denoted by $(3, 2, 1)$-r-COLS(v). The nonexistence of $(3, 2, 1)$-r-COLS(v) for $r \in \{v + 2, v + 3, v + 5\}$ has been proved by Zhang and Xu [*Int. J. Combin. Graph Theory Applic.* 2 no. 2 (2009), 103–109]. In this paper, we show the nonexistence of $(3, 2, 1)$-$(v + 7)$-COLS(v).

1 Introduction

A quasigroup is an ordered pair $(Q, \odot)$, where Q is a set and $\odot$ is a binary operation on Q such that the equations $a \odot x = b$ and $y \odot a = b$ are uniquely solvable for every pair of elements a, b in Q. It is fairly well known (e.g., see [5]) that the multiplication table of a quasigroup defines a Latin square; that is, a Latin square can be viewed as the multiplication table of a quasigroup with the headline and the sideline removed.

If $(Q, \odot)$ is a quasigroup, we may define six binary operations $\odot(1, 2, 3)$, $\odot(1, 3, 2)$, $\odot(2, 1, 3)$, $\odot(2, 3, 1)$, $\odot(3, 1, 2)$ and $\odot(3, 2, 1)$ as follows: $a \odot b = c$ if and only if

$$a \odot (1, 2, 3)b = c, \qquad a \odot (1, 3, 2)c = b, \qquad b \odot (2, 1, 3)a = c,$$
$$b \odot (2, 3, 1)c = a, \qquad c \odot (3, 1, 2)a = b, \qquad c \odot (3, 2, 1)b = a.$$

These six (not necessarily distinct) quasigroups $(Q, \odot(i, j, k))$, where $\{i, j, k\} = \{1, 2, 3\}$, are called the conjugates of $(Q, \odot)$. Since the multiplication table of a quasigroup $(Q, \odot)$ defines a Latin square, say L, then the Latin square defined by

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the multiplication table of its conjugate $(Q, \odot(i,j,k))$ is called the (i,j,k)-conjugate of L.

Two Latin squares of order v, $L = (l_{ij})$ and $M = (m_{ij})$, are said to be *r-orthogonal* if their superposition produces exactly r distinct ordered pairs, that is

$$|\{(l_{ij}, m_{ij}) : 1 \le i,j \le v\}| \;=\; r.$$

Belyavskaya (see [1, 2, 3]) first systematically treated the following question: For which integers v and r does a pair of r-orthogonal Latin squares of order v exist? Evidently, $v \le r \le v^2$, and it is easy to prove that $r \notin \{v+1, v^2-1\}$. Colbourn and Zhu [4], Zhu and Zhang [9, 10] have obtained the following complete result.

Theorem 1.1 [10, Theorem 2.1] *For any integer $v \ge 2$, there exists a pair of r-orthogonal Latin squares of order v if and only if $v \le r \le v^2$ and $r \notin \{v+1, v^2-1\}$ with the exceptions of v and r shown in Table 1.* □

Table 1:

order v	Genuine Exceptions of r
2	4
3	5, 6, 7
4	7, 10, 11, 13, 14
5	8, 9, 20, 22, 23
6	33, 36

Suppose $L = (l_{ij})$ and $M = (m_{ij})$ are r-orthogonal such that M is the (i,j,k)-conjugate of L; then we say that L is (i,j,k)-conjugate *r-orthogonal*, denoted by (i,j,k)-r-COLS(v). If M is the transpose ($(2,1,3)$-conjugate) of L, then L is said to be r-self-orthogonal. The spectrum of r-self-orthogonal Latin squares has almost been completely determined by Xu and Chang [6, 7].

Theorem 1.2 [7, Theorem 6.2] *For any integer $v \ge 2$, there exists an r-SOLS(v) if and only if $v \le r \le v^2$ and $r \notin \{v+1, v^2-1\}$ except for the genuine and possible exceptions listed in Table 2.* □

The determination of the spectrum of $(3,2,1)$-r-COLS(v) may be more difficult. For the nonexistence of $(3,2,1)$-r-COLS(v) we have the following results.

