Some results on incidence coloring, star arboricity and domination number

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Abstract

Two inequalities are established connecting the graph invariants of inci-
dence chromatic number, star arboricity and domination number. Using
these, upper and lower bounds are deduced for the incidence chromatic
number of a graph and further reductions are made to the upper bound
for a planar graph. It is shown that cubic graphs with orders not divis-
ible by four are not 4-incidence colorable. Sharp upper bounds on the
incidence chromatic numbers are determined for Cartesian products of
graphs, and for joins and unions of graphs.

1   Introduction

An incidence coloring separates the whole graph into disjoint independent incidence
sets. Since incidence coloring was introduced by Brualdi and Massey [4], most re-
search has concentrated on determining the minimum number of independent in-
cidence sets, also known as the incidence chromatic number, which can cover the
graph. The upper bound on the incidence chromatic number of planar graphs [7],
cubic graphs [13] and a lot of other classes of graphs were determined [6, 7, 15, 17, 21].
However, for general graphs, only an asymptotic upper bound is obtained [9]. There-
fore, to find an alternative upper bound and lower bound on the incidence chromatic
number for all graphs is the main objective of this paper.

In Section 2, we will establish a global upper bound for the incidence chromatic
number in terms of chromatic index and star arboricity. This result reduces the
upper bound on the incidence chromatic number of the planar graphs. A global
lower bound which involves the domination number will be introduced in Section 3.

Finally, the incidence chromatic number of graphs constructed from smaller graphs will be determined in Section 4.

All graphs in this paper are connected. Let $V(G)$ and $E(G)$ (or V and E) be the vertex-set and edge-set of a graph G, respectively. Let the set of all neighbors of a vertex u be $N_G(u)$(or simply $N(u)$). Moreover, the degree $d_G(u)$(or simply $d(u)$) of u is equal to $|N_G(u)|$ and the maximum degree of G is denoted by $\Delta(G)$ (or simply Δ). An *edge coloring* of G is a mapping $\sigma : E(G) \to C$, where C is a *color-set*, such that adjacent edges of G are assigned distinct colors. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors required to label all edges of G such that adjacent edges received distinct colors. All notations not defined in this paper can be found in the books [3] and [22].

Let $D(G)$ be a digraph induced from G by splitting each edge $uv \in E(G)$ into two opposite arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$. According to [15], incidence coloring of G is equivalent to the coloring of arcs of $D(G)$, where two distinct arcs $\overrightarrow{uv}$ and $\overrightarrow{xy}$ are *adjacent* provided one of the following holds:

(1) $u = x$;
(2) $v = x$ or $y = u$.

Let $A(G)$ be the set of all arcs of $D(G)$. An *incidence coloring* of G is a mapping $\sigma : A(G) \to C$, where C is a *color-set*, such that adjacent arcs of $D(G)$ are assigned distinct colors. The *incidence chromatic number*, denoted by χ_i, is the minimum cardinality of C for which $\sigma : A(G) \to C$ is an *incidence coloring*. An *independent set* of arcs is a subset of $A(G)$ which consists of non-adjacent arcs.

2   Incidence chromatic number and Star arboricity

A *star forest* is a forest whose connected components are stars. The *star arboricity* of a graph G (introduced by Akiyama and Kano [1]), denoted by $st(G)$, is the smallest number of star forests whose union covers all edges of G.

We now establish a connection among the chromatic index, the star arborcity and the incidence chromatic number of a graph. This relation, together with the results by Hakimi et al. [10], provided a new upper bound of the incidence chromatic number of planar graphs, k-degenerate graphs and bipartite graphs.

Theorem 2.1 *Let G be a graph. Then $\chi_i(G) \leq \chi'(G) + st(G)$, where $\chi'(G)$ is the chromatic index of G.*

Proof: We color all the arcs going into the center of a star by the same color. Thus, half of the arcs of a star forest can be colored by one color. Since $st(G)$ is the smallest number of star forests whose union covers all edges of G, half of the arcs of G can be colored by $st(G)$ colors. The uncolored arcs now form a digraph which is an orientation of G. We color these arcs according to the edge coloring of G and this is a proper incidence coloring because edge coloring is more restrictive. Hence $\chi'(G) + st(G)$ colors are sufficient to color all the incidences of G.   □

Remarks: Theorem 2.1 was independently obtained by Yang [23] previously. For further information please consult the webpage [18].

We now obtain the following new upper bounds on the incidence chromatic numbers of planar graphs, a class of k-degenerate graphs and a class of bipartite graphs.

Corollary 2.2 *Let G be a planar graph. Then $\chi_i(G) \leq \Delta + 5$ for $\Delta \neq 6$ and $\chi_i(G) \leq 12$ for $\Delta = 6$.*

Proof: The bound is true for $\Delta \leq 5$, since Brualdi and Massey [4] proved that $\chi_i(G) \leq 2\Delta$. Let G be a planar graph with $\Delta \geq 7$, we have $\chi'(G) = \Delta$ [14, 20]. Also, Hakimi et al. [10] proved that $st(G) \leq 5$. Therefore, $\chi_i(G) \leq \Delta + 5$ by Theorem 2.1. □

While we reduce the upper bound on the incidence chromatic number of planar graphs from $\Delta + 7$ [7] to $\Delta + 5$, Hosseini Dolama and Sopena [6] reduced the bound to $\Delta + 2$ under the additional assumptions that $\Delta \geq 5$ and girth $g \geq 6$.

