AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **52** (2012), Pages 197–204

A construction of the McLaughlin graph from the Hoffman-Singleton graph

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Abstract

In this paper, we give a new construction of the McLaughlin graph from the Hoffman-Singleton graph.

1 Introduction

The sporadic simple group of McLaughlin has a rank 3 permutation representation of degree 275 which can be used to construct a strongly regular graph with parameters $(275, 112, 30, 56)$ (see McLaughlin [10]), which is frequently called the *McLaughlin graph.* A construction of the regular two-graph on 276 points due to Goethals and Seidel [6] leads to the fact that the McLaughlin graph is uniquely determined by its parameters. There are other constructions of the McLaughlin graph. G. Higman uses 23 points and 253 blocks of the Steiner system $\mathrm{S}(4, 7, 23)$ (see, for example, Cameron and van Lint [4, Example 4.15]). Taylor [11] uses 276 equiangular lines in the Leech lattice. Cossident and Penttila [5] use 112 lines and 162 hemisystems in the unitary quadrangle of order $(9, 3)$.

In this paper, we give a new construction of the McLaughlin graph from the Hoffman-Singleton graph. This construction is simple and combinatorial. As a corollary, we also construct the Higman-Sims graph.

2 The Hoffman-Singleton graph

A graph Γ consists of a finite set of *vertices* together with a set of *edges*, where an edge is a 2-subset of the vertex set. We let $\Gamma(x)$ denote the set of vertices adjacent to a vertex x and call it *neighbours* of x. The graph Γ is called a *strongly regular* graph with parameters (v, k, λ, μ) if Γ is k-regular on v vertices, not complete or null, and satisfy that any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. We refer to [4] for this definition.

Let Γ be a strongly regular graph with parameters $(50, 7, 0, 1)$, which is uniquely determined by its parameters (see [9]) and called the *Hoffman-Singleton graph.* The following lemma is proved in Calderbank and Wales [3]. It also occurs in Hafner [8] in terms of a 'biaffine plane'.

Lemma 2.1. (1) *The set $\mathcal{D}$ of all 15-cocliques of Γ has cardinality 100.*

(2) *$\mathcal{D}$ is partitioned into two 50-subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ such that*

(i) *two 15-cocliques of $\mathcal{D}_k$ intersect in 0 or 5 vertices for $k \in \{1, 2\}$ and*

(ii) *a 15-coclique of $\mathcal{D}_1$ and a 15-coclique of $\mathcal{D}_2$ intersect in 3 or 8 vertices.*

Let Γ_k be the graph with vertex set $\mathcal{D}_k$, in which two vertices D and E are adjacent whenever $|D \cap E| = 0$. Then it is also stated in [3] that Γ_k is a Hoffman-Singleton graph. These facts show the following lemma. Let $\mathcal{E}$ be the set of all edges and $\overline{\mathcal{E}}$ the set of all non-edges of Γ.

Lemma 2.2. *Let $k \in \{1, 2\}$. Then*

(1) *each non-edge of Γ is contained in exactly five 15-cocliques of $\mathcal{D}_k$;*

(2) *each vertex of Γ is contained in exactly fifteen 15-cocliques of $\mathcal{D}_k$.*

Proof. (1) Since $|\overline{\mathcal{E}}| = 1050$, we set $\overline{\mathcal{E}} = \{l_1, \ldots, l_{1050}\}$ and let n_i be the number of 15-cocliques of $\mathcal{D}_k$ which contain l_i for $1 \leq i \leq 1050$. Counting, in two ways, the number of pairs (D, l) where $D \in \mathcal{D}_k, l \in \overline{\mathcal{E}}$ and $D \supset l$, we have

$$\sum_{i=1}^{1050} n_i = 5250.$$

Moreover, count the number of pairs $(\{D, E\}, l)$ where $\{D, E\}$ is a 2-subset of $\mathcal{D}_k, l \in \overline{\mathcal{E}}$ and $D \cap E \supset l$. Then we have

$$\sum_{i=1}^{1050} n_i(n_i - 1) = 21000.$$

On the other hand, the Cauchy-Schwarz inequality shows

$$\left(\sum_{i=1}^{1050} n_i\right)^2 \leq 1050 \sum_{i=1}^{1050} n_i^2,$$

with equality if and only if $n_1 = \cdots = n_{1050}$. These yields $n_1 = 5$ and the result follows.

