AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 53–65

Construction of (γ, k)-critical graphs

Michitaka Furuya

Department of Mathematical Information Science
Tokyo University of Science
1–3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601
Japan
michitaka.furuya@gmail.com

Abstract

For a graph G, we let $\gamma(G)$ denote the domination number of G. A graph G is said to be (l, k)-critical if $\gamma(G) = l$ and $\gamma(G - U) < \gamma(G)$ for every $U \subseteq V(G)$ with $|U| = k$. In this paper, we characterize $(2, k)$-critical graphs for $k \geq 1$, and show, for each $l \geq 3$ and each $k \geq 1$, how to construct infinitely many connected graphs which are (l, h)-critical for every $1 \leq h \leq k$.

1 Introduction

In this paper, all graphs are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The cardinality of $V(G)$ is referred to as the *order* of G. For $v \in V(G)$, we denote the *open neighborhood* $N(v)$ of v by $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, the *closed neighborhood* $N[v]$ of v by $N[v] = N(v) \cup \{v\}$, and the degree $d(v)$ of v by $d(v) = |N(v)|$. The maximum of $d(v)$ as v ranges over $V(G)$ is called the *maximum degree* of G and denote by $\Delta(G)$. We say that G is *r-regular* if $d(v) = r$ for all $v \in V(G)$. We let K_n denote the *complete graph* of order n, i.e., the $(n-1)$-regular graph of order n. For terms and symbols not defined here, we refer the reader to [9].

Again let G be a graph. For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{v \in X} N[v]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating set* of G. The minimum cardinality of a dominating set of G is called the *domination number* of G and denoted by $\gamma(G)$. A dominating set of G having cardinality $\gamma(G)$ is called a γ-*set* of G.

It can be observed that while $\gamma(G - v) > \gamma(G) + 1$ is possible for some graphs G and vertices $v \in V(G)$, it is always the case that $\gamma(G - v) \geq \gamma(G) - 1$, that is, deleting a vertex can decrease the domination number by at most one. In 1988, Brigham, Chinn and Dutton [4] began the study of graphs for which $\gamma(G - v) = \gamma(G) - 1$

for every vertex $v \in V(G)$. They defined a vertex v in a graph G to be *domination critical*, or *critical* for short, if $\gamma(G-v) < \gamma(G)$, and the graph G to be *domination critical* or, more simply, *critical*, if every vertex $v \in V(G)$ is critical.

In 2005, Brigham, Haynes, Henning and Rall [5] gave a generalization of this concept. For $k \geq 0$, they defined a graph G to be (γ, k)-*critical* if $\gamma(G-U) < \gamma(G)$ for every $U \subseteq V(G)$ with $|U| = k$. Thus $(\gamma, 1)$-critical graphs are nothing but critical graphs. Note that no graph can be $(\gamma, 0)$-critical. Also note that for $k \geq 1$, if $|V(G)| \leq k$, then G is (γ, k)-critical.

The main purpose of this paper is to bring to an end research on two basic problems concerning (γ, k)-critical graphs: the construction of (γ, k)-critical graphs with given domination number $l \geq 3$, and the characterization of (γ, k)-critical graphs with domination number 2.

For simplicity, we henceforth refer to a (γ, k)-critical graph with domination number l as an (l, k)-critical graph. We first discuss the case where $l \geq 3$. For $l = 3$, Mojdeh, Firoozi and Hasni [13] have shown that there are infinitely many connected $(3, k)$-critical graphs for each odd $k \geq 3$. For $k = 2$, Brigham, Haynes, Henning and Rall [5] and Chen, Fujita, Furuya and Young [8] have shown that there are infinitely many connected $(l, 2)$-critical graphs for each $l \geq 4$. However, for $k \geq 3$ and $l \geq 4$, no result concerning the construction of an infinite family of connected (l, k)-critical graphs has been published (even though, as we shall mention again in Section 6, the existence of such a family seems to be known when k is odd and l is even). In Section 2, we show that there exists such a family of graphs. Before stating the result, we need another definition. The maximum distance between two vertices in a graph G is called the *diameter* of G and denoted by $\mathrm{diam}(G)$. Our result is as follows.

Theorem 1.1 *Let $k \geq 1$ and $l \geq 3$ be integers. Then there exist infinitely many connected graphs G such that G is (l, h)-critical for every h with $1 \leq h \leq k$ and such that* $\mathrm{diam}(G) = 3(l-1)/2$ *or* $3(l-2)/2$ *according as l is odd or even.*

In [5], Brigham, Haynes, Henning and Rall constructed, for each $l \geq 3$, a connected $(l, 2)$-critical graph with $\mathrm{diam}(G) = l - 1$, and asked whether every connected $(l, 2)$-critical graph G must satisfy $\mathrm{diam}(G) \leq l - 1$. Note that Theorem 1.1 shows that the answer is no when $l = 3$ or $l \geq 5$.

We introduce here the following operation of concatenating vertex-disjoint graphs, which is used in [5]. Let H_1 and H_2 be two vertex-disjoint graphs, and let $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. Under this notation, we let $(H_1 \bullet H_2)(x_1, x_2)$ denote the graph obtained from H_1 and H_2 by identifying vertices x_1 and x_2. We call $(H_1 \bullet H_2)(x_1, x_2)$ the *coalescence of H_1 and H_2 via x_1 and x_2*. We write $H_1 \bullet H_2$ for $(H_1 \bullet H_2)(x_1, x_2)$ when there is no need to refer to x_1 or x_2. In proving Theorem 1.1, we make use of the following theorem proved by Mojdeh, Firoozi and Hasni in [13].

Theorem A ([13]) *Let $k \geq 1$ be an integer. Let H_1 and H_2 be vertex-disjoint graphs and let x_i be a non-isolated vertex of H_i for each $i = 1, 2$, and let $G = (H_1 \bullet H_2)(x_1, x_2)$. Then G is (γ, j)-critical for every $1 \leq j \leq k$ if and only if both H_1*

and H_2 are (γ, j)-critical for every $1 \le j \le k$. Furthermore, if G is (γ, j)-critical for every $1 \le j \le k$, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

In Section 3, we prove a refinement of Theorem A.

