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A note on terraces for abelian groups

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Abstract

We adapt a construction for R-sequencings to give new terraces for abelian groups. This enables the construction of terraces for all groups of the form $\mathbb{Z}_2^k \times \mathbb{Z}_t$ where $k \geq 4$ and $t > 5$ is odd. We also present an extendable terrace for $\mathbb{Z}_2^3 \times \mathbb{Z}_5$. When combined with known results this shows that any abelian counterexample to Bailey's Conjecture (which asserts that all groups except the elementary abelian 2-groups of order at least 4 are terraced) has the form $\mathbb{Z}_2^3 \times O$ where $|O|$ is not divisible by 2, 3, 5 or 7.

1 Introduction

Let G be an additively written non-trivial abelian group of order n and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of all of the elements of G. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ by $b_i = a_{i+1} - a_i$ for each i with $1 \leq i \leq n-1$. If $\mathbf{b}$ contains each non-identity element of G exactly once then $\mathbf{a}$ is a *directed terrace* for G and $\mathbf{b}$ is its associated *sequencing*. If $\mathbf{b}$ contains each involution of G exactly once and exactly two occurrences from each set $\{g, -g : 2g \neq 0\}$ then $\mathbf{a}$ is a *terrace* for G and $\mathbf{b}$ is its associated *2-sequencing*. If G has a terrace then it is *terraced*. An abelian group has a directed terrace if and only if it has a single involution [8].

Terraces were formally introduced by Bailey [5] for arbitrary finite groups in connection with the construction of quasi-complete Latin squares. In that paper it is shown that elementary abelian 2-groups of order at least 4 are not terraced and it is conjectured that all other groups are (this is now known as *Bailey's Conjecture*). In this paper we move closer to a proof of Bailey's Conjecture for abelian groups.

Prior to this work it was known that all abelian groups except possibly those of the form $\mathbb{Z}_2^k \times O$ where $k \geq 3$ is odd and O has order coprime to 6 satisfy Bailey's Conjecture [9, 10]. We here show that groups of this form are terraced when $k \geq 5$ or when $|O|$ is divisible by 5 or 7.

In the next section we introduce R-sequencings—interesting objects of study in their own right but for our purposes an intermediate step in the construction—and describe a construction of Wang [12]. This leads to the construction of terraces for many groups. In Section 3 we combine the results here with results in the literature, give an "extendable" terrace for $\mathbb{Z}_2^3 \times \mathbb{Z}_5$, and conclude that finding a terrace for each group of the form $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, where $p > 7$ is prime, is sufficient to prove that Bailey's Conjecture holds for abelian groups.

2 The construction

Again, let G be an additively written non-trivial abelian group of order n. Let $\mathbf{a} = [a_1, a_2, \dots, a_{n-1}]$ be a circular arrangement of all of the non-identity elements of G (we use the convention that circular lists are written in square brackets and subscripts are considered modulo the length of the list). Define $\mathbf{b} = [b_1, b_2, \dots, b_{n-1}]$ by $b_i = a_{i+1} - a_i$ for each i with $1 \leq i \leq n-1$. If $\mathbf{b}$ contains each non-identity element of G exactly once then $\mathbf{a}$ is a *directed rotational terrace* or *directed R-terrace* for G and $\mathbf{b}$ is an *R-sequencing* of G. If a group has an R-sequencing then it is called *R-sequenceable*. R-sequencings were introduced by Ringel in relation to map coloring problems [11]. There is an undirected analogue of R-terraces similar to the definition of a terrace [9] but we do not need that concept here.

Let $\mathbf{b}$ be an R-sequencing of G with the above notation. If $b_i = -a_{i+1}$ for some i then i is a *right match-point* of $\mathbf{b}$. This gives the connection we require between R-sequencings and terraces: if $[a_1, a_2, \dots, a_{n-1}]$ is a directed R-terrace whose associated R-sequencing has a right match-point i then $(0, a_{i+1}, a_{i+2}, \dots, a_{n-1}, a_1, a_2, \dots, a_i)$ is a terrace for G [9].

We need one more embellishment of the directed R-terrace idea. If $[a_1, a_2, \dots, a_{n-1}]$ is a directed R-terrace such that $a_2 + a_{n-1} = a_1$ then it is a *standard directed R^*-terrace* and its associated R-sequencing is a *standard R^*-sequencing.*

The next two results provide the ingredients for the main theorem. We refer the reader to the original papers for the proofs.