(2) Since a vertex x of Γ is contained in exactly 42 non-edges, it follows from (1) that x is contained in exactly fifteen 15-cocliques of $\mathcal{D}_k$. □

Remark 2.3. Let $\{k, l\} = \{1, 2\}$ and for $D \in \mathcal{D}_k$, let

$$n_i = \sharp \{E \in \mathcal{D}_l \mid |D \cap E| = i\} \text{ for } i \in \{3, 8\}.$$

Then by Lemmas 2.1 and 2.2 we have that $n_3 = 35, n_8 = 15$.

3 A construction of the McLaughlin graph

In this section, we construct a unique strongly regular graph with parameters $(275, 112, 30, 56)$, which is called the *McLaughlin graph.*

The following lemma is readily seen from the Hoffman bound (see [1, Proposition 1.3.2]).

Lemma 3.1. *If C is a k-coclique in Γ, then $k \leq 15$, with equality if and only if every vertex outside C has exactly three neighbours in C.*

We refer to Haemers [7, Lemma 3] for the following well-known lemma.

Lemma 3.2. *Let Γ_3 be the graph with vertex set $\mathcal{E}$, in which two vertices are adjacent whenever they are disjoint and there is an edge of $\mathcal{E}$ meeting both of them. Then Γ_3 is a strongly regular graph with parameters* $(175, 72, 20, 36)$.

Γ has girth 5, where the *girth* of Γ is the number of vertices in a shortest cycle in Γ. Therefore we note that, for $e, f \in \mathcal{E}$ with $|e \cap f| = 0$, if there are edges of $\mathcal{E}$ meeting both e and f, then the number of such edges of $\mathcal{E}$ is equal to 1.

The unitary group $U_3(5)$ acting on the vertex set of the McLaughlin graph has orbits of length 50, 50 and 175. Both these orbits of length 50 induce a strongly regular graph with parameters $(50, 7, 0, 1)$ and the orbit of length 175 induces a strongly regular graph with parameters $(175, 72, 20, 36)$ (see Brouwer [2]). Conversely, we are now in a position to construct the McLaughlin graph from the Hoffman-Singleton graph.

Theorem 3.3. *Define Λ to be the graph with vertex set $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{E}$ as follows:*

(i) *Let $k \in \{1, 2\}$. $D, E \in \mathcal{D}_k$ are adjacent whenever $\{D, E\}$ is an edge of Γ_k.*

(ii) *$e, f \in \mathcal{E}$ are adjacent whenever $\{e, f\}$ is an edge of Γ_3.*

(iii) *Let $k \in \{1, 2\}$. $D \in \mathcal{D}_k$ and $e \in \mathcal{E}$ are adjacent whenever $|D \cap e| = 0$.*

(iv) *$D \in \mathcal{D}_1$ and $E \in \mathcal{D}_2$ are adjacent whenever $|D \cap E| = 3$.*

Then Λ is a strongly regular graph with parameters $(275, 112, 30, 56)$.

Proof. It follows from Lemma 3.1 that, for $D \in \mathcal{D}_k$, there are exactly $35(7-3)/2 = 70$ edges of $\mathcal{E}$ disjoint from D, and so by Remark 2.3 we have $|\Lambda(D)| = 7+35+70 = 112$. For $e \in \mathcal{E}$, letting $n_i = \{D \in \mathcal{D}_k \mid |e \cap D| = i\}$ where $i \in \{0, 1\}$, we have $50 = n_0+n_1$, and a simple counting argument shows from Lemma 2.2(2) that $2 \cdot 15 = n_1$, and consequently $n_0 = 20$. Therefore $|\Lambda(e)| = 20 + 20 + 72 = 112$. Thus Λ has valency 112.

We must check that Λ satisfies the conditions $\lambda = 30, \mu = 56$.