We now consider (l, k)-critical graphs with $1 \le l \le 2$. By definition, a graph G is $(1, k)$-critical if and only if G satisfies $|V(G)| \le k$ and $\Delta(G) = |V(G)| - 1$. Note that if a graph G with $\gamma(G) = 2$ satisfies $|V(G)| \le k+1$, then G is clearly $(2, k)$-critical. Thus we focus on $(2, k)$-critical graphs G with $|V(G)| \ge k+2$. An independent set M of edges of a graph G such that every vertex is incident with an edge in M is called a *perfect matching* of G. For $k \in \{1, 2, 3\}$, the following characterization of $(2, k)$-critical graphs is known.

Proposition B ([4, 5, 12]) *Let $k \in \{1, 2, 3\}$, and let G be a graph having order $n \ge k+2$. Then G is $(2, k)$-critical if and only if $k \in \{1, 3\}$, n is even, and $G \simeq K_n - M$, where M is a perfect matching of K_n.*

In Section 4, we extend Proposition B to $(2, k)$-critical graphs with $k \ge 1$, and prove the following proposition.

Proposition 1.2 *Let $k \ge 1$ be an integer, and let G be a graph having order $n \ge k+2$. Then G is $(2, k)$-critical if and only if k is odd, n is even, and $G \simeq K_n - M$, where M is a perfect matching of K_n.*

We add that it was proved in [5] that every r-regular $(\gamma, 2)$-critical graph with $\gamma(G) \ge 2$ satisfies $|V(G)| \le (r+1)(\gamma(G) - 1) + 1$ and, for $k \ge 3$, it was asked in [13] whether the statement that every connected r-regular (γ, k)-critical graph with $\gamma(G) \ge 2$ satisfies $|V(G)| \le (r+1)(\gamma(G) - 1) + k - 1$ is true. In Section 5, we show that the statement is true even if we drop the condition that G is connected.

In passing, we mention that (γ, k)-critical graphs have been studied from other points of view as well (see, for example, [2, 3, 8]). Further, various other types of criticality for the domination number of a graph have been studied ([1, 6, 14, 16]), and variants of the notion of domination have also been studied ([11, 15]).

2 A family of (γ, k)-critical graphs

In this section, we prove Theorem 1.1

We first construct $(3, k)$-critical graphs. For $i = 1, 2$, let $m_i \ge 1$ be an integer, and let $C^{(i)} = x_1^{(i)} x_2^{(i)} \dots x_{5m_i}^{(i)} x_1^{(i)}$ be a cycle of order $5m_i$. For each $1 \le j \le 5$, let $X_j^{(1)} = \{x_l^{(1)} \mid l \equiv j \,(\mathrm{mod}\ 5)\}$ and $X_j^{(2)} = \{x_l^{(2)} \mid l \equiv 2j - 1 \,(\mathrm{mod}\ 5)\}$. Note that $X_1^{(2)} = \{x_l^{(2)} \mid l \equiv 1 \,(\mathrm{mod}\ 5)\}$, $X_2^{(2)} = \{x_l^{(2)} \mid l \equiv 3 \,(\mathrm{mod}\ 5)\}$, $X_3^{(2)} = \{x_l^{(2)} \mid l \equiv 5 \,(\mathrm{mod}\ 5)\}$, $X_4^{(2)} = \{x_l^{(2)} \mid l \equiv 2 \,(\mathrm{mod}\ 5)\}$ and $X_5^{(2)} = \{x_l^{(2)} \mid l \equiv 4 \,(\mathrm{mod}\ 5)\}$. Let $Y = \{y_1, \dots, y_5\}$ and $Z = \{z_1, z_2\}$. Let $E_1 = \{xx' \mid x \in X_j^{(1)}, x' \in X_{j'}^{(2)}, j \ne j'\}$,

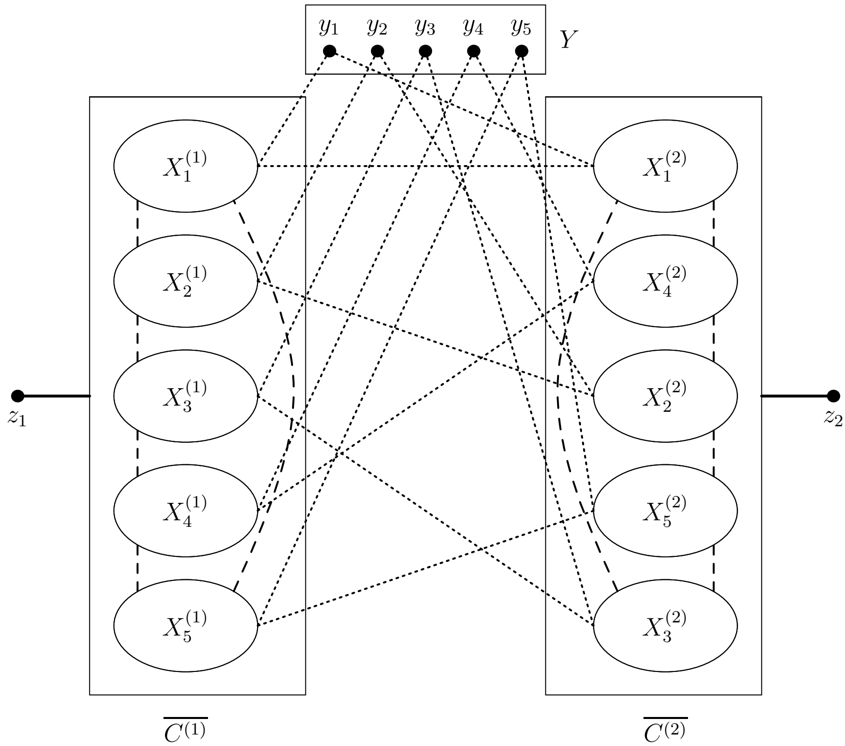

Figure 1: Graph G_{m_1,m_2}

$E_2 = \bigcup_{1\le j\le 5}\{y_jx \mid x \in (V(C^{(1)}) \cup V(C^{(2)})) - (X_j^{(1)} \cup X_j^{(2)})\}$, $E_3 = \bigcup_{i=1,2}\{z_ix \mid x \in V(C^{(i)})\}$. Let G_{m_1,m_2} be the graph defined by

$$V(G_{m_1,m_2}) = V(C^{(1)}) \cup V(C^{(2)}) \cup Y \cup Z$$

and

$$E(G_{m_1,m_2}) = E(\overline{C^{(1)}}) \cup E(\overline{C^{(2)}}) \cup \Big(\bigcup_{1\le j\le 3} E_j\Big).$$

