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The modular product and existential closure

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Abstract

In this article we study the modular product graph operation, denoted by $\Diamond$, with particular emphasis on when the operation preserves the property of a graph being 3-existentially closed. We characterise when $G \Diamond H$ is 3-e.c. provided that H is 3-e.c., and we establish two new infinite classes of 3-e.c. graphs of the form $G \Diamond H$, for which G need not be 3-e.c. itself.

1 Introduction

In 1963, Erdős and Rényi [9] introduced the concept of existential closure, whereby a graph G with vertex set V is said to be n-existentially closed, or n-e.c., if for each proper subset S of V with cardinality $|S| = n$ and each subset T of S, there exists some vertex x not in S that is adjacent to each vertex of T but to none of the vertices of $S \setminus T$.

Since this property was first introduced, only a handful of classes of graphs have been shown to be n-e.c. for arbitrary (but fixed) values of n. Among these classes are sufficiently large Paley graphs [4], a family of strongly regular graphs described by Cameron and Stark [7], as well as graphs arising from affine planes and resolvable designs as described by Baker et al. [2, 3]. Recently, the block intersection graphs of combinatorial designs were considered [10, 13]. Aside from finite designs, in [12, 14] block intersection graphs of infinite designs were also considered. In [14] it was shown that any infinite t-(v, k, λ) design with finite block size behaves similarly to finite designs in the sense that if the block intersection graph is n-e.c., then n is bounded, namely $n \leq t + 1$. However, in [12] an infinite design having infinite block size and whose block intersection graph is isomorphic to the Rado graph R (which was shown by Erdős and Rényi to be asymptotically almost surely n-e.c. for all $n \geq 1$ [9]) was constructed.

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The scarcity of other readily recognised families of n-e.c. graphs for arbitrary n has motivated research into classes of graphs that are n-e.c. for small values of n; however, it is not easy to find explicit examples of such graphs even for $n = 3$. It has been shown that every 3-e.c. graph has at least 24 vertices and examples of 3-e.c. graphs of order 28 have been found [5, 11]. In 2001, Hadamard matrices of order $4m$ with odd $m > 1$ were used to obtain 3-e.c. graphs of order $16m^2$ [6]. Also in 2001, Baker et al. presented new 3-e.c. graphs arising from collinearity graphs of partial planes resulting from affine planes [1]. Recently, another construction of 3-e.c. graphs of order at least p^d for prime $p \geq 7$ and $d \geq 5$ was presented using quadrances [16]. Also it was confirmed that there are only two STS(19) with 3-e.c. block intersection graphs [8, 10].

As part of an effort to find explicit examples of n-e.c. graphs, Bonato and Cameron examined several common binary graph operations to see which operations preserve the n-e.c. property for $n \geq 1$ [5]. They showed that the symmetric difference of two 3-e.c. graphs is a 3-e.c. graph. Baker et al. subsequently introduced another graph construction which is 3-e.c. preserving [1].

In this article, we take a different approach to the construction in [1] that enables us to relax the requirement that the two graphs considered be both 3-e.c. We formulate the construction as a binary non-commutative graph operation denoted by the symbol $\Diamond$ and we determine necessary and sufficient conditions for the graph $G \Diamond H$ to be 3-e.c., given that H itself is a 3-e.c. graph. We then use this operation to construct new classes of 3-e.c. graphs of the form $G \Diamond H$ when G is not necessarily a 3-e.c. graph. In particular, the classes that we consider are those for which G is either a complete multipartite graph or a strongly regular graph. The graph G for which we show that $G \Diamond H$ is 3-e.c. can have as few as four vertices, which represents an improvement in comparison to when G is required to be 3-e.c.

2 The Modular Product and a Characterisation Theorem

If G and H are two graphs, then we let $G \Diamond H$ represent the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, u) and (y, v) are adjacent if

(a) $xy \in E(G)$ and $uv \in E(H)$, or

(b) $xy \notin E(G)$ and $uv \notin E(H)$.