Theorem 1.3 [8, Theorem 1.3] *There exists no $(3,2,1)$-r-COLS(v) with v and r as shown in Table 3.* □

Table 2: Exceptions of r-SOLS(v) for $r \in [v, v^2-1] \setminus \{v+1, v^2-1\}$

order v	Genuine exceptions of r	Possible exceptions of r
2	4	
3	5, 6, 7, 9	
4	6, 7, 8, 10, 11, 12, 13, 14	
5	8, 9, 12, 16, 18, 20, 22, 23	
6	32, 33, 34, 36	
7	46	
12,13,14,15		v^2-5, v^2-4, v^2-3
16,17,18,20		v^2-5, v^2-3
19,21,22,23,24,26		v^2-3

Table 3: Genuine exceptions of $(3,2,1)$-r-COLS(v)

order v	Genuine exceptions of r
2	4
3	5, 6, 7
4	6, 7, 9, 10, 11, 13, 14
5	7, 8, 9, 10, 12, 14, 18, 20, 21, 22, 23
6	8, 9, 11, 13, 31, 32, 33, 34, 36
7	9, 10, 12, 14, 16, 45, 46
8	10, 11, 13, 15, 17, 61

Theorem 1.4 [8, Theorem 3.1] *For any positive integer v, there exist no $(3,2,1)$-r-COLS(v) and $(1,3,2)$-r-COLS(v) for $r \in \{v+2, v+3, v+5\}$.* □

Zhang and Xu [8] conjectured that there exists no $(3,2,1)$-$(v+7)$-COLS(v) for any positive integer v. In this paper, we show that this conjecture is true.

2 Column permutations and cycles

Suppose $L = (l_{ij})$ is a Latin square of order v. We call $\sigma_p = \begin{pmatrix} 1 & 2 & \cdots & v \\ l_{1p} & l_{2p} & \cdots & l_{vp} \end{pmatrix}$ the pth column permutation of L. Generally, we write the permutation into disjoint cycles.

For example, Figure 1 is the multiplication table of a quasigroup of order 5. The first column permutation of the corresponding Latin square is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} = (1)(234)(5)$$

Figure 1: The multiplication table of a quasigroup of order 5

$\odot$	1	2	3	4	5
1	1	2	3	4	5
2	3	4	5	1	2
3	4	5	1	2	3
4	2	3	4	5	1
5	5	1	2	3	4

and the 5th column permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix} = (154)(2)(3).$$

Lemma 2.1 [8, Lemma 2.2] *Suppose $L = (l_{ij})_{v\times v}$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})_{v\times v}$ is the $(3,2,1)$-conjugate of L. Let σ_p and τ_p be the pth column permutations of L and M, respectively:*

$$\sigma_p = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ l_{1p} & l_{2p} & l_{3p} & \cdots & l_{vp} \end{pmatrix},$$

$$\tau_p = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ m_{1p} & m_{2p} & m_{3p} & \cdots & m_{vp} \end{pmatrix}.$$

Then $\tau_p = \sigma_p^{-1}$. ∎

Suppose $L = (l_{ij})_{v\times v}$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})_{v\times v}$ is the $(3,2,1)$-conjugate of L. Let $R = \{(l_{ij}, m_{ij}) : 1 \le i,j \le v\}$; then $|R| = r$. We call R the $(3,2,1)$-*DOP set* (distinct ordered pairs set) of L. Let $\pi_k = (x_1x_2\ldots x_k)$ be a k-cycle in the pth column permutation of L; then we have $l_{x_1,p} = x_2$, $l_{x_2,p} = x_3$, …, $l_{x_k,p} = x_1$. From the definition of $(3,2,1)$-conjugate, we have $m_{x_2,p} = x_1$, $m_{x_3,p} = x_2$, …, $m_{x_1,p} = x_k$. Obviously, there are k distinct ordered pairs, namely, (x_1, x_{k-1}), (x_2, x_k), (x_3, x_1), …, (x_k, x_{k-2}). So we can consider the $(3,2,1)$-DOP set of L column by column. In fact, we always consider the $(3,2,1)$-DOP set of L cycle by cycle.

In this paper, we shall use the notation DOP_{π_k} to denote the set of distinct ordered pairs yielded by a k-cycle $\pi_k = (x_1x_2\ldots x_k)$, that is, $\mathrm{DOP}_{\pi_k} = \{(x_1,x_{k-1}), (x_2,x_k), \ldots, (x_k,x_{k-2})\}$. And we always suppose that every Latin square of order v is based on the set $Q = \{1,2,\ldots,v\}$ and that every cycle is in a column permutation of L.

If $\pi_1 = (x_1)$, then $\mathrm{DOP}_{\pi_1} = \{(x_1,x_1)\}$. If $\pi_2 = (x_1x_2)$, then $\mathrm{DOP}_{\pi_2} = \{(x_1,x_1), (x_2,x_2)\}$. Therefore we get the following lemma.