The graph G_{m_1,m_2} is depicted in Figure 1. In the figure, two solid lines indicate that for each $i = 1,2$, all edges between z_i and $\overline{C^{(i)}}$ are present; dotted lines indicate that no edge between the two sets (or the vertex and the set) joined by a dotted line is present, and all other edges between $\overline{C^{(1)}}$ and $\overline{C^{(2)}}$, Y and $\overline{C^{(1)}}$, and Y and $\overline{C^{(2)}}$ are present; dashed lines indicate that for each $i = 1,2$, all edges inside $\overline{C^{(i)}}$ are present except for a perfect matching between the two sets joined by a dashed line.

Proposition 2.1 *Let $k, m_1, m_2 \ge 1$ be integers such that $m_1, m_2 \ge k$. Then G_{m_1,m_2} is $(3,h)$-critical for every $1 \le h \le k$.*

Proof. First we prove that $\gamma(G_{m_1,m_2}) = 3$. Since $\{x_1^{(1)}, x_1^{(2)}, y_1\}$ is a dominating set of G_{m_1,m_2}, it follows that $\gamma(G_{m_1,m_2}) \leq 3$. Suppose that $\gamma(G_{m_1,m_2}) \leq 2$, and let S be a γ-set of G_{m_1,m_2}. Note that for each $x \in V(C^{(1)}) \cup V(C^{(2)})$, there exist two vertices in $Y \cup Z$ which are not dominated by x. Since $Y \cup Z$ is independent, it follows that $S \cap Z = \emptyset$. Since S dominates Z, this implies $|S \cap V(C^{(1)})| = |S \cap V(C^{(2)})| = 1$. Write $S \cap V(C^{(1)}) = \{x_{j_1}^{(1)}\}$ and $S \cap V(C^{(2)}) = \{x_{j_2}^{(2)}\}$. If $j_2 \equiv j_1 - 1 \pmod 5$ or $j_2 \equiv j_1 + 1 \pmod 5$, then one of the vertices in $N_{C^{(1)}}(x_{j_1}^{(1)})$ is not dominated by S; if $j_2 \equiv j_1 - 2 \pmod 5$ or $j_2 \equiv j_1 + 2 \pmod 5$, then one of the vertices in $N_{C^{(2)}}(x_{j_2}^{(2)})$ is not dominated by S; if $j_2 \equiv j_1 \pmod 5$, then y_j is not dominated by S, where j is the unique integer such that $j \equiv j_1 \pmod 5$ and $1 \leq j \leq 5$. Consequently S is not a dominating set of G_{m_1,m_2}, which is a contradiction. Thus $\gamma(G_{m_1,m_2}) = 3$.

Next we prove that $\gamma(G_{m_1,m_2} - U) \leq 2$ for any $U \subseteq V(G_{m_1,m_2})$ with $1 \leq |U| \leq k$. Let U be a subset of $V(G_{m_1,m_2})$ with $1 \leq |U| \leq k$.

Case 1: $U \cap Z \neq \emptyset$.

Let $i \in \{1, 2\}$ be an integer such that $z_i \in U$. Since $|V(C^{(3-i)})|/2 > |U \cap V(C^{(3-i)})|$, there exist two vertices $x, x' \in V(C^{(3-i)}) - U$ such that $xx' \in E(C^{(3-i)})$. Since $\{x, x'\}$ dominates $V(G_{m_1,m_2}) - \{z_i\}$, $\{x, x'\}$ is a dominating set of $G_{m_1,m_2} - U$. Hence $\gamma(G_{m_1,m_2} - U) \leq 2$.

Case 2: $U \cap Z = \emptyset$ and $U \cap (V(C^{(1)}) \cup V(C^{(2)})) \neq \emptyset$.

Let $i \in \{1, 2\}$ be an integer such that $U \cap V(C^{(i)}) \neq \emptyset$. Since $|V(C^{(i)})| > |U \cap V(C^{(i)})|$, there exist two vertices $x, x' \in V(C^{(i)})$ such that $xx' \in E(C^{(i)})$, $x \in U$ and $x' \notin U$. Let $j (1 \leq j \leq 5)$ be the integer such that $x \in X_j^{(i)}$. Since $|X_j^{(3-i)}| \geq k > |U \cap V(C^{(3-i)})|$, we have $X_j^{(3-i)} - U \neq \emptyset$. Let $x'' \in X_j^{(3-i)} - U$. Since $\{x', x''\}$ dominates $V(G_{m_1,m_2}) - \{x\}$, $\{x', x''\}$ is a dominating set of $G_{m_1,m_2} - U$. Hence $\gamma(G_{m_1,m_2} - U) \leq 2$.

Case 3: $U \cap (Z \cup V(C^{(1)}) \cup V(C^{(2)})) = \emptyset$.

Let $j (1 \leq j \leq 5)$ be an integer such that $y_j \in U$. Note that $U \cap (X_j^{(1)} \cup X_j^{(2)}) = \emptyset$. Take $x \in X_j^{(1)}$ and $x' \in X_j^{(2)}$. Since $\{x, x'\}$ dominates $V(G_{m_1,m_2}) - \{y_j\}$, $\{x, x'\}$ is a dominating set of $G_{m_1,m_2} - U$. Hence $\gamma(G_{m_1,m_2} - U) \leq 2$.

