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Enumeration of area-weighted Dyck paths with restricted height

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Abstract

We derive explicit expressions for q-orthogonal polynomials arising in the enumeration of area-weighted Dyck paths with restricted height.

1 Introduction and Statement of Results

Dyck paths are directed walks on $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the line $y=0$, which have no vertices with negative y-coordinates, and which have steps in the $(1,1)$ and $(1,-1)$ directions [11]. We impose the additional geometrical constraint that the paths have height at most h, that is, they lie between lines $y=0$ and $y=h$. Given a Dyck path π, we define the length $n(\pi)$ to be half the number of its steps, and the area $m(\pi)$ to be the sum of the starting heights of all steps in the $(1,1)$ direction in the path. Actually, $m(\pi)$ is the value of π in the classical lattice of Dyck paths [4, 9], and is equivalent to the number of diamond plaquettes under the Dyck path. An alternative definition of the area is the sum of the heights of all steps in the Dyck path, which evaluates to $n(\pi)+2m(\pi)$ and is equivalent to the number of triangular plaquettes under the Dyck path. The definition used here has the advantage of enabling a more elegant and concise mathematical formulation of our results.

We denote by $u(\pi)$ and $v(\pi)$ the number of vertices in the line $y=0$ (excluding the vertex $(0,0)$) and the number of vertices in the line $y=h$, respectively. Let $\mathcal{D}_h$

be the set of Dyck paths with height at most h, and define the generating function

$$D_h(a,b;q,t) = \sum_{\pi \in \mathcal{D}_h} a^{u(\pi)} b^{v(\pi)} q^{m(\pi)} t^{n(\pi)} \ .$$

The generating function for Dyck paths with restricted height [2, 5] and the generating function for area-weighed Dyck paths [3, 5] have previously been studied. Here we extend these works by combining these properties.

The purpose of this note is to derive the following expression for $D_h(a,b;q,t)$.

Theorem 1. *For $h \geq 0$,*

$$D_h(a,b;q,t) = \frac{\sum\limits_{m=0}^{\infty} (-t)^m q^{m(m-1)} \left((1-b) \begin{bmatrix} h-m \\ m \end{bmatrix}_q + b \begin{bmatrix} h+1-m \\ m \end{bmatrix}_q - (1-b) \begin{bmatrix} h-1-m \\ m-1 \end{bmatrix}_q - b \begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q \right)}{\sum\limits_{m=0}^{\infty} (-t)^m q^{m(m-1)} \left((1-b) \begin{bmatrix} h-m \\ m \end{bmatrix}_q + b \begin{bmatrix} h+1-m \\ m \end{bmatrix}_q - (1-a)(1-b) \begin{bmatrix} h-1-m \\ m-1 \end{bmatrix}_q - (1-a) b \begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q \right)} . \tag{1}$$

Here, we have used the standard notation for q-binomial coefficients [8, 10]; for $n \geq 0$ and $0 \leq k \leq n$ we define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \ , \quad \text{where} \quad (a;q)_n = \prod_{i=0}^{n-1} (1 - a q^i) \ .$$

We extend this definition to integer-valued n, k by letting $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ when $k < 0$ or $k > n$. (This definition is commensurate with the lattice path definition of q-binomial coefficients and is necessary to allow for Theorem 1 to be valid even for $h = 0$.)

For $a = b = 1$, this identity simplifies considerably.

Corollary 2. *For $h \geq 0$,*

$$D_h(1,1;q,t) = \frac{\sum\limits_{m=0}^{\infty} (-t)^m q^{m^2} \begin{bmatrix} h-m \\ m \end{bmatrix}_q}{\sum\limits_{m=0}^{\infty} (-t)^m q^{m(m-1)} \begin{bmatrix} h+1-m \\ m \end{bmatrix}_q} \ .$$

Taking the limit $h \to \infty$, we recover the well-known result [6] that the area-weighted generating function $D(q,t)$ for Dyck paths without height restriction is given by

$$D(q,t) = \frac{\sum\limits_{m=0}^{\infty} \dfrac{(-t)^m q^{m^2}}{(q;q)_m}}{\sum\limits_{m=0}^{\infty} \dfrac{(-t)^m q^{m(m-1)}}{(q;q)_m}} \ .$$

More precisely, $D(q,t)$ as defined here is related to $F(z,q)$ in Eqn. (75) of Chapter V in [6] via $F(z,q) = D(q,qz)$.

2 Proofs

We use as the starting point of our derivation a well-established connection between lattice path enumeration and continued fractions [5].

Proposition 3. *$D_0(a,b;q,t) = b$, $D_1(a,b;q,t) = 1/(1-abt)$, and for $h \geq 2$,*

$$D_h(a,b;q,t) = \cfrac{1}{1-\cfrac{at}{1-\cfrac{qt}{1-\cfrac{q^2t}{1-\cfrac{q^3t}{\ddots \cfrac{q^{h-2}t}{1-bq^{h-1}t}}}}}} . \tag{2}$$

While this can easily be proved by specialising the general theory in [5] to the case at hand, we shall give a direct combinatorial proof.