A graph G is *k-degenerate* [12] if every subgraph of G has a vertex of degree at most k. Coloring problems on k-degenerate graphs [5, 7, 11] are widely studied because of its relative simple structure. In particular, Hosseini Dolama et al.[7] proved that $\chi_i(G) \leq \Delta + 2k - 1$, where G is a k-degenerate graph. Moreover, certain important classes of graph such as outerplanar graphs and planar graphs are 2-degenerate and 5-degenerate graphs, respectively [12]. A *restricted k-degenerate graph* is a k-degenerate graph with the subgraph induced by $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ is a complete graph for every i. We lowered the bound for restricted k-degenerate graph as follow.

Corollary 2.3 *Let G be a restricted k-degenerate graph. Then $\chi_i(G) \leq \Delta + k + 2$.*

Proof: By Vizing's theorem, we have $\chi'(G) \leq \Delta + 1$. Also, the star arboricity of a restricted k-degenerate graph G is less than or equal to $k + 1$ [10]. Hence we have $\chi_i(G) \leq \Delta + k + 2$ by Theorem 2.1. □

Corollary 2.4 *Let B be a bipartite graph with at most one cycle. Then $\chi_i(B) \leq \Delta + 2$.*

Proof: When B is a bipartite graph with at most one cycle, $st(B) \leq 2$ [10]. Also, it is well known that $\chi'(B) = \Delta$. These results together with Theorem 2.1 prove the corollary. □

3   Incidence chromatic number and domination number

A *dominating set* $S \subseteq V(G)$ of a graph G is a set where every vertex not in S has a neighbor in S. The *domination number*, denoted by $\gamma(G)$, is the minimum cardinality of a domination set in G.

A *maximal star forest* is a star forest with maximum number of edges. Let $G = (V, E)$ be a graph, the number of edges of a maximal star forest of G is equal to $|V| - \gamma(G)$ [8]. We now use the domination number to form a lower bound on the incidence chromatic number of a graph. The following proposition reformulates the ideas in [2] and [13].

Proposition 3.1 *Let $G = (V, E)$ be a graph. Then $\chi_i(G) \geq \dfrac{2|E|}{|V| - \gamma(G)}$.*

Proof: Each edge of G is divided into two arcs in opposite directions. The total number of arcs of $D(G)$ is therefore equal to $2|E|$. According to the definition of the adjacency of arcs, an independent set of arcs is a star forest. Thus, a maximal independent set of arcs is a maximal star forest. As a result, the number of color class required is at least $\dfrac{2|E|}{|V| - \gamma(G)}$. $\qquad\square$

Corollary 3.2 *Let $G = (V, E)$ be an r-regular graph. Then $\chi_i(G) \geq \dfrac{r}{1 - \frac{\gamma(G)}{|V|}}$.*

Proof: By Handshaking lemma, we have $2|E| = \sum\limits_{v \in V} d(v) = r|V|$, the result follows from Proposition 3.1. $\qquad\square$

Example 3.1 Corollary 3.2 provides an alternative method to show that a complete bipartite graph $K_{r,r}$, where $r \geq 2$ is not $r + 1$-incidence colorable. As $K_{r,r}$ is a r-regular graph with $\gamma(K_{r,r}) = 2$, we have

$$\chi_i(K_{r,r}) \geq \frac{r}{1 - \frac{\gamma(G)}{|V|}} = \frac{r}{1 - \frac{1}{r}} > r + 1.$$

$\qquad\square$

Example 3.1 can be extended to complete k-partite graph with partite sets of same size.

Example 3.2 Let G be a complete k-partite graph where every partite set is of size j. Thus, the order of G is $kj = n$ and G is a $(n - \frac{n}{k})$-regular graph with $\gamma(G) = 2$. According to Corollary 3.2, we have

$$\chi_i(G) \geq \frac{n - \frac{n}{k}}{1 - \frac{\gamma(G)}{|V|}} = \frac{n - \frac{n}{k}}{1 - \frac{2}{n}}.$$

Simple calculation shows that

$$\frac{n - \frac{n}{k}}{1 - \frac{2}{n}} > n - \frac{n}{k} + 1$$

for $k \geq 3$. Therefore, G is not $(\Delta(G) + 1)$-incidence colorable. $\qquad\square$

Corollary 3.3 *Let $G = (V, E)$ be an r-regular graph. Two necessary conditions for $\chi_i(G) = r + 1$ (also for $\chi(G^2) = r + 1$ [17]) are:*

1. *The number of vertices of G is divisible by $r + 1$.*

2. *If r is odd, then the chromatic index of G is equal to r.*

Proof: We prove 1 only, 2 was proved in [16]. By Corollary 3.2, if G is an r-regular graph and $\chi_i(G) = r + 1$, then $r + 1 = \chi_i(G) \geq \dfrac{r|V|}{|V| - \gamma(G)} \Rightarrow \dfrac{|V|}{r+1} \geq \gamma(G)$. Since the global lower bound of domination number is $\left\lceil \dfrac{|V|}{\Delta + 1} \right\rceil$, we conclude that the number of vertices of G must be divisible by $r + 1$. $\qquad\qquad\square$

4   Graphs Constructed from Smaller Graphs

In this section, the upper bounds on the incidence chromatic number are determined for Cartesian product of graphs and for join and union of graphs, respectively. Also, these bounds can be attained by some classes of graphs [19].

