AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **52** (2012), Pages 133–140

On f-factors in claw-free graphs

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Abstract

Let G be a 2-connected claw-free graph such that $\delta(G) \geq 5$. Then for every function $f : V(G) \to \{1, 2\}$, where $\sum_{x \in V(G)} f(x)$ is even, G has an f-factor.

All graphs considered are assumed to be simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of a vertex in G is denoted by $\delta(G)$. If X and Y are subsets of $V(G)$, we will write $E_G(X, Y)$ and $e_G(X, Y)$ for the set and the number respectively of the edges of G joining X to Y.

For any set X of vertices in G, we define the neighbour set of X in G to be the set of all vertices adjacent to vertices in X; this set is denoted by $N_G(X)$. A set of vertices in G is said to be independent if no two of them are adjacent. The edge analogue of an independent set is a set of edges in G no two of which have a common end.

Let X be a nonempty subset of $V(G)$. The subgraph of G whose vertex set is X and whose edge set is the set of those edges of G that have both ends in X is called the subgraph of G induced by X and is denoted by $G[X]$; we say that $G[X]$ is an induced subgraph of G.

A vertex cut of G is a subset X of $V(G)$ such that $G - X$ is disconnected. A k-vertex cut is a vertex cut of k elements. If $G \ncong K_n$, the connectivity $k(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $k(G)$ to be $n-1$. Thus $k(G) = 0$ if G is either trivial or disconnected. G is said to be k-connected if $k(G) \geq k$.

A graph is called claw-free if it contains no induced subgraph isomorphic to $K_{1,3}$.

Given a function $f : V(G) \to Z^+$, we say that G has an f-factor if there exists a spanning subgraph H of G such that $d_H(x) = f(x)$ for every $x \in V(G)$. If f is the constant function taking the value r, then an f-factor is said to be an r-factor.

A necessary and sufficient condition for a graph to have an f-factor was obtained by Tutte [8] in 1952.

Tutte's f-factor Theorem [8] *A graph G has an f-factor if and only if*

$$q_G(D,S;f)+\sum_{x\in S}(f(x)-d_{G-D}(x))\leq\sum_{x\in D}f(x)$$

for all sets $D,S\subseteq V(G)$, $D\cap S=\varnothing$ where $q_G(D,S;f)$ denotes the number of components C of $(G-D)-S$ such that

$$e_G(V(C),S)+\sum_{x\in V(C)}f(x)$$

is odd. (Sometimes we refer to these as odd components.)

Tutte also noted that

$$q_G(D,S;f)+\sum_{x\in S}(f(x)-d_{G-D}(x))-\sum_{x\in D}f(x)\equiv\sum_{x\in V(G)}f(x)\ (\mathrm{mod}\ 2). \qquad (1)$$

In recent years there have been many results on the family of claw-free graphs (see [5]). Some of them concern the existence of factors in claw-free graphs. Choudoum and Paulraj have proved in [2] the following theorem.

Theorem 1 *Let $k\geq 1$ be an integer and G a connected claw-free graph with $k|V(G)|$ even and $\delta(G)\geq 2k$. Then G contains a k-factor.*

Egawa and Ota improved the minimum degree condition by obtaining the following similar result.

Theorem 2 [3] *Let $k\geq 2$ be an integer and G a connected claw-free graph with $k|V(G)|$ even and $\delta(G)\geq\lceil(9k+12)/8\rceil$. Then G contains a k-factor.*

In [4] the authors considered 2-factors having a small number of components and gave a constructive proof of the following result.

Theorem 3 *Let G be a claw-free graph of order n and minimum degree $\delta(G)\geq 4$. Then we may construct a 2-factor in G having at most $\dfrac{6n}{\delta(G)+2}-1$ components by a polynomial algorithm in $O(n^3)$.*

Some other results on the structure of 2-factors in claw-free graphs can be found in [6] and [7].

The main purpose of this paper is to present the following theorem which fits into the above-mentioned literature.