Proof. The only Dyck path of height zero is the zero step Dyck path. If $h = 0$ then it has weight b, whence $D_0(a,b;q,t) = b$. Let now $h \geq 1$. Except for the zero-step Dyck path with weight 1, every Dyck path of height at most h can be decomposed uniquely into a Dyck path of height at most $(h-1)$ bracketed by a pair of steps into the $(1,1)$ and $(1,-1)$ directions, followed by another Dyck path of height h. The associated generating functions are $atD_{h-1}(1,b;q,qt)$ and $D_h(a,b;q,t)$, respectively. This decomposition leads to the functional-recurrence

$$D_h(a,b;q,t) = 1 + atD_{h-1}(1,b;q,qt)D_h(a,b;q,t) \ .$$

Iterating $D_h(a,b;q,t) = 1/(1-atD_{h-1}(1,b;q,qt))$ gives (2). □

It is clear that the generating function can also be written as a rational function, and from Section 3 in [5] we obtain the following three-term recurrence.

Proposition 4. *For $h \geq 1$,*

$$D_h(a,b;q,t) = \frac{Q_h(0,b;q,t)}{Q_h(a,b;q,t)} \ ,$$

where

$$Q_h(a,b;q,t) = \begin{cases} 1-abt \ , & h=1 \ , \\ 1-at-bqt \ , & h=2 \ , \\ Q_{h-1}(a,1;q,t) - bq^{h-1}tQ_{h-2}(a,1;q,t) & h \geq 3 \ . \end{cases} \tag{3}$$

Proof. The initial conditions follow from $D_1(a,b;q,t) = 1/(1-abt)$ and $D_2(a,b;q,t) = (1-bqt)/(1-at-bqt)$, and the factor $bq^{h-1}t$ in the three-term recurrence is just the final term in the continued fraction (2). More precisely, comparing Eqn. (2) with the h-th convergent of the J-fraction on page 152 of [5], we have $t = z^2$, $a_0 = a$, $a_k = 1$ for $k \geq 1$, $b_k = q^{k-1}$ for $0 \leq k < h$, $b_h = bq^{h-1}$, and $c_k = 0$ for $k \geq 0$. The linear recurrences given on page 152 of [5] then reduce to the recurrence in Eqn. (3). □

We proceed by considering the generating function of the denominators $Q_h(a,b;q,t)$,

$$W(z;a,b;q,t) = \sum_{h=1}^{\infty} Q_h(a,b;q,t) z^h \ .$$

The next proposition expresses $W(z;a,b;q,t)$ in terms of the basic hypergeometric series $\phi(z,q,t) = {}_1\phi_2(q;0,z;q,t)$ [7], i.e.,

$$\phi(z,q,t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} t^n}{(z;q)_n} \ .$$

Proposition 5.

$$W(z;a,b;q,t) = \tfrac{1-at}{tz}(z+b-bz)(1-\phi(z,q,-tz^2)) + (z+b-bz)\phi(z,q,-qtz^2) - b \ . \quad (4)$$

Proof. The recurrence (3) implies that $W(z;a,b;q,t)$ satisfies a functional equation,

$$W(z;a,b;q,t) = (1-abt)z - bqtz^2 + zW(z;a,1;q,t) - z^2 bqtW(qz;a,1;q,t) \ .$$

Letting $b = 1$ and isolating $W(z;a,1,q,t)$ gives

$$W(z;a,1;q,t) = \frac{z}{1-z}(1-at-zqt) - \frac{z^2}{1-z} qtW(qz;a,1;q,t) \ .$$

This functional equation has the structure $W(z) = A(z) + B(z)W(qz)$ which can readily be solved by iteration to give $W(z) = \sum_{n=0}^{\infty} A(q^n z) \prod_{k=0}^{n-1} B(q^k z)$. In this way we find

$$\begin{aligned} W(z;a,1;q,t) =& \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} q^{n^2+n} t^n \, (1-at-zq^{n+1}t)}{(z;q)_{n+1}} \\ =& \frac{1-at}{tz} - 1 - \frac{1-at}{tz}\phi(z,q,-tz^2) + \phi(z,q,-qtz^2) \ . \end{aligned}$$

Inserting this expression into the functional equation gives Eqn. (4). □

Proposition 6. *For $h \geq 1$,*

$$\begin{aligned} Q_h(a,b;q,t) = \sum_{m=0}^{\infty} (-t)^m q^{m(m-1)} \times \Bigg((1-b) \begin{bmatrix} h-m \\ m \end{bmatrix}_q \\ + b \begin{bmatrix} h+1-m \\ m \end{bmatrix}_q - (1-a)(1-b) \begin{bmatrix} h-1-m \\ m-1 \end{bmatrix}_q - (1-a)b \begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q \Bigg) \ . \quad (5) \end{aligned}$$

Proof. We obtain $Q_h(a,b;q,t)$ by extracting the coefficient of z^h in $W(z;a,b;q,t)$. We expand the q-product in the function ϕ with the help of the q-binomial theorem (see page 490 of [1]) to obtain

$$\phi(z,q,tz^2) = 1 + \sum_{m=0}^{\infty} z^m \sum_{n=1}^{\infty} q^{n(n-1)} \begin{bmatrix} m-n-1 \\ n-1 \end{bmatrix}_q t^n \,.$$

Inserting this expansion into (4) and collecting terms with equal powers in z gives Eqn. (5). ☐

Theorem 1 now follows from Propositions 4 and 6 and by checking that the expression gives the correct result also for $h = 0$.

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