Therefore G_{m_1,m_2} is $(3, h)$-critical for every $1 \leq h \leq k$. □

Let $m_1, m_2 \geq 2$ be integers, and let $X_j^{(i)}$, Y, Z and G_{m_1,m_2} be as above. We construct a new graph $G^*_{m_1,m_2}$, which is $(4, k)$-critical, by adding some vertices and edges to $G_{m_1,m_2} - Y$. For a dominating set P of $G_{m_1,m_2} - Y$ with $P \subseteq V(C^{(1)}) \cup V(C^{(2)})$ and $|P| = 3$, we prepare a new vertex v_P and join v_P to all vertices in $(V(C^{(1)}) \cup V(C^{(2)})) - P$ with edges. Apply this operation to every dominating set P of $G_{m_1,m_2} - Y$ such that $P \subseteq V(C^{(1)}) \cup V(C^{(2)})$ and $|P| = 3$, and let $G^*_{m_1,m_2}$ denote the resulting graph. Let Y^* be the set of new vertices (i.e., $Y^* = V(G^*_{m_1,m_2}) - V(G_{m_1,m_2} - Y)$).

Lemma 2.2 *Let Q be a subset of $V(C^{(1)}) \cup V(C^{(2)})$ with $|Q| \leq 2$. Then there exist at least four vertices of Y^* which are not dominated by Q in $G^*_{m_1,m_2}$.*

Proof. We may assume $|Q| = 2$. Set $A = \{a \in (V(C^{(1)}) \cup V(C^{(2)})) - Q \mid Q \cup \{a\}$ dominates $V(C^{(1)}) \cup V(C^{(2)}) \cup Z\}$. For each $a \in A$, it follows from the definition of $G^*_{m_1,m_2}$ that $v_{Q\cup\{a\}}$ is not dominated by Q. Thus it suffices to show that $|A| \geq 4$. Write $Q = \{x, x'\}$, and let $x \in X^{(i)}_j$ and $x' \in X^{(i')}_{j'}$. If $i \neq i'$ and $j = j'$, then Q dominates $V(C^{(1)}) \cup V(C^{(2)}) \cup Z$, and hence $|A| = |(V(C^{(1)}) \cup V(C^{(2)})) - \{x, x'\}| \geq 4$. If $j \neq j'$, then $|A| \geq |X^{(3-i)}_j \cup X^{(3-i')}_{j'}| = m_{3-i} + m_{3-i'} \geq 4$. Thus we may assume $i = i'$ and $j = j'$. Let $j_1, j_2 (1 \leq j_1, j_2 \leq 5)$ be the integers which satisfy $j_1 \equiv j + i \pmod 5$ and $j_2 \equiv j - i \pmod 5$. Then we get $|A| \geq |X^{(3-i)}_{j_1} \cup X^{(3-i)}_{j_2}| \geq 4$, as desired. □

Proposition 2.3 *Let $k, m_1, m_2 \geq 1$ be integers such that $m_1, m_2 \geq k + 1$. Then $G^*_{m_1,m_2}$ is $(4, h)$-critical for every $1 \leq h \leq k$.*

Proof. First we prove that $\gamma(G^*_{m_1,m_2}) = 4$. Since $\{x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2\}$ is a dominating set of $G^*_{m_1,m_2}$, $\gamma(G^*_{m_1,m_2}) \leq 4$. We let S be a γ-set of $G^*_{m_1,m_2}$, and show that $|S| \geq 4$. We may assume that $|S \cap (V(C^{(1)}) \cup V(C^{(2)}))| \leq 3$. Since $Y^* \cup Z$ is independent, all vertices in Y^* which are not dominated by $S \cap (V(C^{(1)}) \cup V(C^{(2)}))$ belong to S. Hence if $|S \cap (V(C^{(1)}) \cup V(C^{(2)}))| \leq 2$, then it follows from Lemma 2.2 that $|S| \geq |S \cap Y^*| \geq 4$. Consequently we may assume $|S \cap (V(C^{(1)}) \cup V(C^{(2)}))| = 3$. Also we may assume that $S \cap (V(C^{(1)}) \cup V(C^{(2)}))$ dominates $V(C^{(1)}) \cup V(C^{(2)}) \cup Z$. Then $|S| \geq |(S \cap (V(C^{(1)}) \cup V(C^{(2)}))) \cup \{v_{S\cap(V(C^{(1)})\cup V(C^{(2)}))}\}| = 4$. Thus $\gamma(G^*_{m_1,m_2}) = 4$.

Next we prove that $\gamma(G^*_{m_1,m_2} - U) \leq 3$ for any $U \subseteq V(G^*_{m_1,m_2})$ with $1 \leq |U| \leq k$. Let U be a subset of $V(G^*_{m_1,m_2})$ with $1 \leq |U| \leq k$.

Case 1: $U \cap Z \neq \emptyset$.

Let $i \in \{1, 2\}$ be an integer such that $z_i \in U$. Since $|V(C^{(3-i)})|/3 > |U \cap V(C^{(3-i)})|$, there exist three vertices $x, x', x'' \in V(C^{(3-i)}) - U$ such that $xx', x'x'' \in E(C^{(3-i)})$. Then $\{x, x', x''\}$ dominates $V(C^{(1)}) \cup V(C^{(2)}) \cup (Z - \{z_i\})$. Since $\{x, x', x''\}$ does not dominate z_i, it follows from the definition of $G^*_{m_1,m_2}$ that $\{x, x', x''\}$ also dominates Y^*. Hence $\{x, x', x''\}$ is a dominating set of $G^*_{m_1,m_2} - U$. Consequently $\gamma(G^*_{m_1,m_2} - U) \leq 3$.

Case 2: $U \cap Z = \emptyset$ and $U \cap (V(C^{(1)}) \cup V(C^{(2)})) \neq \emptyset$.

