AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **52** (2012), Pages 235–242

# A note on the Caro-Tuza bound on the independence number of uniform hypergraphs*

Béla Csaba[†]

*Department of Mathematics*
*Western Kentucky University*
*1906 College Heights Boulevard*
*Bowling Green, KY 42101*
*U.S.A.*
b.csaba@bham.ac.uk

Thomas A. Plick[‡] Ali Shokoufandeh

*Department of Computer Science*
*Drexel University*
*3141 Chestnut Street*
*Philadelphia, PA 19104*
*U.S.A.*
tap42@drexel.edu ashokouf@cs.drexel.edu

**Abstract**

We show some consequences of Caro and Tuza's [*J. Graph Theory* 15 (1991), 99–107] lower bound on the independence number of a $K$-uniform hypergraph $H$. This bound has the form $C_K \cdot \sum_{i=1}^{n}(d_i+1)^{-1/(K-1)}$, where $C_K$ is a constant depending only on $K$, and $d_1, \ldots, d_n$ are the degrees of the vertices in $H$. We improve on the best known bounds for $C_K$: in particular, we prove that $C_3 \geq \sqrt{\pi}/2$ and that $C_K \geq \exp(-\gamma/(K-1))$ for $K \geq 3$, where $\gamma$ is the Euler-Mascheroni constant.

## 1 Introduction

A theorem due independently to Caro [1] and to Wei [6] states that for a graph $G = (V, E)$ whose vertices have respective degrees $d_1, d_2, \ldots, d_n$, the independence

* The authors were partially supported by United States NSF grant #0803670 under the IIS Division and United States Office of Naval Research grant ONRN000140410363.

† This author was partially supported by the Hungarian Scientific Research Fund (OTKA) grant K76099; author now at School of Mathematics, University of Birmingham, B15 2TT, U.K.

‡ Corresponding author.

number $\alpha(G)$ obeys

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

Caro and Tuza [2] later generalized this result to $K$-uniform hypergraphs, and Thiele [5] generalized this result further to general hypergraphs. Caro and Tuza proved:

**Theorem 1 (Caro-Tuza)** *For $K \geq 2$, let $H = (V, E)$ be a $K$-uniform hypergraph whose vertices $v_1, v_2, \ldots, v_n$ have respective degrees $d_1, d_2, \ldots, d_n$. Then $\alpha(H) \geq \sum_{i=1}^{n} p_K(d_i)$, where*

$$p_K(d) = \prod_{j=1}^{d} \left(1 - \frac{1}{(K-1)j + 1}\right).$$

We first consider the case $K = 3$. With the identity

$$p_3(d) = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2d}{2d+1} = \frac{4^d \cdot (d!)^2}{(2d+1)!}$$

and a lengthy computation involving Stirling's approximation, it is straightforward to show $p_3(d) \sim \frac{\sqrt{\pi}}{2\sqrt{d}}$. In Section 2 we prove an even stronger statement:

**Theorem 2**

$$\frac{\sqrt{\pi}}{2\sqrt{d+1}} < p_3(d) < \frac{\sqrt{\pi}}{2\sqrt{d+1/2}}.$$

This point does not seem to have been made in the literature. As a consequence, we have

**Theorem 3** *Let $H = (V,E)$ be a 3-uniform hypergraph whose $n$ vertices $v_1, v_2, \ldots, v_n$ (with $n > 0$) have respective degrees $d_1, d_2, \ldots, d_n$. Then*

$$\alpha(H) > \frac{\sqrt{\pi}}{2} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{d_i + 1}}.$$

($\frac{\sqrt{\pi}}{2} = 0.886\ldots$)

In Section 3 we generalize these results to $K$-uniform hypergraphs with $K > 3$, obtaining the following analogue to Theorem 3:

**Theorem 4** *Let $H$ be a $K$-uniform hypergraph for $K \geq 3$, and let $d_1, d_2, \ldots, d_n$ be the degrees of its vertices, with $n > 0$. Then*

$$\alpha(H) > e^{-\gamma/(K-1)} \cdot \sum_{i=1}^{n} \frac{1}{(d_i + 1)^{1/(K-1)}},$$

*where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.*

The coefficient $e^{-\gamma/(K-1)}$ is an improvement on the value $1 - 1/K$ from Spencer's [4] result

$$\alpha(H) \geq \frac{n}{\delta^{1/(K-1)}} \cdot \left(1 - \frac{1}{K}\right),$$

where $\delta \geq 1$ is the average vertex degree in $H$.