It so happens that $G \Diamond H$ is the complement of a construction that was introduced by Vizing in 1974 [17]. In keeping with [15, 18], we shall refer to $G \Diamond H$ as the modular product of G and H.

Unless stated otherwise, we shall generally assume that the graph G has a loop at each vertex and also that H is 3-e.c. When describing the graph $G \Diamond H$, for each vertex $x \in V(G)$ let H_x be the subgraph of $G \Diamond H$ that is isomorphic to H and consists of all vertices of the form (x, u) where $u \in V(H)$. Since the vertices of H_x can be considered to be indexed by $V(G)$, we will often use the notation u_x to denote the

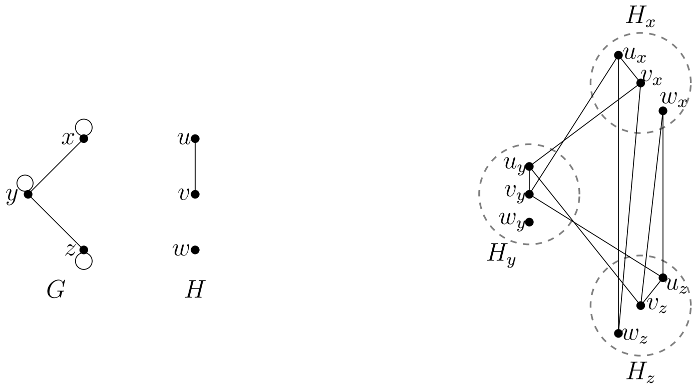

Figure 1: $G \Diamond H$.

vertex (x, u). Two vertices $u_x = (x, u) \in H_x$ and $v_y = (y, v) \in H_y$ will be said to be congruent if $u = v$; otherwise they are incongruent. An example of $G \Diamond H$ is illustrated in Figure 1, for $G = K_{1,2}$ and $H = \overline{K_{1,2}}$.

It can be easily deduced that $\overline{G \Diamond H} = G \Diamond \overline{H}$ where $\overline{G}$ is the simple complement of G (the complement of a loop is a non-loop and the complement of a non-loop remains a non-loop). Also, note that when G has a loop at every vertex, $G \Diamond H$ is isomorphic to the graph $G(H)$ as described in [1] in which the following theorem was proved:

Theorem 2.1 *[1] If the graphs G and H are both 3-e.c., then the graph $G \Diamond H$ is also 3-e.c.*

We devote the remainder of this section to the development and proof of a characterisation of 3-e.c. graphs of the form $G \Diamond H$ where H is 3-e.c. but G is not necessarily so. This characterisation will help us to find smaller 3-e.c. graphs by simplifying the process of checking when $G \Diamond H$ is 3-e.c.

For a graph G, given a set $S \subset V(G)$ and a subset T of S, we say a vertex $x \in V(G) \setminus S$ is a T-solution with respect to S if x is adjacent to every vertex in T and to none in $S \setminus T$. A solution for S is said to exist if there is a T-solution for every $T \in P(S)$, where $P(S)$ denotes the power set of S. Observe that if a solution exists for every n-subset of V, then G is n-e.c.

We say a graph G is weakly n-existentially closed, or n-w.e.c., if for any set S with $|S| = n$ and any $T \subseteq S$, there exists a vertex in $V(G)$ that is adjacent to each vertex in T and to no vertex in $S \setminus T$, or there exists a vertex that is adjacent to each vertex in $S \setminus T$ and to no vertex in T. Such a vertex is called a weak T-solution with respect to S. Note that $K_1 \Diamond H = H$ and it can easily be confirmed that if $|V(G)| \in \{2, 3\}$, then G cannot be 3-w.e.c. So we henceforth assume that $|V(G)| \geq 4$.