Lemma 2.2 *Let v be a positive integer. Suppose L is a Latin square of order v and π_k is a k-cycle ($k = 1, 2$) in a column permutation of L. Suppose that DOP$_{\pi_k}$ is the set of distinct ordered pairs yielded by π_k. Then DOP$_{\pi_k}$ is contained in $\{(i,i) : 1 \le i \le v\}$.* ∎

Lemma 2.3 [8, Lemma 2.1] *Suppose R is the $(3,2,1)$-DOP set of a $(3,2,1)$-r-COLS(v). Then $\{(i,i) : 1 \le i \le v\} \subset R$.* $\square$

Let $L = (l_{ij})_{v\times v}$ be a Latin square and $M = (m_{ij})_{v\times v}$ be the $(3,2,1)$-conjugate of L. Let $P = \{(l_{ij}, m_{ij}) : l_{ij} \ne m_{ij}, 1 \le i, j \le v\}$. From Lemma 2.3 we have $|P| = r - v$. The nonexistence of $(3,2,1)$-$(v+7)$-COLS(v) means that $|P| \ne 7$ for any Latin square. If $P \ne \emptyset$, then from Lemma 2.2 we know that P must be produced by k-cycles with $k \ge 3$.

Lemma 2.4 *Let v and k be positive integers. Suppose $L = (l_{ij})_{v\times v}$ is a Latin square and $M = (m_{ij})_{v\times v}$ is the $(3,2,1)$-conjugate of L. Let π_k and π'_k be k-cycles in column permutations of L and DOP_{π_k} be the set of distinct ordered pairs yielded by π_k. Then we have the following results:*

(1) If $3 \le k \le 6$, then $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi_k}| \ne k-1$.

(2) If $\mathrm{DOP}_{\pi_5} \ne \mathrm{DOP}_{\pi'_5}$, then $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi'_5}| \le 3$.

(3) $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_4}| \le 2$.

(4) $|\mathrm{DOP}_{\pi_3} \cap \mathrm{DOP}_{\pi'_3}| = 0$.

(5) $|\mathrm{DOP}_{\pi_4} \cap \mathrm{DOP}_{\pi'_4}| = 0, 2, 4$.

(6) $|\mathrm{DOP}_{\pi_4} \cap \mathrm{DOP}_{\pi_3}| \le 1$.

Proof. (1) Suppose $\pi_6 = (x_1x_2x_3x_4x_5x_6)$ and $\pi_k = (y_1y_2\dots y_k)$. Then

$$\mathrm{DOP}_{\pi_6} = \{(x_1,x_5),(x_2,x_6),(x_3,x_1),(x_4,x_2),(x_5,x_3),(x_6,x_4)\}$$

and

$$\mathrm{DOP}_{\pi_k} = \{(y_1,y_{k-1}),(y_2,y_k),(y_3,y_1),\dots,(y_k,y_{k-2})\}.$$

Case 1: $k=3$. If (y_1,y_2), $(y_2,y_3) \in \mathrm{DOP}_{\pi_6}$, it is easy to see $(y_3,y_1) \in \mathrm{DOP}_{\pi_6}$. Likewise, for any two pairs in DOP_{π_3}, if they are contained in DOP_{π_6}, then the third pair in DOP_{π_3} is also contained in DOP_{π_6}. Therefore $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi_3}| \ne 2$.

Case 2: $k=4$. If $(y_1,y_3) \in \mathrm{DOP}_{\pi_6}$, then we have $(y_3,y_1) \notin \mathrm{DOP}_{\pi_6}$, so $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi_4}| \le 2$.

Case 3: $k=5$. Without loss of generality, we may suppose $x_6 \notin \{y_1,y_2,y_3,y_4,y_5\}$. Then we have $(x_2,x_6) \notin \mathrm{DOP}_{\pi_5}$ and $(x_6,x_4) \notin \mathrm{DOP}_{\pi_5}$. If $(x_1,x_5),(x_3,x_1) \in \mathrm{DOP}_{\pi_6}$, then $(x_5,x_3) \notin \mathrm{DOP}_{\pi_5}$. So there is at least one element which is contained in $\{(x_1,x_5),(x_3,x_1),(x_5,x_3)\}$ but not contained in DOP_{π_5}. Hence $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi_5}| \ne 4$.

Case 4: $k=6$. If $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi'_6}| \ge 5$, that is, DOP_{π_6} and $\mathrm{DOP}_{\pi'_6}$ have at least five common pairs (elements), it is easy to see $\mathrm{DOP}_{\pi_6} = \mathrm{DOP}_{\pi'_6}$. Therefore $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi'_6}| \ne 5$.

(2) If $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi'_5}| \ge 4$, that is, DOP_{π_5} and $\mathrm{DOP}_{\pi'_5}$ have at least four common pairs, it is easily seen that the fifth pair of DOP_{π_5} is also a common pair, which implies that $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi'_5}| = 5$. Then the result follows.