(I) Let $\{k, l\} = \{1, 2\}$, and for distinct $D_1, D_2 \in \mathcal{D}_k$, let

$$n_{ij} = \sharp \{D \in \mathcal{D}_l \mid |D_1 \cap D| = i \text{ and } |D_2 \cap D| = j\}$$

where $i, j \in \{3, 8\}$. Then the following three equations hold:

$$50 = |\mathcal{D}_k| = \sum_{i,j} n_{ij},$$
$$\sharp\{(x, D) \in D_1 \times \mathcal{D}_l \mid x \in D\} =$$
$$15 \cdot 15 = \sum_{i,j} i n_{ij},$$
$$\sharp\{(x, D) \in D_2 \times \mathcal{D}_l \mid x \in D\} =$$
$$15 \cdot 15 = \sum_{i,j} j n_{ij}.$$

Moreover, letting $m = |D_1 \cap D_2| \in \{0, 5\}$, we have the following equation:

$$\sharp\{(x, y, D) \in D_1 \times D_2 \times \mathcal{D}_l \mid \{x, y\} \subset D\} =$$
$$\big(15^2 - m - (15 - m) \cdot 3\big) \cdot 5 + 15m = \sum_{i,j} ij n_{ij}.$$

Therefore the four equations yield that $(n_{33}, n_{38}, n_{83}, n_{88}) = (20, 15, 15, 0)$ or $(25, 10, 10, 5)$ according as $m = 0$ or 5.

Next, let

$$m_{ij} = \sharp\{e \in \mathcal{E} \mid |D_1 \cap e| = i \text{ and } |D_2 \cap e| = j\}$$

where $i, j \in \{0, 1\}$. Then the following three equations hold:

$$175 = |\mathcal{E}| = \sum_{i,j} m_{ij},$$
$$\sharp\{(x, e) \in D_1 \times \mathcal{E} \mid x \in e\} =$$
$$15 \cdot 7 = \sum_{i,j} i m_{ij},$$
$$\sharp\{(x, e) \in D_2 \times \mathcal{E} \mid x \in e\} =$$
$$15 \cdot 7 = \sum_{i,j} j m_{ij}.$$

Moreover, the following equation holds:

$$\sharp\{(x, y, e) \in D_1 \times D_2 \times \mathcal{E} \mid \{x, y\} \subseteq e\} =$$
$$(15 - m) \cdot 3 + 7m = \sum_{i,j} ij m_{ij}.$$

Therefore the four equations show that $m_{00} = 10$ or 30 according as $m = 0$ or 5. Hence

$$|\Lambda(D_1) \cap \Lambda(D_2)| = \begin{cases} 0 + 20 + 10 = 30 & \text{if } m = 0, \\ 1 + 25 + 30 = 56 & \text{if } m = 5. \end{cases}$$

(II) Let $k \in \{1, 2\}$, and for distinct $e, f \in \mathcal{E}$, let

$$n_{ij} = \sharp\,\{D \in \mathcal{D}_k \mid |e \cap D| = i \text{ and } |f \cap D| = j\}$$

where $i, j \in \{0, 1\}$. Then the following three equations hold:

$$\begin{aligned} 50 = |\mathcal{D}_k| &= \sum_{i,j} n_{ij}, \\ \sharp\,\{(x, D) \in e \times \mathcal{D}_k \mid x \in D\} &= \\ 2 \cdot 15 &= \sum_{i,j} i n_{ij}, \\ \sharp\,\{(x, D) \in f \times \mathcal{D}_k \mid x \in D\} &= \\ 2 \cdot 15 &= \sum_{i,j} j n_{ij}. \end{aligned}$$

Moreover, we have $\sharp\,\{(x, y, D) \in e \times f \times \mathcal{D}_k \mid \{x, y\} \subset D\} = n_{11}$. (i) If $\{e, f\}$ is an edge of Γ_3, then it follows from the note on Lemma 3.2 that $n_{11} = (2^2 - 1) \cdot 5$, and so we obtain $n_{00} = 5$. Hence $|\Lambda(e) \cap \Lambda(f)| = 5 + 5 + 20 = 30$.

(ii) If $|e \cap f| = 1$, then there are exactly two elements of $e \times f$ for which the unordered pairs lie in $\mathcal{E}$. Therefore it follows that $n_{11} = (2^2 - 1 - 2) \cdot 5 + 1 \cdot 15$, and so we obtain $n_{00} = 10$. Furthermore e and f have exactly 36 common neighbours in Γ_3. Hence $|\Lambda(e) \cap \Lambda(f)| = 10 + 10 + 36 = 56$.

(iii) If $|e \cap f| = 0$ and there exists no edge of $\mathcal{E}$ meeting both e and f, then it easily follows that $n_{11} = 2^2 \cdot 5$, and so we obtain $n_{00} = 10$. Furthermore e and f have exactly 36 common neighbours in Γ_3. Hence $|\Lambda(e) \cap \Lambda(f)| = 10 + 10 + 36 = 56$.