For a graph G and a vertex $x \in V(G)$ we define $N[x] = \{y \in V(G) | \; xy \in E(G)\}$, and for a set A of vertices we let $N[A] = \bigcup_{x \in A} N[x]$ and $N'[A] = \bigcap_{x \in A} N[x]$. Also, for a set of vertices $A \subseteq V(G \Diamond H)$ and for each $a \in V(G)$, we let $A_a = \{u_a \in V(H_a) |$ there is some $x \in V(H)$ such that $u_x \in A\}$.

With these notations, note that a graph G is 3-w.e.c. if and only if for every 3-subset $A \subset V(G)$, the following two items hold

(1) $N'[A] \neq \emptyset$ or $V(G) \setminus N[A] \neq \emptyset$, and

(2) for every vertex $t \in A$, $N[t] \setminus N[A \setminus \{t\}] \neq \emptyset$ or $N'[A \setminus \{t\}] \setminus N[t] \neq \emptyset$.

We are now ready to state and prove a characterisation theorem.

Theorem 2.2 *Let G be a graph with $|V(G)| \geq 4$ and with loops at every vertex of $V(G)$ and let H be a 3-e.c. graph. The graph G is 3-w.e.c. if and only if $G \Diamond H$ is 3-e.c.*

Proof Suppose that H is 3-e.c. and G is 3-w.e.c. In order to show that $G \Diamond H$ is 3-e.c., for an arbitrary set of three vertices $S = \{u_x, v_y, w_z\} \subset V(G \Diamond H)$ we show that there exists a T-solution for each $T \in P(S)$. Note that since $\overline{G \Diamond H} = G \Diamond \overline{H}$, then if there is a T-solution in $G \Diamond H$ for $|T| = 0$ (1, resp.), then there is a T-solution for $|T| = 3$ (2, resp.). To see this, suppose that there is an $\emptyset$-solution in $G \Diamond H$, and since $\overline{H}$ is 3-e.c., there is an $\emptyset$-solution in $G \Diamond \overline{H}$ and hence in $\overline{G \Diamond H}$, too. This implies that there is an S-solution in $G \Diamond H$. A similar argument holds for the case $|T| = 1$.

Let $A = \{x, y, z\}$ and $B = \{u, v, w\}$. So $1 \leq |A|, |B| \leq 3$. If $|A| = 1$, then since H is 3-e.c., there exists an S-solution. Now we consider the remaining possibilities for B and A.

Case 1. Suppose that $|A| = 3$. First consider the case $T = \emptyset$. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is an S_a-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S. If $a \in N'[A]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S.

If $T = \{u_x\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[x] \setminus N[A \setminus \{x\}]$, then if t is an S_a-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. If $a \in N'[A \setminus \{x\}] \setminus N[x]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. Similar arguments hold for $T \in \{\{v_y\}, \{w_z\}\}$.

Case 2. Next suppose that $|A| = 2$. We argue this case in two subcases depending on whether the vertices of S are congruent or incongruent.

Case 2.a. First suppose that the vertices of S are incongruent; $S = \{u_x, v_y, w_y\}$.

We first consider the case $T = \emptyset$. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is an S_a-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S. If $a \in N'[A]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S.

If $T = \{u_x\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[x] \setminus N[y]$, then if t is an S_a-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. If $a \in N[y] \setminus N[x]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S.

If $T = \{v_y\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[y] \setminus N[x]$, then if t is a $\{u_a, v_a\}$-solution with respect to S_a, t_a is a $\{v_y\}$-solution with respect to S. If $a \in N[x] \setminus N[y]$, then if t is a $\{w_a\}$-solution with respect to S_a, t_a is a $\{v_y\}$-solution with respect to S. A similar argument holds for $T = \{w_y\}$.