Let $i \in \{1, 2\}$ be an integer such that $U \cap V(C^{(i)}) \neq \emptyset$. Since $|V(C^{(i)})| > |U \cap V(C^{(i)})|$, there exist two vertices $x, x' \in V(C^{(i)})$ such that $xx' \in E(C^{(i)})$, $x \in U$ and $x' \notin U$. Let $j (1 \leq j \leq 5)$ be the integer such that $x \in X^{(i)}_j$. Since $|X^{(3-i)}_j| - 2 \geq k - 1 \geq |U \cap V(C^{(3-i)})|$, we have $|X^{(3-i)}_j - U| \geq 2$. Let $u, u' \in X^{(3-i)}_j - U$. Then $\{x', u, u'\}$ dominates $((V(C^{(1)}) \cup V(C^{(2)})) - \{x\}) \cup Z$, and $\{x', u, u'\}$ also dominates Y^* because it does not dominate x. Hence $\{x', u, u'\}$ is a dominating set of $G^*_{m_1,m_2} - U$. Consequently $\gamma(G^*_{m_1,m_2} - U) \leq 3$.

Case 3: $U \cap (Z \cup V(C^{(1)}) \cup V(C^{(2)})) = \emptyset$.

In this case, we have $U \subseteq Y^*$. Take $v \in U$, and write $v = v_P$, where P is a subset of $V(C^{(1)}) \cup V(C^{(2)})$ with $|P| = 3$ such that P dominates $V(C^{(1)}) \cup V(C^{(2)}) \cup Z$. Then P dominates $Y^* - \{v_P\}$, and hence P is a dominating set of $G^*_{m_1,m_2} - U$. Consequently $\gamma(G^*_{m_1,m_2} - U) \leq 3$.

Therefore $G^*_{m_1,m_2}$ is $(4,h)$-critical for every $1 \leq h \leq k$. □

Note that for each $m_1 \geq 2$ and each $m_2 \geq 2$, we have

$$\mathrm{diam}(G_{m_1,m_2}) = \mathrm{diam}(G^*_{m_1,m_2}) = 3.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let k, l be as in Theorem 1.1. If k is odd, then let G_1 be a graph with diameter 3 which is $(3,h)$-critical for every $1 \leq h \leq k$; if k is even, then let G_1 be a graph with diameter 3 which is $(4,h)$-critical for every $1 \leq h \leq k$. Also let $d = 3(l-1)/2$ and $m = (l-1)/2$ or $d = 3(l-2)/2$ and $m = (l-2)/2$ according as l is odd or even. For each $2 \leq i \leq m$, let G_i be a graph with diameter 3 which is $(3,h)$-critical for every $1 \leq h \leq k$. For each $1 \leq i \leq m$, let z_i, $z_{i'}$ be vertices of G_i which are at dstance three apart. Let G be the graph obtained by concatenating $G_1, \ldots, G_m$ by letting G_{i-1} and G_i coalesce via z'_{i-1} and z_i for each $2 \leq i \leq m$. Then $\mathrm{diam}(G) = \sum_{1 \leq i \leq m} \mathrm{diam}(G_i) = d$. Further by Theorem A, G is (γ,h)-critical for every $1 \leq h \leq k$, and $\gamma(G) = \gamma(G_1) + \sum_{2 \leq i \leq m}(\gamma(G_i) - 1) = l$. Since Propositions 2.1 and 2.3 show that there are infinitely many candidates for G_i for each i, this yields the desired conclusion. □

3 Coalescence of two graphs

In this section, we prove a theorem about coalescence. We start with the following corollary of Theorem A.

Proposition 3.1 ([5]) *Let H_1 and H_2 be vertex-disjoint graphs, and let x_i be a non-isolated vertex of H_i for each $i = 1, 2$. Then $(H_1 \bullet H_2)(x_1, x_2)$ is $(\gamma,1)$-critical and $(\gamma,2)$-critical if and only if both H_1 and H_2 are $(\gamma,1)$-critical and $(\gamma,2)$-critical.*

Note that Proposition 3.1 does not give a necessary and sufficient condition for $H_1 \bullet H_2$ to be $(\gamma,2)$-critical. We prove the following modification of Proposition 3.1.

Proposition 3.2 *Let H_i, x_i be as in Proposition 3.1. Then $(H_1 \bullet H_2)(x_1, x_2)$ is $(\gamma,2)$-critical if and only if*

(i) *both H_1 and H_2 are $(\gamma,2)$-critical, and*

(ii) *for some $i \in \{1,2\}$, H_i is critical and $\gamma(H_{3-i} - x_{3-i}) < \gamma(H_{3-i})$.*

Actually we prove the following more general result.

Theorem 3.3 *Let $k \geq 1$ be an integer. Let H_1 and H_2 be vertex-disjoint graphs and let x_i be a vertex of H_i with $d_{H_i}(x_i) \geq k-1$ for each $i = 1, 2$, and let $G = (H_1 \bullet H_2)(x_1, x_2)$. Then G is (γ, k)-critical if and only if*

(i) *for any nonnegative integers k_1, k_2 with $k_1 + k_2 = k$, H_1 is (γ, k_1)-critical or H_2 is (γ, k_2)-critical, and*

(ii) *for each $i = 1, 2$, $\gamma(H_i - (U \cup \{x_i\})) < \gamma(H_i)$ for every $U \subseteq V(H_i - x_i)$ with $|U| \leq k-1$.*

Further if G is (γ, k)-critical, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

We observe that from condition (i) in Theorem 3.3, it follows that H_1 and H_2 are (γ, k)-critical (because no graph is $(\gamma, 0)$-critical). Thus Theorem 3.3 implies Theorem A (to deduce the "only if" part of Theorem A, note that under the notation of Theorem A, if G is (γ, j)-critical for every $1 \leq j \leq k$, then by induction on j, we see from Theorem 3.3 that each H_i is (γ, j)-critical and satisfies $d_{H_i}(x_i) \geq j+1$, because in general, a (γ, j)-critical graph cannot have a vertex with degree precisely equal to j).

We prove Theorem 3.3 using the following two lemmas (it is likely that the second lemma is also already known, but we include its proof for the convenience of the reader).