Lemma 1 [6] *The group $\mathbb{Z}_2^k$ has a standard directed R^*-terrace for all $k \geq 4$.*

Lemma 2 [9] *For all odd $t > 5$, the group $\mathbb{Z}_t$ has a standard directed R^*-terrace whose associated R^*-sequencing has a match-point in an odd position.*

Proof Note. The only aspect not explicitly covered in [9] is the parity of the match-point. That paper gives R*-sequencings for $\mathbb{Z}_{2r+1}$, where $r \geq 3$, that have "right match-point i, where

$$i = \begin{cases} (2r+3)/3 & \text{when } r \equiv 0 \pmod 3 \\ (4r-1)/3 & \text{when } r \equiv 1 \pmod 3 \\ (4r-5)/3 & \text{when } r \equiv 2 \pmod 6 \\ (2r-1)/3 & \text{when } r \equiv 5 \pmod 6 \text{ and } r > 5. \end{cases}$$

The standard R*-sequencing of $\mathbb{Z}_{11}$ ... has right match-point 7." It is routine to check that each of these values is odd. □

We now move Wang's direct product construction for R*-sequencings. We restrict ourselves to the case that is of immediate use to us; Wang's theorem is more general than the version given here.

Theorem 1 [12] *Let G be an abelian group of even order m with a standard directed R*-terrace and H an abelian group of odd order with a standard directed R*-terrace. Then $G \times H$ has a standard directed R*-terrace.*

Proof Construction. The construction is taken directly from [12] with only minor notational changes. The reader is referred to that paper for a proof of its correctness.

Let $[a_1, a_2, \ldots, a_{m-1}]$ be a standard directed R*-terrace for G. Let H have order $2l+1$ and let $[c_1, c_2, \ldots, c_{2l}]$ be a standard directed R*-terrace for H. We define two $m \times (2l+1)$ matrices A and C built from these R*-terraces:

$$A = \begin{pmatrix} \cdot & a_1 & a_1 & \cdots & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & a_1 & a_1 & \cdots & a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_2 & a_2 & \cdots & a_2 & a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m-1} & a_{m-1} & a_{m-1} & \cdots & a_{m-1} & a_{m-1} & a_{m-1} & \cdots & a_{m-1} \end{pmatrix}$$

$$C = \begin{pmatrix} \cdot & c_3 & c_5 & \cdots & c_{2l-1} & c_1 & c_2 & c_4 & \cdots & c_{2l-2} & c_{2l} \\ 0 & c_2 & c_4 & \cdots & c_{2l-2} & c_{2l} & c_1 & c_3 & \cdots & c_{2l-3} & c_{2l-1} \\ 0 & c_3 & c_5 & \cdots & c_{2l-1} & c_1 & c_2 & c_4 & \cdots & c_{2l-2} & c_{2l} \\ 0 & -c_3 & -c_5 & \cdots & -c_{2l-1} & -c_1 & -c_2 & -c_4 & \cdots & -c_{2l-2} & -c_{2l} \\ 0 & c_3 & c_5 & \cdots & c_{2l-1} & c_1 & c_2 & c_4 & \cdots & c_{2l-2} & c_{2l} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -c_3 & -c_5 & \cdots & -c_{2l-1} & -c_1 & -c_2 & -c_4 & \cdots & -c_{2l-2} & -c_{2l} \\ 0 & c_2 & c_4 & \cdots & c_{2l-2} & c_{2l} & c_1 & c_3 & \cdots & c_{2l-3} & c_{2l-1} \\ c_2 & c_4 & c_6 & \cdots & c_{2l} & c_1 & c_3 & c_5 & \cdots & c_{2l-1} & 0 \end{pmatrix}$$

The sequence of $m(2l+1)-1$ pairs (x_i, y_i) obtained by reading down the columns of A and C, taking the entry from A to be x_i and the entry from C to be y_i, is a standard directed R*-terrace for $G \times H$. □

We now combine these results to produce terraces for many groups.

Theorem 2 *The group $\mathbb{Z}_2^k \times \mathbb{Z}_t$ is terraced for all $k \geq 4$ and odd $t > 5$.*

Proof. Let $\mathbf{a}$ be a standard directed R*-terrace for $\mathbb{Z}_2^k$ (which exists by Lemma 1) and let $\mathbf{c} = [c_1, c_2, \ldots c_{t-1}]$ be a standard directed R*-terrace for $\mathbb{Z}_t$ whose associated R*-sequencing $[d_1, d_2, \ldots, d_{t-1}]$ has a right match-point in an odd position i (which exists by Lemma 2). Apply Theorem 1 to these to obtain a directed R*-terrace $\mathbf{e} = [e_1, e_2, \ldots e_{2^k t-1}]$ for $\mathbb{Z}_2^k \times \mathbb{Z}_t$ with associated R*-sequencing $\mathbf{f} = [f_1, f_2, \ldots, f_{2^k t-1}]$.