Case 2.b. Now suppose that S contains congruent vertices; $S = \{u_x, u_y, w_y\}$. In this case, the only difference with Case 2.a. is in finding a $\{w_y\}$-solution. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is a $\{u_a\}$-solution with respect to S_a, t_a is a $\{w_y\}$-solution with respect to S. If $a \in N'[A]$, then if t is a $\{w_a\}$-solution with respect to S_a, t_a is a $\{w_y\}$-solution with respect to S.

Note that for any 3-subset $S \subset V(G\Diamond H)$, and any $a \in V(G)$, since H_a is isomorphic to H and hence is 3-e.c., then for each $T' \subseteq S_a$ there exists a T'-solution with respect to S_a. Observe for each case considered in this argument, the solutions found are in $V(G\Diamond H) \setminus S$. As there is a solution for an arbitrary set of three vertices of $G\Diamond H$, we conclude that $G\Diamond H$ is 3-e.c.

To prove the converse implication, suppose that $G\Diamond H$ is 3-e.c. but G is not 3-w.e.c. Assume that $A = \{x, y, z\} \subset G$ for which there is no weak T-solution for some $T \subseteq A$. Let $S = \{u_x, u_y, u_z\}$.

As an initial case, suppose that there is no weak $\emptyset$-solution. If every vertex of G is in the neighbourhood of at least one and at most two of the vertices in A, then every vertex of $G\Diamond H$ is adjacent to at least one and at most two of the vertices in S, and so there is no vertex of $G\Diamond H$ that is an S-solution with respect to S.

Now suppose that there is no weak T-solution for some $T \subseteq A$ with $|T| = 1$. Without loss of generality suppose that $N[x] \setminus N[\{y, z\}] = \emptyset$ and $N[\{y, z\}] \setminus N[x] = \emptyset$. So, any vertex in $N[x]$ is also in $N[\{y, z\}]$ and any vertex in $N[\{y, z\}]$ is also in $N[x]$. These imply that any vertex in $N[x]$ is in $N[y]$ or $N[z]$ and any vertex in $V(G) \setminus N[x]$ is in at most one of $N[y]$ and $N[z]$. Thus any vertex of $G\Diamond H$ that is adjacent to u_x is also adjacent to u_y or to u_z and so there is no $\{u_x\}$-solution with respect to S.

In each case we establish the contradition that the graph $G\Diamond H$ is not 3-e.c., and the argument is complete. □

Theorem 2.1 now becomes a corollary of Theorem 2.2.

Proof of Theorem 2.1 Since G is 3-e.c., it is also 3-w.e.c. □

In general, given graphs G and H such that H is 3-e.c., in order to determine whether or not $G\Diamond H$ is 3-e.c., we need to examine the existence of $8\binom{|V(G)||V(H)|}{3}$ T-solutions. However, by applying Theorem 2.2, we only need to examine if G is 3-w.e.c., and hence at most $8\binom{|V(G)|}{3}$ sets would need to be compared with the empty set.

Having shown that the modular product can produce a 3-e.c. graph given a 3-w.e.c. graph and a 3-e.c. graph, we now find graphs G that are 3-w.e.c. We focus our attention on cases in which G is either a complete multipartite graph or a strongly regular graph.

3 Weakly 3-e.c. Complete Multipartite Graphs

In this section we show that most of the complete multipartite graphs are 3-w.e.c.

Theorem 3.1 *The complete i-partite graph $K_{\ell_1,\ell_2,\ldots,\ell_i}$ with $\ell_j \geq 2$ for all $j \in \{1, 2, \ldots, i\}$ is 3-w.e.c.*

Proof Let X and Y be two distinct parts in the obvious partition of $K_{\ell_1,\ell_2,\ldots,\ell_i}$. Consider a set of vertices $A = \{x, y, z\}$ of $K_{\ell_1,\ell_2,\ldots,\ell_i}$. If all three vertices of A are in the same part, say $A \subseteq X$, any vertex in Y is a weak $\emptyset$-solution with respect to A. Also $x \in N[x] \setminus N[\{y, z\}]$, and similarly $y \in N[y] \setminus N[\{x, z\}]$ and $z \in N[z] \setminus N[\{x, y\}]$, and so there exists a weak T-solution for any $T \subset A$ with $|T| = 1$.