## 2 The case of 3-uniform hypergraphs

In this section we will write $p(\cdot)$ for $p_3(\cdot)$. Recall Wallis' formula,

$$\frac{\pi}{2} = \prod_{i=1}^{\infty} \frac{2 \cdot \lceil i/2 \rceil}{2 \cdot \lfloor i/2 \rfloor + 1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{10}{11} \cdots.$$

Let $W_k$ denote the product of the first $k$ fractions in the formula as written above. Thus $W_1 = \frac{2}{1}$, $W_2 = \frac{2}{1} \cdot \frac{2}{3}$, etc.

**Remark 5** *Normally Wallis' formula is given as*

$$\prod_{i=1}^{\infty} \frac{(2i)^2}{(2i-1)(2i+1)},$$

*giving partial products $W_2, W_4, \ldots$, but we will find the prior version more convenient. The difference in grouping does not change its convergence properties, since*

$$\lim_{k\to\infty} \frac{W_{2k+1}}{W_{2k}} = 1.$$

**Proposition 6** *$W_k < \frac{\pi}{2}$ for even $k$, and $W_k > \frac{\pi}{2}$ for odd $k$.*

**Proof:** Let $k$ be even. The next two factors in the product have the forms $\frac{i}{i-1}$ and $\frac{i}{i+1}$; their product is greater than 1. The same is true for the next two, the next two after them, etc. Letting $R$ be the product of *all* these pairs of factors, we have $R > 1$. Since $W_k \cdot R = \frac{\pi}{2}$, it follows that $W_k < \frac{\pi}{2}$.

Similarly, when $k$ is odd, each subsequent pair of factors has the form $\frac{i-1}{i}$ and $\frac{i+1}{i}$, whose product is less than 1. Thus the product of the remaining pairs, $R$, is less than 1. Since $W_k \cdot R = \frac{\pi}{2}$, we have $W_k > \frac{\pi}{2}$. □

**Proof of Theorem 2:** Recall

$$p(d) = \frac{2 \cdot 4 \cdot 6 \cdots (2d)}{1 \cdot 3 \cdot 5 \cdots (2d+1)}.$$

Squaring and multiplying by $2d + 1$, we obtain

$$p(d)^2 \cdot (2d+1) \quad = \quad \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2d) \cdot (2d)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2d-1) \cdot (2d-1) \cdot (2d+1)} = W_{2d} < \frac{\pi}{2}$$

by Proposition 6. Similarly,

$$p(d)^2 \cdot (2d+2) = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2d) \cdot (2d) \cdot (2d+2)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2d+1) \cdot (2d+1)} = W_{2d+1} > \frac{\pi}{2}.$$

Thus

$$\frac{\pi}{2(2d+2)} < p(d)^2 < \frac{\pi}{2(2d+1)}$$
$$\frac{\sqrt{\pi}}{2\sqrt{d+1}} < p(d) < \frac{\sqrt{\pi}}{2\sqrt{d+1/2}}.$$

□

Theorem 2 makes it obvious that $p(d) \sim \frac{\sqrt{\pi}}{2\sqrt{d}}$.

## 3 The case of $K$-uniform hypergraphs, $K > 3$

In this section we will develop analogues of Theorems 2 and 3 that apply to $K$-uniform hypergraphs with $K > 3$. It is known (see [3]) that for each $K > 1$, some positive constant $c_K$ satisifies

$$\alpha(H) \geq c_K \cdot \sum_{i=1}^{n} \frac{1}{(d_i+1)^{1/(K-1)}}.$$

Recall that we defined

$$p_K(d) = \prod_{i=1}^{d} \frac{(K-1)i}{(K-1)i+1};$$

this is the formula found by Caro and Tuza in [2]. In Section 2 we considered the case $K = 3$. For general $K \geq 2$, we have

**Proposition 7**

$$\lim_{d\to\infty} p_K(d) \cdot (d+1)^{1/(K-1)} = e^{h(K-1)},$$

*where*

$$h(z) = \frac{1}{z} \cdot \sum_{i=2}^{\infty} \frac{(-1)^i \cdot \zeta(i)}{i} \cdot \left(\frac{1}{z^{i-1}} - 1\right)$$