(3) Suppose $\pi_4 = (x_1x_2x_3x_4)$ and $\pi_5 = (y_1y_2y_3y_4y_5)$. Then we get $\mathrm{DOP}_{\pi_4} = \{(x_1,x_3),(x_2,x_4),(x_3,x_1),(x_4,x_2)\}$ and $\mathrm{DOP}_{\pi_5} = \{(y_1,y_4),(y_2,y_5),(y_3,y_1),(y_4,y_2),(y_5,y_3)\}$. It is easily seen that $(x_1,x_3) \notin \mathrm{DOP}_{\pi_5}$ if $(x_3,x_1) \in \mathrm{DOP}_{\pi_5}$. So there are at least two pairs of DOP_{π_4} not contained in DOP_{π_5}.

(4) Suppose $\pi_3 = (x_1x_2x_3)$ is in the pth column permutation and $\pi'_3 = (y_1y_2y_3)$ is in the qth column permutation. Then $\mathrm{DOP}_{\pi_3} = \{(x_1,x_2),(x_2,x_3),(x_3,x_1)\}$ and $\mathrm{DOP}_{\pi'_3} = \{(y_1,y_2),(y_2,y_3),(y_3,y_1)\}$. If $|\mathrm{DOP}_{\pi_3} \cap \mathrm{DOP}_{\pi'_3}| \neq 0$, without loss of generality, we may suppose $(x_1,x_2) \in \mathrm{DOP}_{\pi_3} \cap \mathrm{DOP}_{\pi'_3}$. On the other hand, since π'_3 is a 3-cycle, $l_{x_1,p} = x_2$, $l_{x_2,p} = x_3$ and L is a Latin square, we have $l_{x_1,q} = y_3$, $l_{x_2,q} = x_1$ and $l_{y_3,q} = x_2$ as shown in Figure 2. From the definition of $(3,2,1)$-conjugate, we have $m_{x_1,q} = x_2$, $m_{x_2,q} = y_3$ and $m_{y_3,q} = x_1$. Hence $\mathrm{DOP}_{\pi'_3} = \{(x_1,y_3),(x_2,x_1),(y_3,x_2)\}$, which contradicts $(x_1,x_2) \in \mathrm{DOP}_{\pi'_3}$.

Figure 2: The multiplication tables of two 3-cycles

	p	q
x_1	x_2	y_3
x_2	x_3	x_1
x_3	x_1	
y_3		x_2

L

	p	q
x_1	x_3	x_2
x_2	x_1	y_3
x_3	x_2	
y_3		x_1

M

(5) Suppose $\mathrm{DOP}_{\pi_4} = \{(x_1,x_3),(x_2,x_4),(x_3,x_1),(x_4,x_2)\}$ and $\mathrm{DOP}_{\pi'_4} = \{(y_1,y_3),(y_2,y_4),(y_3,y_1),(y_4,y_2)\}$. It is easily seen that $(x_1,x_3) \in \mathrm{DOP}_{\pi'_4}$ if $(x_3,x_1) \in \mathrm{DOP}_{\pi'_4}$. Then the result follows.

(6) Suppose $\mathrm{DOP}_{\pi_4} = \{(x_1,x_3),(x_2,x_4),(x_3,x_1),(x_4,x_2)\}$ and $\mathrm{DOP}_{\pi_3} = \{(y_1,y_2),(y_2,y_3),(y_3,y_1)\}$. If $(y_1,y_2) \in \mathrm{DOP}_{\pi_4}$, it is easy to see that $(y_2,y_3),(y_3,y_1) \notin DOP_{\pi_4}$. Then the result follows. $\square$

3 Main result

In this section, we focus our attention on the nonexistence of $(3,2,1)$-$(v+7)$-COLS(v). Suppose $L = (l_{ij})_{v\times v}$ is a Latin square and $M = (m_{ij})_{v\times v}$ is the $(3,2,1)$-conjugate of L. Let $R = \{(l_{ij},m_{ij}) : 1 \leq i,j \leq v\}$ be the $(3,2,1)$-DOP set of L and $P = \{(l_{ij},m_{ij}) : l_{ij} \neq m_{ij}, 1 \leq i,j \leq v\}$. It is obvious, from Lemma 2.3, that the existence of $(3,2,1)$-$(v+7)$-COLS(v) is equivalent to $|P| = 7$. Every cycle considered is in a column permutation of L. Since $|\mathrm{DOP}_{\pi_k}| = k$, the length of every cycle considered in the following is less than 8. To prove the nonexistence of $(3,2,1)$-$(v+7)$-COLS(v), we need some "ingredients" provided in the following lemmas.