If x is a vertex in a part, say $x \in X$, and y and z are in another part, say $\{y, z\} \subseteq Y$, then $x \in N'[A]$ and so there is a weak $\emptyset$-solution with respect to A. Also note that since $\ell_j \geq 2$, then there exists a vertex $r \in X \setminus \{x\}$, and hence $r \in N'[\{y, z\}] \setminus N[x]$. Also, $z \in N'[\{x, z\}] \setminus N[y]$, and $y \in N'[\{x, y\}] \setminus N[z]$.

It now remains to consider the case when each vertex in A is in a distinct part. Suppose that x', y' and z' are vertices of $V(K_{\ell_1,\ell_2,\ldots,\ell_i}) \setminus A$ and in the same parts as x, y and z respectively. We have $x \in N'[A]$ and so there exists a weak $\emptyset$-solution with respect to A. Also $x' \in N'[\{y, z\}] \setminus N[x]$, $y' \in N'[\{x, z\}] \setminus N[y]$, and $z' \in N'[\{x, y\}] \setminus N[z]$ and so is a weak T-solution for any $T \subset A$ with $|T| = 1$. So, A has a weak solution and the graph $K_{\ell_1,\ell_2,\ldots,\ell_i}$ is 3-w.e.c. □

It follows from Theorem 3.1 that every bipartite graph $K_{\ell,m}$ with $\ell, m \geq 2$ is 3-w.e.c. The only remaining bipartite graphs to consider are of the form $K_{1,m}$ with $m \geq 3$. Let $A = \{x, y, z\} \subset V(K_{1,m})$. We will show that there is a weak solution for A. If all the vertices of A are in the same part, then the argument is similar to the corresponding case in the proof of Theorem 3.1. Now without loss of generality suppose that x is the singleton part, and y, z and r are in the part with m vertices. So, $x \in N'[A]$, $y \in N'[\{x, y\}] \setminus N[z]$, $z \in N'[\{x, z\}] \setminus N[y]$ and $r \in N[x] \setminus N[\{y, z\}]$ and so $K_{1,m}$ is 3-w.e.c.

We have shown that $K_{2,2} \Diamond H$ and $K_{1,3} \Diamond H$ are 3-e.c. if H is 3-e.c., thereby producing two non-isomorphic 3-e.c. graphs of order $4|V(H)|$. Since the smallest 3-e.c. graph that is known to date has order 28 [11], this order of $4|V(H)|$ is much smaller than $28|V(H)|$ if both graphs were required to be 3-existentially closed (as was required in [1]).

4 Weakly 3-e.c. Strongly Regular Graphs

A k-regular graph G in which each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours is called a strongly regular graph; we say that G is a $\mathrm{SRG}(v,k,\lambda,\mu)$ with $v=|V(G)|$. In this section we recognise a few classes of strongly regular graphs that possess the 3-w.e.c. adjacency property.

Theorem 4.1 *The empty graph G with $|V(G)|\geq 4$ is 3-w.e.c.*

Proof Let $A=\{x,y,z\}\subset V(G)$ and $t\in V(G)\setminus A$. Obviously, $t\in V(G)\setminus N[A]$ which establishes the existence of a weak $\emptyset$-solution. Also $x\in N[x]\setminus N[\{y,z\}]$, $y\in N[y]\setminus N[\{x,z\}]$, and $z\in N[z]\setminus N[\{x,y\}]$ which establish the existence of a weak T-solution with respect to A for any set $T\subset A$ with $|T|$=1. □

By Theorem 4.1, in addition to the two 3-e.c. graphs $K_{2,2}\Diamond H$ and $K_{1,3}\Diamond H$, we obtain $\overline{K_4}\Diamond H$ as another 3-e.c. graph on $4|V(H)|$ vertices. We now characterise another family of 3-w.e.c. strongly regular graphs.