Theorem 4 *Let G be a 2-connected claw-free graph such that $\delta(G) \geq 5$. Then for every function $f : V(G) \to \{1, 2\}$, where $\sum_{x\in V(G)} f(x)$ is even, G has an f-factor.*

For the proof of Theorem 4, we shall need the following lemma.

Lemma 1 *Let G be a graph and let the function $f : V(G) \to \{1, 2\}$ satisfy $\sum_{x\in V(G)} f(x)$ is even. Suppose that there exist D, $S \subseteq V(G)$, $D \cap S = \varnothing$ such that*

$$q_G(D, S; f) + \sum_{x\in S}(f(x) - d_{G-D}(x)) \geq \sum_{x\in D} f(x) + 2. \tag{2}$$

If S is minimal with respect to (2), then

(a) $E_G(S, S) = \varnothing$;

(b) For every $x \in S$, $N_{G-D}(x)$ contains $d_{G-D}(x)$ elements belonging to $d_{G-D}(x)$ different odd components of $(G - D) - S$.

Proof. Define $W = (G - D) - S$, $L = G[S]$.

(a) Suppose that $E_G(S, S) \neq \varnothing$, that is to say S is not an independent set. Then there exists a vertex $u \in S$ such that $d_L(u) \geq 1$.

Define $S' = S - \{u\}$. Then

$$\begin{aligned} q_G(D, S'; f) &\geq q_G(D, S; f) - (d_{G-D}(u) - d_L(u)) \\ &\geq q_G(D, S; f) - d_{G-D}(u) + 1 \end{aligned}$$

and

$$\sum_{x\in S'}(f(x) - d_{G-D}(x)) \geq \sum_{x\in S}(f(x) - d_{G-D}(x)) - (f(u) - d_{G-D}(u)).$$

Hence

$$q_G(D, S'; f) + \sum_{x\in S'}(f(x) - d_{G-D}(x)) \geq q_G(D, S; f) + \sum_{x\in S}(f(x) - d_{G-D}(x)) - f(u) + 1,$$

and by using (2), we have

$$\begin{aligned} q_G(D, S'; f) + \sum_{x\in S'}(f(x) - d_{G-D}(x)) &\geq \sum_{x\in D} f(x) + 2 - f(u) + 1 \\ &\geq \sum_{x\in D} f(x) + 1 \text{ since } f(u) \leq 2. \end{aligned} \tag{3}$$

But using (1), (3) yields

$$q_G(D, S'; f) + \sum_{x\in S'}(f(x) - d_{G-D}(x)) \geq \sum_{x\in D} f(x) + 2,$$

which contradicts the minimality of S with respect to (2). So S is an independent set in G.

(b) Suppose that there exists $u \in S$, such that $N_{G-D}(u)$ contains at least two elements belonging to the same odd component of W or $N_{G-D}(u)$ contains at least one element belonging to an even component of W.

Define $S^{'} = S - \{u\}$. Then

$$q_G(D, S^{'}; f) + \sum_{x \in S'} (f(x) - d_{G-D}(x)) \geq \sum_{x \in D} f(x) + 2 \tag{4}$$

since

$$q_G(D, S^{'}; f) \geq q_G(D, S; f) - (d_{G-D}(u) - 1),$$

$$\begin{aligned} \sum_{x \in S'} (f(x) - d_{G-D}(x)) &\geq \sum_{x \in S} (f(x) - d_{G-D}(x)) - (f(u) - d_{G-D}(u)) \\ &\geq \sum_{x \in S} (f(x) - d_{G-D}(x)) + d_{G-D}(u) - 2 \text{ since } f(u) \leq 2 \end{aligned}$$

and by using (1). But (4) contradicts the minimality of S with respect to (2). So this case cannot occur. □

Proof of Theorem 4:

Suppose that G does not have an f-factor. Then by Tutte's f-factor Theorem and (1), there exists $D, S \subseteq V(G)$, $D \cap S = \varnothing$ such that

$$q_G(D, S; f) + \sum_{x \in S} (f(x) - d_{G-D}(x)) \geq \sum_{x \in D} f(x) + 2. \tag{5}$$

We assume that S is minimal with respect to (5).