(III) Let $\{k, l\} = \{1, 2\}$, and for $D_1 \in \mathcal{D}_k$ and $D_2 \in \mathcal{D}_l$, let

$$n_{ij} = \sharp\,\{D \in \mathcal{D}_k \mid |D_1 \cap D| = i \text{ and } |D_2 \cap D| = j\}$$

where $i \in \{0, 5, 15\}$, $j \in \{3, 8\}$, and furthermore let

$$m_{ij} = \sharp\,\{e \in \mathcal{E} \mid |D_1 \cap e| = i \text{ and } |D_2 \cap e| = j\}$$

where $i, j \in \{0, 1\}$. Then in a similar way to the case (I), we have the following six equations:

$$\begin{aligned} 50 &= \sum_{i,j} n_{ij}, & 175 &= \sum_{i,j} m_{ij}, \\ 225 &= \sum_{i,j} i n_{ij}, & 105 &= \sum_{i,j} i m_{ij}, \\ 225 &= \sum_{i,j} j n_{ij}, & 105 &= \sum_{i,j} j m_{ij}. \end{aligned}$$

Moreover, letting $m = |D_1 \cap D_2| \in \{3, 8\}$, we have the following two equations:

$$\big(15^2 - m - (15 - m) \cdot 3\big) \cdot 5 + 15m = \sum_{i,j} ij n_{ij},$$

$$(15-m)\cdot 3+7m=\sum_{i,j} ijm_{ij}.$$

Thus the four equations with respect to the n_{ij} yield that $(n_{03}, n_{08}, n_{53}, n_{58}) = (4, 3, 30, 12)$ or $(7, 0, 28, 14)$ according as $m = 3$ or 8, and furthermore the four equations with respect to m_{ij}s show that $m_{00} = 22$ or 42 according as $m = 3$ or 8. Hence

$$|\Lambda(D_1)\cap\Lambda(D_2)| = \begin{cases} 4+4+22=30 & \text{if } m=3,\\ 7+7+42=56 & \text{if } m=8.\end{cases}$$

(IV) Let $\{k, l\} = \{1, 2\}$, and for $e \in \mathcal{E}$ and $D \in \mathcal{D}_k$, let

$$n_{ij} = \sharp\{E \in \mathcal{D}_k \mid |e\cap E| = i \text{ and } |D\cap E| = j\}$$

where $i \in \{0, 1\}$, $j \in \{0, 5, 15\}$, and furthermore let

$$m_{ij} = \sharp\{E \in \mathcal{D}_l \mid |e\cap E| = i \text{ and } |D\cap E| = j\}$$

where $i \in \{0, 1\}, j \in \{3, 8\}$. Then in a similar way to the preceding arguments, we have the following six equations:

$$50=\sum_{i,j} n_{ij}, \qquad 50=\sum_{i,j} m_{ij},$$
$$30=\sum_{i,j} in_{ij}, \qquad 30=\sum_{i,j} im_{ij},$$
$$225=\sum_{i,j} jn_{ij}, \qquad 225=\sum_{i,j} jm_{ij}.$$

Moreover, letting $m = |e \cap D| \in \{0, 1\}$, we have following two equations:

$$\sharp\{(x,y,E)\in e\times D\times\mathcal{D}_k \mid \{x,y\}\subset E\} =$$
$$(2\cdot 15-m-(2-m)\cdot 3)\cdot 5+15m=\sum_{i,j} ijn_{ij},$$
$$\sharp\{(x,y,E)\in e\times D\times\mathcal{D}_l \mid \{x,y\}\subset E\} =$$
$$(2\cdot 15-m-(2-m)\cdot 3)\cdot 5+15m=\sum_{i,j} ijm_{ij}.$$

Therefore direct calculations show that $(n_{00}, m_{03}) = (1, 11)$ or $(4, 16)$ according as $m = 0$ or 1.

We finally let n denote the number of edges f of $\mathcal{E}$ such that $\{e, f\}$ is an edge of Γ_3 and $|D \cap f| = 0$, and let Ξ denote the induced subgraph of Γ on 35 vertices outside D.