*and $\zeta$ is the Riemann zeta function.*

Thus for fixed $K$,

$$p_K(d) \sim \frac{e^{h(K-1)}}{d^{1/(K-1)}}.$$

**Proof:** We can rewrite $p_K(d) \cdot (d+1)^{1/(K-1)}$ as $\prod_{i=1}^{d} a_i$, where

$$\begin{aligned} a_i &= \frac{(K-1)i}{(K-1)i+1} \cdot \left(\frac{i+1}{i}\right)^{1/(K-1)} \\ &= \left(1 + \frac{1}{(K-1)i}\right)^{-1} \cdot \left(1 + \frac{1}{i}\right)^{1/(K-1)}. \end{aligned}$$

Then

$$\begin{aligned} \log a_i &= \frac{1}{K-1} \cdot \log\left(1 + \frac{1}{i}\right) - \log\left(1 + \frac{1}{(K-1)i}\right) \\ &= \frac{1}{K-1} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \\ &= \frac{1}{K-1} \cdot \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j}, \end{aligned}$$

since the terms for $j = 1$ cancel out. These were the troublesome terms, since they belong to the harmonic series, which does not converge.

The next step is to calculate $\sum_{i=1}^{\infty} \log a_i$, but we must be careful. The series for $\log a_i$ converges absolutely when $i > 1$ but *not* when $i = 1$. We must therefore pay closer attention to that series.

$$\begin{aligned} \sum_{i=1}^{\infty} \log a_i &= \sum_{i=1}^{\infty} \left( \frac{1}{K-1} \cdot \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \\ &= \sum_{i=1}^{\infty} \left( \sum_{j=2}^{\infty} \left( \frac{(-1)^{j+1}}{(K-1) \cdot j \cdot i^j} - \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \right) \\ &= \sum_{j=2}^{\infty} \left( \frac{(-1)^{j+1}}{(K-1) \cdot j} - \frac{(-1)^{j+1}}{j \cdot (K-1)^j} \right) \\ &\qquad + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left( \frac{(-1)^{j+1}}{(K-1) \cdot j \cdot i^j} - \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \\ &= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \left( \frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \\ &\qquad + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} \left( \frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \qquad (1) \\ &= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \left( \frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \\ &\qquad + \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} \left( \frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \qquad (2) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=2}^{\infty}\left(\sum_{i=1}^{\infty}\frac{(-1)^{j+1}}{j\cdot i^j}\left(\frac{1}{K-1}-\frac{1}{(K-1)^j}\right)\right)\\
&= \sum_{j=2}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{1}{K-1}\cdot\sum_{i=1}^{\infty}\frac{1}{i^j}-\frac{1}{(K-1)^j}\cdot\sum_{i=1}^{\infty}\frac{1}{i^j}\right)\\
&= \frac{1}{K-1}\cdot\sum_{j=2}^{\infty}\frac{(-1)^{j+1}\cdot\zeta(j)}{j}\left(1-\frac{1}{(K-1)^{j-1}}\right)\\
&= h(K-1).
\end{aligned}
$$

We have repeatedly used the identity $\sum_{k=1}^{\infty}(x_k+y_k) = (\sum_{k=1}^{\infty} x_k) + (\sum_{k=1}^{\infty} y_k)$, which is valid whenever the sums $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ are finite. To get from (1) to (2), we used the fact that the double sum is absolutely convergent (in fact, $\sum_{i=2}^{\infty}\sum_{j=2}^{\infty} 1/(j\cdot i^j) = 1-\gamma$).

Exponentiating, we conclude $\prod_{i=1}^{\infty} a_i = \exp(h(K-1))$. □

As was the case for $K=3$, our asymptotic bound also yields a lower bound:

**Proposition 8** *For $K \geq 3$,*

$$p_K(d)\cdot(d+1)^{1/(K-1)} > e^{h(K-1)}.$$

Thus

$$p_K(d) > \frac{e^{h(K-1)}}{(d+1)^{1/(K-1)}}.$$

**Proof:** We will show that each factor $a_i$ (from the proof of Proposition 7) is less than 1; as a result, the partial products will decrease toward the limit given in Proposition 7. The statement $a_i < 1$ is equivalent to

$$\frac{(K-1)i}{(K-1)i+1}\cdot\left(\frac{i+1}{i}\right)^{1/(K-1)} \quad < \quad 1,$$

which in turn is equivalent to

$$\left(\frac{(K-1)i}{(K-1)i+1}\right)^{K-1}\cdot\frac{i+1}{i} \quad < \quad 1$$

and

$$((K-1)i)^{K-1}\cdot(i+1) \quad < \quad ((K-1)i+1)^{K-1}\cdot i.$$

This latter statement is easily shown to be true. The left-hand side equals $(K-1)^{K-1}\cdot i^K + (K-1)^{K-1}\cdot i^{K-1}$. The expansion of the right-hand side will contain (among others) the terms $(K-1)^{K-1}\cdot i^K$ and $\binom{K-1}{1}\cdot(K-1)^{K-2}\cdot i^{K-2}\cdot i$. Thus the right-hand side exceeds the left-hand side. □