Define W=$(G - D) - S$, let Q be the set of odd components of $(G - D) - S$ and let E be the set of components of $(G - D) - S$ which are not odd. Note that for every $x \in S$, we have $|N_G(x) \cap V(W)| \leq 2$, since G is a claw-free graph and by using Lemma 1(b).

Moreover, define

$$Q_i = \{C \in Q | e_G(V(C), S) = i\}, \quad |Q_i| = q_i,$$

$$E_i = \{C \in E | e_G(V(C), S) = i\}, \quad |E_i| = e_i, \text{ for } i = 0, 1, 2, \dots .$$

We also define

$$Q_1^1 = \{C \in Q_1 || V(C)| = 1\}, \quad |Q_1^1| = q_1^1,$$

$$Q_1^2 = \{C \in Q_1 || V(C)| \geq 2\}, \quad |Q_1^2| = q_1^2,$$

$$Q_1^{1,1} = \{C \in Q_1^1 | d_{G-D}(x) = 1, \text{ where } N_{G-D}(V(C)) = \{x\}\}, \quad |Q_1^{1,1}| = q_1^{1,1},$$

$$Q_1^{1,2} = \{C \in Q_1^1 | d_{G-D}(x) = 2, \text{ where } N_{G-D}(V(C)) = \{x\}\}, \quad |Q_1^{1,2}| = q_1^{1,2}.$$

Now by using Lemma 1, we have

$$E_1 \cup E_2 \cup \cdots = \varnothing \text{ and } E_G(S, S) = \varnothing$$

since S is minimal with respect to (5). So (5) implies

$$q_0 + 2|S| - q_2 - 2q_3 - \cdots > |D|. \tag{6}$$

Let $L_1, L_2, \ldots, L_{q_1^{1,2}}$ be the elements of $Q_1^{1,2}$ and let $V(L_i) = \{u_i\}$ for $i = 1, 2, \ldots, q_1^{1,2}$. Define $T_1 = \{u_1, u_2, \ldots, u_{q_1^{1,2}}\}$ and $K = N_{G-D}(T_1)$.
Clearly

$$|K| \leq q_1^{1,2}. \tag{7}$$

Let also $R_1, R_2, \ldots, R_{q_1^2}$ be the elements of Q_1^2. For every R_i, we choose a vertex v_i of R_i, such that $N_G(v_i) \cap S = \varnothing$ and $|N_G(v_i) \cap D| \geq 1$. Clearly such vertices exist since G is 2-connected.

Define $T_2 = \{v_1, v_2, \ldots, v_{q_1^2}\}$.

Furthermore let $M_1, \ldots, M_{q_0}, M_{q_0+1}, \ldots, M_{q_0+e_0}$ be the elements of $Q_0 \cup E_0$. For every element M_i of $Q_0 \cup E_0$ we choose two edges $e_i, e_i^{'}$ belonging to $E_G(V(M_i), D)$ and not having an element of D as a common end-vertex. Clearly such a pair of edges exists since G is 2-connected. Define

$$B = \{e_1, e_1^{'} \ldots, e_{q_0}, e_{q_0}^{'}, e_{q_0+1}, e_{q_0+1}^{'}, \ldots, e_{q_0+e_0}, e_{q_0+e_0}^{'}\}.$$