Lemma 3.4 ([5]) *For any vertex-disjoint graphs H_1 and H_2, we have $\gamma(H_1) + \gamma(H_2) - 1 \leq \gamma(H_1 \bullet H_2) \leq \gamma(H_1) + \gamma(H_2)$.*

Lemma 3.5 *Let H_1 and H_2 be graphs. Let $x_i \in V(H_i)$ for each $i = 1, 2$, and suppose that x_1 is a critical vertex of H_1 or x_2 is a critical vertex of H_2. Then $\gamma((H_1 \bullet H_2)(x_1, x_2)) = \gamma(H_1) + \gamma(H_2) - 1$.*

Proof. Let $G = (H_1 \bullet H_2)(x_1, x_2)$. In view of Lemma 3.4, it suffices to show that $\gamma(G) \leq \gamma(H_1) + \gamma(H_2) - 1$. Without loss of generality, we may assume that x_1 is a critical vertex of H_1. Let S_1 and S_2 be γ-sets of $H_1 - x$ and H_2, respectively. Then $|S_1| \leq \gamma(H_1) - 1$ and $|S_2| = \gamma(H_2)$. If $x_2 \notin S_2$, then let $S = S_1 \cup S_2$; if $x_2 \in S_2$, then let $S = S_1 \cup (S_2 - \{x_2\}) \cup \{x\}$. Then S is a dominating set of G, and $|S| \leq (\gamma(H_1) - 1) + \gamma(H_2)$. Hence $\gamma(G) \leq \gamma(H_1) + \gamma(H_2) - 1$. □

Proof of Theorem 3.3. Let k, H_i, x_i, G be as in Theorem 3.3, and x be denote the vertex in G arising from the identification of x_1 and x_2. First we assume that G is (γ, k)-critical, and show that $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$, and (i) and (ii) hold.

Claim 3.1 *The vertex x is critical in G.*

Proof. For each $i = 1, 2$, we can choose a subset U_i of $N_{H_i}(x_i)$ with $|U_i| = k-1$ because $d_{H_i}(x_i) \geq k-1$. Since G is (γ, k)-critical, we have

$$\gamma(H_i - (U_i \cup \{x_i\})) + \gamma(H_{3-i} - x_{3-i}) = \gamma(G - (U_i \cup \{x\})) \leq \gamma(G) - 1 \qquad (3.1)$$

for each i. Let S be a γ-set of $G-(U_1 \cup U_2 \cup \{x\})$. Then it follows from (3.1) that

$$\begin{aligned}
|S| &= \gamma(G-(U_1 \cup U_2 \cup \{x\})) \\
&= \sum_{i=1,2} \gamma(H_i - (U_i \cup \{x_i\})) \\
&\leq \sum_{i=1,2} ((\gamma(G)-1) - \gamma(H_{3-i} - x_{3-i})) \\
&= 2\gamma(G) - (\gamma(H_1 - x_1) + \gamma(H_2 - x_2)) - 2 \\
&= 2\gamma(G) - \gamma(G-x) - 2.
\end{aligned}$$

On the other hand, $S \cup \{x\}$ is a dominating set of G, and hence $|S| \geq \gamma(G)-1$. Consequently $2\gamma(G)-\gamma(G-x)-2 \geq \gamma(G)-1$, which means that $\gamma(G)-\gamma(G-x)-1 \geq 0$. Hence x is a critical vertex of G. □

Claim 3.2 *For each $i=1,2$, x_i is a critical vertex of H_i.*

Proof. Recall that removing a vertex can decrease the domination number at most by one. Let S be a γ-set of $G-x$. By Claim 3.1,

$$|S| = \gamma(G) - 1. \tag{3.2}$$

Hence $|S| \leq \gamma(H_1)+\gamma(H_2)-1$ by Lemma 3.4. Since $S \cap V(H_i)$ is a dominating set of $H_i - x_i$ for each $i=1,2$, this implies that we have $|S \cap V(H_i)| = \gamma(H_i)-1$ for $i=1$ or $i=2$. Without loss of generality, we may assume that $|S \cap V(H_1)| = \gamma(H_1)-1$, which means that x_1 is a critical vertex of H_1. Then $\gamma(G)=\gamma(H_1)+\gamma(H_2)-1$ by Lemma 3.5, and hence it follows from (3.2) that $|S| \leq \gamma(H_1)+\gamma(H_2)-2$. Since $|S \cap V(H_1)| = \gamma(H_1)-1$ and $S \cap V(H_2)$ is a dominating set of $H_2 - x_2$, this forces $|S \cap V(H_2)| = \gamma(H_2)-1$. Consequently x_2 is a critical vertex of H_2. □

Note that $\gamma(G)=\gamma(H_1)+\gamma(H_2)-1$ by Claim 3.2 and Lemma 3.5. We now prove that (ii) holds. By symmetry, it suffices to consider only the case where $i=1$. Let U be a subset of $V(H_1)-\{x_1\}$ with $|U| \leq k-1$. We show that $\gamma(H_1-(U \cup \{x_1\})) < \gamma(H_1)$. Let U' be a subset of $N_{H_2}(x_2)$ with $|U'| = k-1-|U|$. Let S be a γ-set of $G-(U \cup U' \cup \{x\})$. Since G is (γ,k)-critical, we get $|S| \leq \gamma(G)-1 = \gamma(H_1)+\gamma(H_2)-2$. Since $(S \cap V(H_2)) \cup \{x_2\}$ is a dominating set of H_2, $|S \cap V(H_2)| \geq \gamma(H_2)-1$. Consequently $|S \cap V(H_1)| \leq \gamma(H_1)-1$. Since $S \cap V(H_1)$ is a dominating set of $H_1-(U \cup \{x_1\})$, $\gamma(H_1-(U \cup \{x_1\})) < \gamma(H_1)$, as desired.