In the case $m = 0$, put $e = \{x_1, x_2\}$ and by Lemma 3.1 we have $|\Xi(x_i)\setminus\{x_j\}| = 3$ for $i \neq j$. Therefore set $\Xi(x_1) \setminus \{x_2\} = \{y_1, y_2, y_3\}$ and $\Xi(x_2) \setminus \{x_1\} = \{z_1, z_2, z_3\}$. Since Γ satisfies the condition $\lambda = 0$, $y_1, y_2, y_3, z_1, z_2, z_3$ are mutually distinct. From Lemma 3.1 again, set

$$\Xi(y_i)\setminus\{x_1\}=\{y_{i1},y_{i2},y_{i3}\} \text{ and } \Xi(z_i)\setminus\{x_2\}=\{z_{i1},z_{i2},z_{i3}\}$$

for $i \in \{1, 2, 3\}$, and set

$$X = \bigcup_{i,j=1}^{3} \{\{y_i, y_{ij}\}, \{z_i, z_{ij}\}\}.$$

Since Γ has girth 5, we see that $|X| = 18$ and, for each $f \in X$, $\{e, f\}$ is an edge of Γ_3, and so $n = |X| = 18$. Hence $|\Lambda(e) \cap \Lambda(D)| = 1 + 11 + 18 = 30$.

In the case $m = 1$, put $e = \{x_1, x_2\}(x_1 \notin D, x_2 \in D)$ and by Lemma 3.1 we have $|\Xi(x_1)| = 4$ and $|\Gamma(x_2) \setminus \{x_1\}| = 6$. Therefore set $\Xi(x_1) = \{y_1, y_2, y_3, y_4\}$ and $\Gamma(x_2) \setminus \{x_1\} = \{z_1, \ldots, z_6\}$. Since Γ satisfies the condition $\lambda = 0$, $y_1, \ldots, y_4, z_1, \ldots, z_6$ are mutually distinct. By Lemma 3.1 again, set

$$\Xi(y_i) \setminus \{x_1\} = \{y_{i1}, y_{i2}, y_{i3}\} \text{ for } i \in \{1, \ldots, 4\} \text{ and}$$
$$\Xi(z_i) = \{z_{i1}, z_{i2}, z_{i3}, z_{i4}\} \text{ for } i \in \{1, \ldots, 6\}.$$

Moreover, we set

$$X = \{\{y_i, y_{ij}\} \mid 1 \le i \le 4 \text{ and } 1 \le j \le 3\} \cup \{\{z_i, z_{ij}\} \mid 1 \le i \le 6 \text{ and } 1 \le j \le 4\}.$$

Since Γ has girth 5, we see that $|X| = 36$ and, for each $f \in X$, $\{e, f\}$ is an edge of Γ_3, and so $n = |X| = 36$. Hence $|\Lambda(e) \cap \Lambda(D)| = 4 + 16 + 36 = 56$. This completes the proof. □

As a corollary, we can also construct a unique strongly regular graph with parameters $(100, 22, 0, 6)$ which is called the *Higman-Sims graph*. Although the procedure has been known (see [2, 8]), it seems easier to us in this setting.

Corollary 3.4. *Define Σ to be the graph with vertex set $\mathcal{D}_1 \cup \mathcal{D}_2$ as follows:*

(i) *Let $k \in \{1, 2\}$. $D, E \in \mathcal{D}_k$ are adjacent whenever $\{D, E\}$ is an edge of Γ_k.*

(ii) *$D \in \mathcal{D}_1$ and $E \in \mathcal{D}_2$ are adjacent whenever $|D \cap E| = 8$.*

Then Σ is a strongly regular graph with parameters $(100, 22, 0, 6)$.

Proof. For $D \in \mathcal{D}_k$, by Remark 2.3, $|\Sigma(D)| = 7 + 15 = 22$. For distinct $D, E \in \mathcal{D}_k$, by the case (I) in the proof of Theorem 3.3, we have

$$|\Sigma(D) \cap \Sigma(E)| = \begin{cases} 0 + 0 = 0 & \text{if } |D \cap E| = 0, \\ 1 + 5 = 6 & \text{if } |D \cap E| = 5. \end{cases}$$

For $D \in \mathcal{D}_1$ and $E \in \mathcal{D}_2$, by the case (III) in the proof of Theorem 3.3, we have

$$|\Sigma(D) \cap \Sigma(E)| = \begin{cases} 0 + 0 = 0 & \text{if } |D \cap E| = 8, \\ 3 + 3 = 6 & \text{if } |D \cap E| = 3. \end{cases}$$

The result follows. □

Acknowledgements

The author is indebted to Professor M. Kitazume for valuable discussions which led to the main result.

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(Received 29 May 2011; revised 8 Oct 2011)