We now consider the set

$$Y = E_G(T_1 \cup T_2, D) \cup E_G(S - K, D) \cup B.$$

We note that every vertex belonging to D is the end-vertex of at most two edges belonging to the above set, since G is a claw-free graph. So

$$\begin{aligned}
2|D| &\geq e_G(T_1 \cup T_2, D) + e_G(S - K, D) + |B| \\
&\geq |T_2| + (\delta - 1)|T_1| + \sum_{x \in S-K} d_G(x) - \sum_{x \in S-K} d_{G-D}(x) + |B| \text{ where } \delta(G) = \delta \\
&\geq q_1^2 + (\delta - 1)q_1^{1,2} + \delta|S - K| - (q_1 + 2q_2 + \cdots - 2|K|) + 2q_0 + 2e_0 \\
&\geq q_1^2 + (\delta - 1)q_1^{1,2} + \delta|S| - \delta|K| - q_1 - 2q_2 - \cdots + 2|K| + 2q_0 + 2e_0 \\
&\geq q_1^2 + (\delta - 1)q_1^{1,2} + \delta|S| - q_1 - 2q_2 - \cdots + (2 - \delta)|K| + 2q_0 + 2e_0 \\
&\geq q_1^2 + (\delta - 1)q_1^{1,2} + \delta|S| - q_1 - 2q_2 - \cdots + (2 - \delta)q_1^{1,2} + 2q_0 + 2e_0 \text{ by (7)} \\
&\geq q_1^2 + q_1^{1,2} + \delta|S| - q_1 - 2q_2 - \cdots + 2q_0 + 2e_0 \\
&\geq \delta|S| - q_1^{1,1} - 2q_2 - 3q_3 - \ldots + 2q_0 + 2e_0 \text{ since } q_1 = q_1^{1,1} + q_1^{1,2} + q_1^2.
\end{aligned}$$

Hence

$$\begin{aligned}\frac{4}{\delta}|D| &\geq 2|S| - \frac{2}{\delta}q_1^{1,1} - \frac{4}{\delta}q_2 - \frac{6}{\delta}q_3 - \cdots + \frac{4}{\delta}q_0 + \frac{4}{\delta}e_0 \\ &\geq 2|S| - \frac{2}{\delta}q_1^{1,1} - \sum_{i\geq 2}\frac{2i}{\delta}q_i + \frac{4}{\delta}q_0 + \frac{4}{\delta}e_0. \qquad (8)\end{aligned}$$

Now let $Z_1, Z_2, \ldots, Z_{q_1^{1,1}}$ be the elements of $Q_1^{1,1}$ and let $V(Z_i) = \{z_i\}$ for $i = 1, 2, \ldots, q_1^{1,1}$. Define $T_3 = \{z_1, z_2, \ldots, z_{q_1^{1,1}}\}$. Clearly

$$e_G(z, D) \geq \delta - 1, \text{ for every } z \in T_3.$$

On the other hand since G is a claw-free graph we also have

$$\begin{aligned}2|D| &\geq |B \cup E_G(T_3, D)| \\ &\geq 2q_0 + 2e_0 + (\delta - 1)q_1^{1,1}.\end{aligned}$$

Thus

$$\frac{\delta-4}{\delta}|D| \geq \frac{\delta-4}{\delta}q_0 + \frac{\delta-4}{\delta}e_0 + \frac{(\delta-1)(\delta-4)}{2\delta}q_1^{1,1} \quad \text{since } \delta > 4. \qquad (9)$$

From (8) and (9), we can obtain

$$\begin{aligned}|D| &\geq q_0 + e_0 + 2|S| - \sum_{i\geq 2}\frac{2i}{\delta}q_i + \left(\frac{(\delta-1)(\delta-4)}{2\delta} - \frac{2}{\delta}\right)q_1^{1,1} \qquad (10) \\ &\geq q_0 + e_0 + 2|S| - \sum_{i\geq 2}\frac{2i}{\delta}q_i + \frac{\delta-5}{2}q_1^{1,1} \\ &\geq q_0 + e_0 + 2|S| - \sum_{i\geq 2}\frac{2i}{\delta}q_i \quad \text{since } \delta \geq 5. \qquad (11)\end{aligned}$$

But $\delta > 4$, so $2\delta - 4 > \delta$, that is, $2(\delta - 2) > \delta$; and $i(\delta - 2) > \delta$ for $i \geq 2$. Thus $i\delta - \delta > 2i$ and so $i - 1 > 2i/\delta$. Hence (11) yields

$$|D| > q_0 + e_0 + 2|S| - \sum_{i\geq 2}(i-1)q_i,$$

contradicting (6). □

We next show that the conditions of the hypothesis of Theorem 4 are, in some sense, best possible. We will prove this first for the connectivity condition, by describing a family of graphs G having slightly lower connectivity and not having the properties implied by the theorem although the minimum degree of G is arbitrarily high.