In order to prove (i), let k_1, k_2 be nonnegative integers such that $k_1+k_2=k$ and, by way of contradiction, suppose that H_1 is not (γ,k_1)-critical and H_2 is not (γ,k_2)-critical. For each i, let U_i be a subset of $V(H_i)$ with $|U_i|=k_i$ such that $\gamma(H_i-U_i) \geq \gamma(H_i)$. In view of (ii), we have $x_i \notin U_i$ for each i, which means that $G-(U_1 \cup U_2) = ((H_1-U_1) \bullet (H_2-U_2))(x_1,x_2)$. Consequently by Lemma 3.4, $\gamma(G-(U_1 \cup U_2)) \geq \gamma(H_1-U_1)+\gamma(H_2-U_2)-1 \geq \gamma(H_1)+\gamma(H_2)-1 = \gamma(G)$. This contradicts the assumption that G is (γ,k)-critical. Therefore (i) holds.

Next, conversely, we assume that (i) and (ii) hold, and show that G is (γ, k)-critical. Let U be a subset of $V(G)$ with $|U| = k$. For each $i = 1, 2$, let $U_i = U \cap (V(H_i) - \{x_i\})$ and $k_i = |U_i|$.

Case 1: $x \in U$.

For each $i = 1, 2$, let S_i be a γ-set of $H_i - (U_i \cup \{x_i\})$. Let $S = S_1 \cup S_2$. Then S is a dominating set of $G - U$. By (ii), $|S_i| \leq \gamma(H_i) - 1$ for each $i = 1, 2$, and hence $|S| \leq \gamma(H_1) + \gamma(H_2) - 2$. In view of Lemma 3.4, this implies $|S| \leq \gamma(G) - 1$. Consequently $\gamma(G - U) < \gamma(G)$.

Case 2: $x \notin U$.

By (i), without loss of generality, we may assume that H_1 is (γ, k_1)-critical. Then $k_1 \neq 0$ and $\gamma(H_1 - U_1) \leq \gamma(H_1) - 1$. Since $|U_2| = k - k_1 \leq k - 1$, it follows from (ii) that $\gamma(H_2 - (U_2 \cup \{x_2\})) \leq \gamma(H_2) - 1$. Let S_1 and S_2 be γ-sets of $H_1 - U_1$ and $H_2 - (U_2 \cup \{x_2\})$, respectively. If $x_1 \notin S_1$, then let $S = S_1 \cup S_2$; if $x_1 \in S_1$, then let $S = (S_1 - \{x_1\}) \cup \{x\} \cup S_2$. Then S is a dominating set of $G - U$. We also have $|S| = |S_1| + |S_2| \leq \gamma(H_1) + \gamma(H_2) - 2$, and hence $|S| \leq \gamma(G) - 1$ by Lemma 3.4. Consequently $\gamma(G - U) < \gamma(G)$.

Therefore G is a (γ, k)-critical graph. This completes the proof of Theorem 3.3. □

4 Characterization of $(2, k)$-critical graphs

In this section, we focus on $(2, k)$-critical graphs, and prove Proposition 1.2. We make use of the following results in our proof.

Lemma 4.1 ([13]) *Let $k \geq 1$ be an odd integer and $n \geq 2$ be an even integer. Then $K_n - M$ is $(2, k)$-critical, where M is a perfect matching of K_n.*

Lemma 4.2 ([7]) *If a graph G is not critical, then G has a vertex x such that $\gamma(G - x) = \gamma(G)$.*

Proof of Proposition 1.2. Let k, n, G be as in Proposition 1.2. We proceed by induction on k. In view of Proposition B, we may assume that $k \geq 4$ and the proposition holds for smaller values of k. The "if" part of the proposition follows from Lemma 4.1. Thus it suffices to show, under the assumption that G is $(2, k)$-critical, that k is odd, n is even, and $G \simeq K_n - M$, where M is a perfect matching of K_n.

Suppose that G is not critical. Then by Lemma 4.2, there exists $x \in V(G)$ such that $\gamma(G - x) = 2$. Since G is $(2, k)$-critical, we have $\gamma((G - x) - U) < 2$ for every $U \subseteq V(G - x)$ with $|U| = k - 1$. Hence $G - x$ is 2-$(\gamma, k-1)$-critical. By the induction assumption, this implies that $k-1$ is odd, $n-1$ is even, and $G - x \simeq K_{n-1} - M$, where M is a perfect matching of K_{n-1}. Write $V(G - x) = \{x_1, \ldots, x_{(n-1)/2}, y_1, \ldots, y_{(n-1)/2}\}$ so that $x_i y_i \notin E(G)$ for each i. Since $\gamma(G) = 2$, there exists $y \in V(G - x)$ such that

$xy \notin E(G)$. We may assume $y = y_{(n-1)/2}$. Then $\gamma(G - \{x_1, \ldots, x_{k/2}, y_1, \ldots, y_{k/2}\}) \geq 2$, which contradicts the assumption that G is $(2, k)$-critical.

Thus G is critical. By the induction assumption, this implies that n is even, and $G \simeq K_n - M$, where M is a perfect matching of K_n. Write $V(G) = \{x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}\}$ so that $x_i y_i \notin E(G)$ for each i. If k is even, then $\gamma(G - \{x_1, \ldots, x_{k/2}, y_1, \ldots, y_{k/2}\}) = 2$, which contradicts the assumption that G is $(2, k)$-critical. Consequently k is odd, which completes the proof of Proposition 1.2. □

5 Regular (γ, k)-critical graphs

In this section, we consider regular (γ, k)-critical graphs. We first state a lemma.

Lemma 5.1 ((i)[10], (ii)[13])

(i) *If G is a critical graph having order $(\Delta(G)+1)(\gamma(G)-1)+1$, then G is regular.*

(ii) *Let $k \geq 1$ be an integer, and let G be a (γ, k)-critical graph. Then $|V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + k$.*

The following proposition shows that the answer to the question in [13] mentioned toward at the end of Section 1 is affirmative.