Lemma 3.1 *Let r and $v \geq 9$ be integers. Suppose $L = (l_{ij})$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})$ is the $(3,2,1)$-conjugate of L. If π_7 is a 7-cycle in the pth column permutation of L, then $r \neq v + 7$.*

Proof. Suppose $\pi_7 = (x_1x_2x_3x_4x_5x_6x_7)$. Then $\mathrm{DOP}_{\pi_7} = \{(x_1,x_6),(x_2,x_7),(x_3,x_1),(x_4,x_2),(x_5,x_3),(x_6,x_4),(x_7,x_5)\}$, as shown in Figure 3, and $|\mathrm{DOP}_{\pi_7}| = 7$. If $r = v+7$ (or $|P| = 7$), then we have $\mathrm{DOP}_{\pi_7} = P$. On the other hand, there must be an integer q such that $m_{x_7,q} = x_1$. From the definition of $(3,2,1)$-conjugate, we have $l_{x_1,q} = x_7$. Since $P = \mathrm{DOP}_{\pi_7}$ contains all pairs (i,j) $(i \neq j)$ in the $(3,2,1)$-DOP set of L, we must have $m_{x_1,q} = x_5$. Similarly, we have a series of results: $l_{x_5,q} = x_1, m_{x_5,q} = x_6, l_{x_6,q} = x_5, m_{x_6,q} = x_3, l_{x_3,q} = x_6, m_{x_3,q} = x_4, l_{x_4,q} = x_3$. Since $m_{x_4,p} = x_3$ and $m_{x_7,q} = x_1$, $m_{x_4,q}$ cannot be any value in $\{x_1,x_3\}$. Hence the pair $(l_{x_4,q}, m_{x_4,q})$ is not contained in P, which leads to a contradiction. ∎

Figure 3: The multiplication tables of a 7-cycle

	p	q
x_1	x_2	x_7
x_2	x_3	
x_3	x_4	x_6
x_4	x_5	x_3
x_5	x_6	x_1
x_6	x_7	x_5
x_7	x_1	

L

	p	q
x_1	x_7	x_5
x_2	x_1	
x_3	x_2	x_4
x_4	x_3	
x_5	x_4	x_6
x_6	x_5	x_3
x_7	x_6	x_1

M

Lemma 3.2 *Let r and $v \geq 9$ be integers. Suppose $L = (l_{ij})$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})$ is the $(3,2,1)$-conjugate of L. If there exists no k-cycle for $k \geq 7$ and π_6 is a 6-cycle in the pth column permutation of L, then $r \neq v + 7$.*

Proof. Suppose $\pi_6 = (x_1x_2x_3x_4x_5x_6)$. Then $\mathrm{DOP}_{\pi_6} = \{(x_1,x_5),(x_2,x_6),(x_3,x_1),(x_4,x_2),(x_5,x_3),(x_6,x_4)\}$ and $|\mathrm{DOP}_{\pi_6}| = 6$. If $r = v+7$ (or $|P| = 7$), then there must exist a new k-cycle $(3 \leq k \leq 6)$, π_k, such that $|\mathrm{DOP}_{\pi_6} \cup \mathrm{DOP}_{\pi_k}| = 7$. Therefore $|\mathrm{DOP}_{\pi_6} \cap \mathrm{DOP}_{\pi_k}| = k - 1$, which is impossible from Lemma 2.4(1). ∎

Lemma 3.3 *Let r and $v \geq 9$ be integers. Suppose $L = (l_{ij})$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})$ is the $(3,2,1)$-conjugate of L. If there exists no k-cycle for $k \geq 6$ and π_5 is a 5-cycle in the pth column permutation of L, then $r \neq v + 7$.*

Proof. Suppose $\pi_5 = (x_1x_2x_3x_4x_5)$. Then $\mathrm{DOP}_{\pi_5} = \{(x_1,x_4),(x_2,x_5),(x_3,x_1),(x_4,x_2),(x_5,x_3)\}$ and $|\mathrm{DOP}_{\pi_5}| = 5$. If $r = v+7$ (or $|P| = 7$), then there must exist another k-cycle ($3 \le k \le 5$) π_k in the qth column permutation of L such that $6 \le |\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_k}| \le 7$.