We construct such graphs as follows. We start from a copy of K_n and n copies of K_m, where n is an even integer and $n, m \geq 6$. Let $\{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the copy of K_n and let $\{H_1, H_2, \ldots, H_n\}$ be the set of copies of K_m. We choose vertex z_i from every such copy H_i of K_m, where $i = 1, 2, \ldots, n$. We add the independent edges joining u_i to z_i for $1 \le i \le n$.

The resulting family of graphs G are claw-free, $k(G) = 1$, $\delta(G) \ge 5$ and the minimum degree can be arbitrarily high depending on the values of n and m. Now consider a function $f : V(G) \to \{1, 2\}$ such that

$$f(x) = 1 \text{ if } x \in \{u_1, u_2, \ldots, u_{n-1}\} \cup \{z_1, z_2, \ldots, z_{n-1}\}$$

and

$$f(x) = 2 \text{ if } x \in V(G) - (\{u_1, u_2, \ldots, u_{n-1}\} \cup \{z_1, z_2, \ldots, z_{n-1}\}).$$

Clearly G does not have an f-factor since if we let $D = \{u_1, u_2, \ldots, u_{n-1}\}$ and $S = \{u_n\}$,

$$q_G(D, S; f) + \sum_{x \in S}(f(x) - d_{G-D}(x)) > \sum_{x \in D} f(x)$$

because

$$q_G(D, S; f) = n, \sum_{x \in S}(f(x) - d_{G-D}(x)) = 1 \text{ and } \sum_{x \in D} f(x) = n - 1.$$

We will next show that the minimum degree condition is also best possible. We will describe a family of graphs G having slightly lower minimum degree and not having the property implied by the theorem. We construct such graphs G as follows.

We start from a cycle $z_1 z_2 \ldots z_{n-1} z_n z_1$, where $3n/2$ is an even integer; and n copies of K_2. Let $\{H_1, H_2, \ldots, H_n\}$ be the set of the above copies. We join vertex z_i to the vertices of H_i and H_{i+1}, for all values of i, where $i = 1, 2, \ldots, n$. Here the subscripts are considered modulo n.

We also add vertices $w_1, w_2, \ldots, w_{\frac{n}{2}}$ and we join vertex w_j to the vertices of H_{2j-1} and H_{2j} for all values of j, where $j = 1, 2, \ldots, n/2$.

The resulting family of graphs G are claw-free, $k(G) = 2$ and $\delta(G) = 4$. Now consider the function $f : V(G) \to \{1, 2\}$ such that

$$f(x) = 1 \text{ if } x \in \{z_1, z_2, \ldots, z_n\} \cup \{w_1, w_2, \ldots, w_{\frac{n}{2}}\}$$

and

$$f(x) = 2 \text{ if } \ x \in V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n).$$

Clearly G does not have an f-factor since if we let

$$D = \{z_1, z_2, \ldots, z_n\} \cup \{w_1, w_2, \ldots, w_{\frac{n}{2}}\} \text{ and } S = V(G) - D,$$

then

$$q_G(D, S; f) + \sum_{x \in S}(f(x) - d_{G-D}(x)) > \sum_{x \in D} f(x)$$

because

$$q_G(D, S; f) = 0, \ \sum_{x \in S}(f(x) - d_{G-D}(x)) = 2n \text{ and } \sum_{x \in D} f(x) = 3n/2.$$

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(Received 1 Mar 2011; revised 14 July 2011)