Theorem 4.2 *If G is a SRG(v,k,λ,μ) such that*

(i) $v\geq\max\{3k-\lambda-2\mu+2, 3k-3\mu+4\}$ *and*

(ii) $k\geq\max\{2\lambda+3,\lambda+\mu+2,2\mu+1\}$,

then G is 3-w.e.c.

Proof Let $A=\{x,y,z\}\subset V(G)$. We first show that there is a weak $\emptyset$-solution with respect to A.

If at least two pairs of the vertices of A are adjacent, then $N'[A]\neq\emptyset$ and so there is a weak $\emptyset$-solution with respect to A.

If only one pair of the vertices in A is adjacent, say x and y, then x and y have λ common neighbours, whereas x and z (resp. y and z) have μ common neighbours. Considering that the degree of each vertex is k, then by the principle of inclusion and exclusion we have $|N[A]|=3k-\lambda-2\mu+1+|N'[A]|$. Now if $|N'[A]|\neq\emptyset$, then clearly there is a weak $\emptyset$-solution with respect to A. Otherwise $|N'[A]|=\emptyset$ and $|N[A]|=3k-\lambda-2\mu+1$, and since $v>3k-\lambda-2\mu+1$ by (i), then there is a vertex in $V(G)\setminus A$ that is not a neighbour of x, y, or z; hence $V(G)\setminus N[A]\neq\emptyset$ and there is a weak $\emptyset$-solution with respect to A.

Finally, if no pair of the vertices in A is adjacent, then since G is k-regular and since every pair of the vertices of A have μ common neighbours, then by the principle of inclusion and exclusion $|N[A]|=3k-3\mu+3+|N'[A]|$. Again, if $|N'[A]|\neq\emptyset$, then there is a weak $\emptyset$-solution with respect to A. Otherwise $|N'[A]|=\emptyset$ and $|N[A]|=3k-3\mu+3$, and since $v>3k-3\mu+3$ by (i), then there exists a vertex in $V(G)\setminus A$ that is non-adjacent to every vertex in A; hence $V(G)\setminus N[A]\neq\emptyset$ which establishes the existence of a weak $\emptyset$-solution with respect to A.

Now it only remains to show that there is a weak T-solution with respect to A for any $T\subset A$ with $|T|=1$. Without loss of generality let $t=x$. If x is adjacent to both y and z, then x and y (resp. x and z) have λ common neighbours. Since $\deg(x) = k$ and $k > 2\lambda + 2$ by (ii), then there exists a vertex different from y and z which is adjacent to x and non-adjacent to both y and z. This implies that $N[x] \setminus N[\{y,z\}] \neq \emptyset$.

If x is adjacent to one of y or z, say y, then x and y have λ common neighbours, and x and z have μ common neighbours. Again, since $\deg(x) = k$ and $k > \lambda + \mu + 1$ by (ii), then there exists a vertex different from y which is adjacent to x and non-adjacent to both y and z, and hence $N[x] \setminus N[\{y,z\}] \neq \emptyset$. The case that x is non-adjacent to both y and z can be argued similarly.

So, there is a weak solution for A and G is 3-w.e.c. □

Note that the graphs that satisfy the conditions of Theorem 4.2 tend to be sparse. Examples of such graphs are the Petersen graph (a SRG$(10,3,0,1)$), the Clebsch graph (a SRG$(16,5,0,2)$), the Hoffman-Singleton graph (a SRG$(50,7,0,1)$), the Gewirtz graph (a SRG$(56,10,0,2)$), the M22 graph (a SRG$(77,16,0,4)$), the Brouwer-Haemers graph (a SRG$(81,20,1,6)$), the Higman-Sims graph (a SRG$(100, 22,0,6)$), the Local McLaughlin graph (a SRG$(162,56,10,24)$), and the $n \times n$ square rook's graph (a SRG$(n^2, 2n-2, n-2, 2)$) for large enough n. Next we present a family of 3-w.e.c. that are dense.