(1) $k = 5$. Let the 5-cycle in the qth column permutation be $\pi_5' = (y_1y_2y_3y_4y_5)$. Then $\mathrm{DOP}_{\pi_5'} = \{(y_1, y_4), (y_2, y_5), (y_3, y_1), (y_4, y_2), (y_5, y_3)\}$. From Lemma 2.4(2), we know that $|\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_5'}| \ge 7$. If $\{x_1, x_2, x_3, x_4, x_5\} = \{y_1, y_2, y_3, y_4, y_5\}$ and $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_5'}| = 3$, that is, they have three common pairs. Without loss of generality, we may suppose $\{(x_1, x_4), (x_2, x_5), (x_3, x_1)\} = \{(y_1, y_4), (y_2, y_5), (y_3, y_1)\} \subset \mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_5'}$. It is not difficult to verify that $\pi_5 = \pi_5'$. However, two such cycles cannot exist in two column permutations of L.

If $\{x_1, x_2, x_3, x_4, x_5\} \ne \{y_1, y_2, y_3, y_4, y_5\}$, we shall suppose $x_5 \notin \{y_1, y_2, y_3, y_4, y_5\}$ and $y_5 \notin \{x_1, x_2, x_3, x_4, x_5\}$. Then $\{(y_1, y_4), (y_3, y_1), (y_4, y_2)\} = \{(x_1, x_4), (x_3, x_1), (x_4, x_2)\}$. It is easy to see that $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$ and $x_4 = y_4$. Obviously, two such cycles cannot exist in two column permutations of L.

(2) $k = 4$ (only one 5-cycle). Suppose $\pi_4 = (y_1y_2y_3y_4)$; then $\mathrm{DOP}_{\pi_4} = \{(y_1, y_3), (y_2, y_4), (y_3, y_1), (y_4, y_2)\}$ and $6 \le |\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_4}| \le 7$. From Lemma 2.4(3), we have $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_4}| = 2$ (or $|\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_4}| = 7$). Without loss of generality, we shall suppose $(y_1, y_3) = (x_3, x_1)$ and $(y_2, y_4) = (x_2, x_4)$. Then we have $P = \{(x_1, x_4), (x_2, x_5), (x_3, x_1), (x_4, x_2), (x_5, x_3), (x_1, x_3), (x_2, x_4)\}$. Note that there exists an integer r such that $m_{x_5,r} = x_1$ as shown in Figure 4; then we can get $l_{x_1,r} = x_5$, $m_{x_1,r} = x_3$, and $l_{x_3,r} = x_1$. Since $m_{x_3,r}$ cannot be any value in $\{x_1, x_3, x_4\}$, the pair $(l_{x_3,r}, m_{x_3,r})$ is not contained in P, which leads to a contradiction.

Figure 4: The multiplication tables of a 5-cycle and a 4-cycle

	p	q	r
x_1	x_2	x_4	x_5
x_2	x_3	x_1	
x_3	x_4	x_2	x_1
x_4	x_5	x_3	
x_5	x_1		

L

	p	q	r
x_1	x_5	x_2	x_3
x_2	x_1	x_3	
x_3	x_2	x_4	
x_4	x_3	x_1	
x_5	x_4		x_1

M

(3) $k = 3$ (only one 5-cycle and no 4-cycle). Suppose $\pi_3 = (y_1y_2y_3)$ such that $6 \le |\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_3}| \le 7$. Then $\mathrm{DOP}_{\pi_3} = \{(y_1, y_2), (y_2, y_3), (y_3, y_1)\}$.

When $|\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_3}| = 7$, we have $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_3}| = 1$ and $\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_3} = P$. Without loss of generality, we may suppose $(y_1, y_2) = (x_1, x_4) \in \mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_3}$. If $y_3 = x_2$ or x_3, it is easily seen that $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_3}| = 2$, for $l_{x_4,q} = x_5$, $y_3 \ne x_5$. We now discuss the case of $y_3 \notin \{x_1, x_2, x_3, x_4, x_5\}$. Note that there is an integer r such that $m_{y_3,r} = x_1$ as shown in Figure 5. Then we have $l_{x_1,r} = y_3$. Since $m_{x_1,r}$ cannot be any value in $\{y_3, x_1\}$, the pair $(l_{x_4,r}, m_{x_4,r})$ is not contained in P, which leads to a contradiction.