Since for a $K$-uniform hypergraph $H$ we have $\alpha(H) \geq \sum_{i=1}^{n} p_K(d_i)$, Theorem 1 and Proposition 8 demonstrate

**Theorem 9** *For $K \geq 3$, let $H = (V, E)$ be a $K$-uniform hypergraph whose $n$ vertices $v_1, v_2, \ldots, v_n$ (with $n > 0$) have respective degrees $d_1, d_2, \ldots, d_n$. Then*

$$\alpha(H) > e^{h(K-1)} \cdot \sum_{i=1}^{n} \frac{1}{(d_i + 1)^{1/(K-1)}}.$$

**Proof of Theorem 4:** The identity $\sum_{i=2}^{\infty}(-1)^i \cdot \zeta(i)/i = \gamma$ lets us rewrite $h(z)$ as

$$\frac{1}{z} \cdot \left[\left(\sum_{i=2}^{\infty} \frac{(-1)^i \cdot \zeta(i)}{i \cdot z^{i-1}}\right) - \gamma\right].$$

The parenthesized summation is an alternating series whose terms are descending in magnitude; thus the sign of the limit is the same as the sign of its first term, which is positive. We thereby have

$$h(K-1) > -\frac{\gamma}{K-1},$$

which together with Theorem 9 proves Theorem 4. □

## 4 Conclusion

We have shown that for any non-empty $K$-uniform hypergraph $H$ with vertex degrees $d_1, \ldots, d_n$, the statement

$$\alpha(H) \geq c_K \cdot \sum_{i=1}^{n} \frac{1}{(d_i + 1)^{1/(K-1)}}$$

holds true when we set $c_3 = \sqrt{\pi}/2$ and $c_K = e^{-\gamma/(K-1)}$ for $K > 3$. The value $e^{-\gamma/(K-1)}$ is a close approximation to the true value given by Propositions 7 and 8; it would be nice to know whether the exact value is expressible in a closed form.

It is also of interest to determine how close to optimal these constants are; since the case of a hypergraph with no edges (every degree is zero) forces $c_K \leq 1$, we can alternatively consider hypergraphs where every degree is at least some constant $D$ and try to bound the corresponding constant $c_{K,D}$. In this case, we can show fairly easily that the optimal value of $c_{K,D}$ is at most $e + o(1)$, where the $o(1)$ term depends only on $K$. Consider a complete $K$-uniform hypergraph $H$ on $n$ vertices. Clearly $\alpha(H) = K - 1$. On the other hand, each vertex has degree $\binom{n-1}{K-1}$, so that it must hold true that

$$c_{K,D} \cdot \frac{n}{(\binom{n-1}{K-1} + 1)^{1/(K-1)}} \leq K - 1.$$

Hold $K$ constant and let $n$ go to $\infty$. Then $\binom{n-1}{K-1}$ behaves as $n^{K-1}/(K-1)!$, so that the left-hand side is asymptotic to $c_{K,D} \cdot (K-1)!^{1/(K-1)}$. From Stirling's formula it is easily seen that

$$K!^{1/K} \to \frac{K}{e},$$

and $c_{K,D} \leq e + o(1)$.

## References

[1] Yair Caro, New results on the independence number, Technical report, Tel-Aviv University, 1979.

[2] Yair Caro and Zsolt Tuza, Improved lower bounds on $k$-independence, *J. Graph Theory* 15 (1991), 99–107.

[3] Kunal Dutta, Dhruv Mubayi and C.R. Subramanian, New lower bounds for the independence number of sparse graphs and hypergraphs, (2011), to appear.

[4] Joel Spencer, Turán's theorem for $k$-graphs, *Discrete Math.* 2(2) (1972), 183–186.

[5] Torsten Thiele, A lower bound on the independence number of arbitrary hypergraphs, *J. Graph Theory* 30(3) (1999), 213–221.

[6] V. K. Wei, A lower bound on the stability number of a simple graph, Technical Report (Memo) No. 81–11217–9, Bell Labs, 1981.

(Received 20 July 2011; revised 20 Oct 2011)