Theorem 4.3 *If G is a SRG$(v, v-2, v-4, v-2)$ with $v \geq 4$, then G is 3-w.e.c.*

Proof Note that since G is $(v-2)$-regular, for each set of three vertices of G at least two pairs of the vertices are adjacent. Let $A = \{x,y,z\} \subset V(G)$ be a set of three vertices, and without loss of generality suppose that x is adjacent to both y and z. Since $\deg(x) = v-2$, there exists a vertex $r \in V(G) \setminus A$ such that $rx \notin E(G)$ and $\{ry, rz\} \subseteq E(G)$. Note that $x \in N'[A]$ and so there is a weak $\emptyset$-solution with respect to A. It only remains to show that there is a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$. We will consider two cases depending on whether or not $yz \in E(G)$.

As a first case, suppose that $yz \notin E(G)$. So $y \in N'[\{x,y\}] \setminus N[z]$, $z \in N'[\{x,z\}] \setminus N[y]$ by symmetry, and $r \in N'[\{y,z\}] \setminus N[x]$. Second, suppose that $yz \in E(G)$ and without loss of generality let $t = x$. We have $r \in N'[\{y,z\}] \setminus N[x]$, and by symmetry similar arguments establish the cases $t = y$ and $t = z$. So there is a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$.

So, there is a weak solution for A and G is 3-w.e.c. □

Note that for each even $v \geq 4$, SRG$(v, v-2, v-4, v-2)$ is the complement of a perfect matching on v vertices.

5 Discussion

Now that we are able to recognise some classes of graphs G that are 3-w.e.c. and hence enabling us to construct new 3-e.c. graphs $G \Diamond H$ given that H is 3-e.c., in this section we discuss some graphs G for which G is not 3-w.e.c. In Theorem 3.1 we showed that $K_{2,2}$ is 3-w.e.c. By observing that $K_{2,2}$ is isomorphic to C_4, it is natural to ask which values of m result in 3-w.e.c. C_m. As it happens $m = 4$ is unique in this regard.

Proposition 5.1 *The cycle C_m of order m is 3-w.e.c. if and only if $m = 4$.*

Proof Suppose that we have labelled the vertices of C_m in the clockwise order by $1, 2, \ldots, m$. The graph C_4 is isomorphic to $K_{2,2}$ for which we have shown $K_{2,2}$ is 3-w.e.c. If $m \neq 4$, then for $A = \{1, 2, 3\}$ there is no weak $\{2\}$-solution with respect to A because $N'[\{1, 3\}] \setminus N[2] = \emptyset$ and $N[2] \setminus N[\{1, 3\}] = \emptyset$. □

It is also natural to ask whether it might be possible to use the modular product to obtain graphs that are 4-e.c.

Proposition 5.2 *If G and H are two graphs such that $|V(G)| \geq 2$ and H is 4-e.c., then $G \Diamond H$ cannot be 4-e.c.*

Proof Let H be a 4-e.c. graph, and let G be any graph. Consider $S = \{u_x, u_y, v_x, v_y\}$ a set of four vertices of $G \Diamond H$ such that $x \neq y$, u_x and u_y are congruent, and v_x and v_y are also congruent. For $T = \{u_x, u_y, v_x\}$ there is no T-solution. □

We conclude this section with two open problems that warrant further investigation.

Problem 5.1 *Other than graph complementation, no graph operation has yet been found that preserves the n-e.c. property for $n \geq 4$. Find an n-e.c. preserving (binary) graph operation for $n = 4$ and then for higher values of n.*

Problem 5.2 *Produce n-e.c. graphs using some graph operations such that none of the graph(s) in the operation needs to be n-e.c.*

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