Figure 5: The multiplication tables of a 5-cycle and a 3-cycle

L	p	q	r
x_1	x_2	x_4	y_3
x_2	x_3		
x_3	x_4		
x_4	x_5	y_3	
x_5	x_1		
y_3		x_1	

M	p	q	r
x_1	x_5	y_3	
x_2	x_1		
x_3	x_2		
x_4	x_3	x_1	
x_5	x_4		
y_3		x_4	x_1

When $|\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_3}| = 6$, we have $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi_3}| = 2$. Without loss of generality, we may suppose $(y_1, y_2) = (x_1, x_4)$ and $(y_2, y_3) = (x_4, x_2)$. There must exist another 3-cycle $\pi'_3 = (z_1 z_2 z_3)$ in the rth column permutation such that $|\mathrm{DOP}_{\pi_5} \cup \mathrm{DOP}_{\pi_3} \cup \mathrm{DOP}_{\pi'_3}| = 7$. Since $|\mathrm{DOP}_{\pi_3} \cap \mathrm{DOP}_{\pi'_3}| = 0$ from Lemma 2.4(4), we get $|\mathrm{DOP}_{\pi_5} \cap \mathrm{DOP}_{\pi'_3}| = 2$. Likewise, we may suppose $(z_1, z_2) = (x_2, x_5)$ and $(z_2, z_3) = (x_5, x_3)$. Then $P = \{(x_1, x_4), (x_2, x_5), (x_3, x_1), (x_4, x_2), (x_5, x_3), (x_2, x_1), (x_3, x_2)\}$. Note that there exists an integer s such that $m_{x_5,s} = x_1$ as shown in Figure 6. Similarly, we have a series of results: $l_{x_1,s} = x_5$, $m_{x_1,s} = x_3$, $l_{x_3,s} = x_1$, $m_{x_3,s} = x_4$, $l_{x_4,s} = x_3$, $m_{x_4,s} = x_2$, $l_{x_2,s} = x_4$. Since $m_{x_2,s}$ cannot be any value in $\{x_2, x_4\}$, the pair $(l_{x_2,s}, m_{x_2,s})$ is not contained in P, which leads to a contradiction. The proof is now complete. ∎

Figure 6: The multiplication tables of a 5-cycle and two 3-cycles

L	p	q	r	s
x_1	x_2	x_4		x_5
x_2	x_3	x_1	x_5	x_4
x_3	x_4		x_2	x_1
x_4	x_5	x_2		x_3
x_5	x_1		x_3	

M	p	q	r	s
x_1	x_5	x_2		x_3
x_2	x_1	x_4	x_3	
x_3	x_2		x_5	x_4
x_4	x_3	x_1		x_2
x_5	x_4		x_2	x_1

Lemma 3.4 *Let r and $v \geq 9$ be integers. Suppose $L = (l_{ij})$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})$ is the $(3,2,1)$-conjugate of L. If there exists no k-cycle for $k \geq 5$ and π_4 is a 4-cycle in the pth column permutation of L, then $r \neq v + 7$.*

Proof. Suppose $\pi_4 = (x_1 x_2 x_3 x_4)$. Then we obtain $\mathrm{DOP}_{\pi_4} = \{(x_1, x_3), (x_2, x_4), (x_3, x_1), (x_4, x_2)\}$ and $|\mathrm{DOP}_{\pi_4}| = 4$. If $r = v + 7$, then there must exist a k-cycle π_k $(3 \leq k \leq 4)$ in the qth column permutation such that $5 \leq |\mathrm{DOP}_{\pi_4} \cup \mathrm{DOP}_{\pi_k}| \leq 7$.

(1) $k=4$. Let the 4-cycle in the qth column permutation be $\pi'_4=(y_1y_2y_3y_4)$ such that $5\le|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}|\le 7$. From Lemma 2.4(5), we get $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}|=6$. We may suppose $\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}=\{(x_1,x_3),\ (x_2,x_4),\ (x_3,x_1),\ (x_4,x_2),\ (y_2,y_4),\ (y_4,y_2)\}$. Then there must still exist a new i-cycle in the rth column permutation such that $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}\cup\mathrm{DOP}_{\pi_i}|=7$. Hence $|\mathrm{DOP}_{\pi_i}\cap(\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4})|=i-1$. However, this is impossible.

In fact, if $i=4$, we may suppose the 4-cycle in the rth column permutation is $\pi''_4=(z_1z_2z_3z_4)$. Then $\mathrm{DOP}_{\pi''_4}=\{(z_1,z_3),(z_2,z_4),(z_3,z_1),(z_4,z_2)\}$. If $(z_1,z_3)\notin\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}$, then $(z_3,z_1)\notin\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}$. So $|\mathrm{DOP}_{\pi''_4}\cap(\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4})|\ne 3$. If $i=3$, let the 3-cycle in the rth column permutation be $\pi_3=(z_1z_2z_3)$, then $\mathrm{DOP}_{\pi_3}=\{(z_1,z_2),(z_2,z_3),(z_3,z_1)\}$. If $|\mathrm{DOP}_{\pi_3}\cap(\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4})|\ge 1$, without loss of generality, we may suppose that $(z_1,z_2)\in\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}$. Obviously, neither (z_2,z_3) nor (z_3,z_1) is contained in $\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4}$. So $|\mathrm{DOP}_{\pi_3}\cap(\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi'_4})|$ $\ne 2$.