Proposition 5.2 *Let $k \geq 2$ be an integer, and let G be an r-regular (γ, k)-critical graph with $\gamma(G) \geq 2$. Then $|V(G)| \leq (r + 1)(\gamma(G) - 1) + k - 1$.*

Proof. Let $n = |V(G)|$. Suppose that $n \geq (r + 1)(\gamma(G) - 1) + k$. Then, applying Lemma 5.1(ii) with k replaced by $k-1$, we see that G is not $(\gamma, k-1)$-critical. Hence there exists $U \subseteq V(G)$ with $|U| = k-1$ such that $\gamma(G-U) \geq \gamma(G)$. Let $H = G-U$. Take $x \in V(H)$. Since G is (γ, k)-critical, $\gamma(H - x) = \gamma(G - (U \cup \{x\})) < \gamma(G) \leq \gamma(H)$. Since $\gamma(H - x) \geq \gamma(H) - 1$, this forces $\gamma(H) = \gamma(G)$. Since x is arbitrary, we also see that H is critical. By Lemma 5.1(ii), this implies that

$$n - (k - 1) = |V(H)| \leq (\Delta(H) + 1)(\gamma(H) - 1) + 1 \leq (r + 1)(\gamma(G) - 1) + 1.$$

Since $\gamma(H) = \gamma(G) \geq 2$, this, together with the assumption that $n \geq (r+1)(\gamma(G) - 1) + k$, implies that $|V(H)| = (\Delta(H) + 1)(\gamma(H) - 1) + 1$ and $\Delta(H) = r$. Since H is critical, it follows from Lemma 5.1(i) that H is r-regular. Since G is r-regular, this means that H is the union of some components of G. But then $\gamma(H) < \gamma(G)$, which contradicts the earlier assertion that $\gamma(H) = \gamma(G)$. Thus $n \leq (r+1)(\gamma(G)-1)+k-1$. □

6 Conclusion

In Theorem 1.1, we have shown that there exist infinitely many (l, k)-critical graphs for each $k \geq 1$ and each $l \geq 3$ and, in Proposition 1.2, we have given a characterization of $(2, k)$-critical graphs for all $k \geq 3$. Our proof of Theorem 1.1 is based on an operation called coalescence and, in Theorem 3.3, we have proved a result concerning coalescence, which we believe will be useful for further research.

We conclude this paper by presenting two open problems related to Theorem 1.1. Theorem 1.1 does not yield a graph which is (γ, k)-critical but not (γ, h)-critical for some h with $1 \leq h \leq k-1$. On the other hand, we see from Lemma 4.1 that when $k \geq 3$ is odd, there exist infinitely many graphs G with $\gamma(G) = 2$ such that G is (γ, h)-critical for every odd h with $1 \leq h \leq k$ but not (γ, k)-critical for any even h with $2 \leq h \leq k-1$ and, as we touched on in the paragraph preceding the statement of Theorem 1.1, a modification of Lemma 4.1 shows that when $k \geq 3$ is odd and $l \geq 2$ is even, there exist infinitely many graphs G with $\gamma(G) = l$ such that G is (γ, h)-critical for every odd h with $1 \leq h \leq k$ but not (γ, h)-critical for any even h with $2 \leq h \leq k-1$. We pose the following question.

Problem 1 *Let k, l be integers with $k \geq 2$ and $l \geq 3$. For which $I \subseteq \{1, \ldots, k\}$, do there exist infinitely many connected graphs G with $\gamma(G) = l$ such that G is (γ, h)-critical for every $h \in I$ but not (γ, h)-critical for any $h \in \{1, \ldots, k\} - I$?*

In view of the question in [5] mentioned in the paragraph following the statement of Theorem 1.1, the following question also naturally arises.

Problem 2 *For $l \geq 3$, what is the best upper bound for the diameter of a connected $(l, 2)$-critical graph?*

Acknowledgments

The author would like to thank the referees for constructive comments.

References

[1] N. Ananchuen and M.D. Plummer, Matching properties in domination critical graphs, *Discrete Math.* **277** (2004), 1–13.

[2] N. Ananchuen and M.D. Plummer, Matchings in 3-vertex-critical graphs: the even case, *Networks* **45** (2005), 210–213.

[3] N. Ananchuen and M.D. Plummer, Matchings in 3-vertex-critical graphs: the odd case, *Discrete Math.* **307** (2007), 1651–1658.

[4] R.C. Brigham, P.Z. Chinn and R.D. Dutton, Vertex domination-critical graphs, *Networks* **18** (1988), 173–179.

[5] R.C. Brigham, T.W. Haynes, M.A. Henning and D.F. Rall, Bicritical domination, *Discrete Math.* **305** (2005), 18–32.

[6] T. Burton and D.P. Sumner, Domination dot-critical graphs, *Discrete Math.* **306** (2006), 11–18.

[7] J.R. Carrington, F. Harary and T.W. Haynes, Changing and unchanging the domination number of a graph, *J. Combin. Math. Combin. Comput.* **9** (1991), 57–63.

[8] X. Chen, S. Fujita, M. Furuya and M.Y. Sohn, Constructing connected bicritical graphs with edge-connectivity 2, *Discrete Appl. Math.* to appear.

[9] R. Diestel, *"Graph Theory" (4th edition)*, Graduate Texts in Mathematics **173**, Springer (2010).

[10] J. Fulman, D. Hanson and G. MacGillivray, Vertex domination-critical graphs, *Networks* **25** (1995), 41–43.

[11] X. Hou and M. Edwards, Paired domination vertex critical graphs, *Graphs Combin.* **24** (2008), 453–459.

[12] D.A. Mojdeh and P. Firoozi, Characteristics of $(\gamma, 3)$-critical graphs, *Appl. Anal. Discrete Math.* **4** (2010), 197–206.

[13] D.A. Mojdeh, P. Firoozi and R. Hasni, On connected (γ, k)-critical graphs, *Australas. J. Combin.* **46** (2010), 25–35.

[14] J.B. Phillips, T.W. Haynes and P.J. Slater, A generalization of domination critical graphs, *Util. Math.* **58** (2000), 129–144.

[15] N.J. Rad, Bicritical total domination, *Australas. J. Combin.* **47** (2010), 217–226.

[16] D.P. Sumner and P. Blitch, Domination critical graphs, *J. Combin. Theory Ser. B* **34** (1983), 65–76.

(Received 19 Apr 2011; revised 5 Nov 2011, 25 Jan 2012)