As i is odd there are successive elements $(0, c_{i+1})$ and $(0, c_i)$ in $\mathbf{e}$ obtained from the second half of the first two rows of A and C in the construction of $\mathbf{e}$. More specifically, we have $e_j = (0, c_{i+1})$ and $e_{j+1} = (0, c_i)$ for $j = 2m(l + i)$. Now, $f_j = e_{j+1} - e_j = (0, -d_i) = (0, c_{i+1})$. Consider the sequence

$$(e_{j+1}, e_{j+2}, \ldots, e_{2^k p-1}, e_1, e_2, \ldots, e_j, (0, 0)).$$

This is a terrace for $\mathbb{Z}_2^k \times \mathbb{Z}_t$ as $\mathbf{f}$ contains each non-identity element of $G \times H$ once and the differences generated by the new sequence are the same as those in $\mathbf{f}$ except that $(0, c_{i+1})$ is replaced by $(0, -c_{i+1}) = -(0, c_{i+1})$. This does not violate the definition of a terrace. $\square$

3 Final remarks

Theorems 3 and 4 are powerful results for constructing terraces for abelian groups. In this section we use these along with the new terraces constructed here to make the progress towards Bailey's Conjecture claimed in Section 1.

Theorem 3 [3, 4] *Let G be a group with a normal subgroup N. If N has odd order and G/N is terraced then G is terraced. Alternatively, if N has odd index and N is terraced then G is terraced.*

The second result deals with terraces that have a stronger property: A terrace $(a_1, a_2, \ldots, a_n)$ for an abelian group of order n is *extendable* if $a_1 = 0$, $a_n = 2a_2$ and $a_{j-1} + a_{j+1} = a_j$ for some $j \geq 5$.

Theorem 4 [9, 10] *If G is an abelian group with a subgroup V isomorphic to $\mathbb{Z}_2^2$ and G/V has an extendable terrace then G has an extendable terrace.*

In [9, Theorem 17] it is mistakenly claimed that V may also be isomorphic to $\mathbb{Z}_4$ in Theorem 4 (the word "extendable" is not used in that paper; an extendable terrace is equivalent to a "standard R*-2-sequencing with i as a right match-point for some $1 \leq i \leq m - 3$ [where m is the order of the group]"). The $\mathbb{Z}_4$ version of the construction is only used in one place in that paper and it is shown in [10] that that piece of the argument can be avoided when forming terraces for the groups claimed to be terraced in [9].

Extendable terraces are known to exist for

- abelian 2-groups of order at least 8, except those that are elementary abelian,
- cyclic groups of order at least 7, except those of order twice an odd number,

- $\mathbb{Z}_2^3 \times \mathbb{Z}_3$.

From this it follows, using Theorems 3 and 4, that all abelian groups satisfy Bailey's Conjecture except possibly those of the form $\mathbb{Z}_2^k \times O$ when $k \geq 3$ is odd and $|O|$ is not divisible by 2 or 3 [9, 10].

We give here an extendable terrace (with $j = 14$) for $\mathbb{Z}_2^3 \times \mathbb{Z}_5$, found using a heuristic algorithm based on ideas from [1] implemented in GAP [7]. Parentheses and commas from the direct product notation are omitted for brevity:

$$(0000, 1111, 1114, 0014, 1014, 0010, 0001, 0112, 1003, 1104,$$
$$1011, 0110, 1013, 1101, 0113, 0011, 0012, 1102, 1012, 0114,$$
$$1002, 1010, 1004, 0013, 1100, 1112, 1001, 1103, 0102, 1110,$$
$$0101, 0111, 1113, 0103, 0100, 0004, 0104, 1000, 0003, 0002).$$

This means that we can also rule out counterexamples to Bailey's Conjecture with $|O|$ divisible by 5. Theorem 2, in conjunction with Theorem 3, allows us to eliminate many more possible counterexamples:

Theorem 5 *Let $k > 3$ be odd. Then $\mathbb{Z}_2^k \times O$ is terraced for all non-trivial abelian groups O of odd order.*

Proof. If $|O|$ is divisible by 3 or by 5 then the result holds by the above discussion. Otherwise, let p be a prime divisor of $|O|$ with $p > 5$. Let N be a normal subgroup of $\mathbb{Z}_2^k \times O$ isomorphic to $\mathbb{Z}_2^k \times \mathbb{Z}_p$ and use Theorem 2 to construct a terrace for N. The quotient group G/N has odd order and so we can apply Theorem 3 to obtain the result. □

Therefore the only possible abelian counterexamples to Bailey's Conjecture have the form $\mathbb{Z}_2^3 \times O$ where O is a non-trivial abelian group of odd order with $|O|$ not divisible by 3 or 5. Further, using Theorem 3, to show that no counterexample exists with $|O|$ divisible by a given prime p it is sufficient to find a terrace for $\mathbb{Z}_2^3 \times \mathbb{Z}_p$. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is terraced [2], hence we can also rule out counterexamples of the form $\mathbb{Z}_2^3 \times O$ with $|O|$ divisible by 7.

Constructing a terrace for $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ for each prime $p > 7$ is now sufficient to complete the proof of the conjecture for abelian groups.

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