(2) $k=3$ (only one 4-cycle). Suppose the 3-cycle in the qth column permutation is $\pi_3=(y_1y_2y_3)$ such that $5\le|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}|\le 7$. From Lemma 2.4(6), $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}|\ge 6$ holds.

When $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}|=7$, then $P=\{(x_1,x_3),(x_2,x_4),(x_3,x_1),(x_4,x_2),(y_1,y_2),(y_2,y_3),(y_3,\ y_1)\}$. If $\{y_1,y_2,y_3\}\nsubseteq\{x_1,x_2,x_3,x_4\}$, we may suppose $y_3\notin\{x_1,x_2,x_3,x_4\}$. Note that there must exist an integer r such that $m_{y_3,r}=y_1$ as shown in Figure 7, and then $l_{y_1,r}=y_3$. Since $m_{y_1,r}$ cannot be any value in $\{y_1,y_3\}$, the pair $(l_{y_1,r},m_{y_1,r})$ is not contained in P, which leads to a contradiction.

Figure 7: The multiplication tables of a 4-cycle and a 3-cycle 1

	q	r
y_1	y_2	y_3
y_2	y_3	
y_3	y_1	

L

	q	r
y_1	y_3	
y_2	y_1	
y_3	y_2	y_1

M

If $\{y_1,y_2,y_3\}\subseteq\{x_1,x_2,x_3,x_4\}$, we may suppose $x_4\notin\{y_1,y_2,y_3\}$. For $l_{x_1,p}=x_2$, it must be that $l_{x_1,q}=x_3$, $l_{x_2,q}=x_1$ and $l_{x_3,q}=x_2$, as shown in Figure 8. Then $m_{x_3,q}=x_1$, $m_{x_2,q}=x_3$ and $m_{x_1,q}=x_2$. But the pair (x_1,x_3) produced by π_3 is contained in DOP_{π_4}, which contradicts the fact that $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}|=7$. For the case of x_1 (or x_2, x_3) $\notin\{y_1,y_2,y_3\}$, following the above discussion, we can also have a similar contradiction.

When $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}|=6$, then there must exist another 3-cycle π'_3 such that $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}\cup\mathrm{DOP}_{\pi'_3}|=7$. However, since $|\mathrm{DOP}_{\pi'_3}\cap\mathrm{DOP}_{\pi_4}|\le 1$ from Lemma 2.4(6) and $|\mathrm{DOP}_{\pi_3}\cap\mathrm{DOP}_{\pi'_3}|=0$ from Lemma 2.4(4), we have $|\mathrm{DOP}_{\pi_4}\cup\mathrm{DOP}_{\pi_3}\cup\mathrm{DOP}_{\pi'_3}|\ge 8$. The proof is then complete. $\square$

Figure 8: The multiplication tables of a 4-cycle and a 3-cycle 2

$$L = \begin{array}{c|cc} & p & q \\ \hline x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_4 & x_2 \\ x_4 & x_1 & \end{array} \qquad M = \begin{array}{c|cc} & p & q \\ \hline x_1 & x_4 & x_2 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & \end{array}$$

Lemma 3.5 *Let r and $v \geq 9$ be integers. Suppose $L = (l_{ij})$ is a $(3,2,1)$-r-COLS(v) and $M = (m_{ij})$ is the $(3,2,1)$-conjugate of L. If there exists no k-cycle for $k \geq 4$, then $r \neq v + 7$.*

Proof. If $r \neq v$, then there must exist 3-cycles from Lemma 2.2. Since $|\mathrm{DOP}_{\pi_3} \cap \mathrm{DOP}_{\pi'_3}| = 0$ from Lemma 2.4(4), we have $|\cup_{i=1}^{k} \mathrm{DOP}_{\pi_{3i}}| = 3k$, where π_{3i} $(1 \leq i \leq k)$ are 3-cycles. Obviously, $3k \neq 7$ holds for any integer k. ∎

From Theorem 1.3, there exists no $(3,2,1)$-$(v+7)$-COLS(v) for any positive integer $v \leq 8$. Combining Lemmas 3.1– 3.5, we have the following theorem.

Theorem 3.6 *There exists no $(3,2,1)$-$(v+7)$-COLS(v) for any positive integer v.* ∎

If L is a $(3,2,1)$-r-COLS(v), it is easy to see that its transpose, L^T, is a $(1,3,2)$-r-COLS(v). So, we have the following theorem.

Theorem 3.7 *There exists no $(1,3,2)$-$(v+7)$-COLS(v) for any positive integer v.* ∎

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