We start by proving the following theorem about union of graphs, where the graphs may not disjoint.

Theorem 4.1 *Let G_1 and G_2 be graphs. Then $\chi_i(G_1 \cup G_2) \leq \chi_i(G_1) + \chi_i(G_2)$.*

Proof: If some edge $e \in E(G_1) \cap E(G_2)$, then we delete it from either one of the edge set. This process will not affect $I(G_1 \cup G_2)$, hence, we assume $E(G_1) \cap E(G_2) = \varnothing$. Let σ be a $\chi_i(G_1)$-incidence coloring of G_1 and λ be a $\chi_i(G_2)$-incidence coloring of G_2 using different color set. Then ϕ is a proper $(\chi_i(G_1) + \chi_i(G_2))$-incidence coloring of $G_1 \cup G_2$ with $\phi(\overrightarrow{uv}) = \sigma(\overrightarrow{uv})$ when $uv \in E(G_1)$ and $\phi(\overrightarrow{uv}) = \lambda(\overrightarrow{uv})$ when $uv \in E(G_2)$.
$\qquad\qquad\square$

The following example revealed that the upper bound given in Theorem 4.1 is sharp.

Example 4.1 Let n be an even integer and not divisible by 3. Let G_1 be a graph with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $E(G_1) = \{u_{2i-1}u_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$. Moreover, let G_2 be another graph with $V(G_2) = V(G_1)$ and $E(G_2) = \{u_{2i}u_{2i+1} \mid 1 \leq i \leq \frac{n}{2}\}$. Then, it is obvious that $\chi_i(G_1) = \chi_i(G_2) = 2$ and $G_1 \cup G_2 = C_n$ where n is not divisible by 3. Therefore, $\chi_i(C_n) = 4 = \chi_i(G_1) + \chi_i(G_2)$. $\qquad\square$

The *Cartesian product* of graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(G_2)$ or (2) $v = v'$ and $uu' \in E(G_1)$.

Theorem 4.2 *Let G_1 and G_2 be graphs. Then $\chi_i(G_1 \square G_2) \leq \chi_i(G_1) + \chi_i(G_2)$.*

Proof: Let $|V(G_1)| = m$ and $|V(G_2)| = n$. $G_1 \square G_2$ is a graph with mn vertices and two types of edges: from conditions (1) and (2) respectively. The edges of type (1) form a graph consisting of n disjoint copies of G_1, hence its incidence chromatic number equal to $\chi_i(G_1)$. Likewise, the edges of type (2) form a graph with incidence chromatic number $\chi_i(G_2)$. Consequently, the graph $G_1 \square G_2$ is equal to the union of the graphs from (1) and (2). By Theorem 4.1, we have $\chi_i(G_1 \square G_2) \leq \chi_i(G_1) + \chi_i(G_2)$. $\square$

We demonstrate the upper bound given in Theorem 4.2 is sharp by the following example.

Example 4.2 Let $G_1 = G_2 = C_3$, it follows that $G_1 \square G_2$ is a 4-regular graph. If $G_1 \square G_2$ is 5-incidence colorable, then the chromatic number of its square is equal to 5 [17]. However, all vertices in $G_1 \square G_2$ are of distance at most 2. Therefore, $G_1 \square G_2$ is not 5-incidence colorable and the bound derived in Theorem 4.2 is attained. $\square$

Finally, we consider the incidence chromatic number of the join of graphs.

Theorem 4.3 *Let G_1 and G_2 be graphs with $|V(G_1)| = m$ and $|V(G_2)| = n$, where $m \geq n \geq 2$. Then $\chi_i(G_1 \vee G_2) \leq \min\{m + n, \max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2\}$.*

Proof: On the one hand, we have $\chi_i(G_1 \vee G_2) \leq m + n$. On the other hand, the disjoint graphs G_1 and G_2 can be colored by $\max\{\chi_i(G_1), \chi_i(G_2)\}$ colors, and all other arcs in between can be colored by $m+2$ new colors. Therefore, $\max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2$ is also an upper bound for $\chi_i(G_1 \vee G_2)$. $\square$

We utilize the following example to show that the upper bound in Theorem 4.3 is sharp.

Example 4.3 Let $G_1 = K_m$ and $G_2 = K_n$. Then the upper bound $m+n$ is attained since $G_1 \vee G_2 \cong K_{m+n}$. On the other hand, let G_1 be the complement of K_m and G_2 be the complement of K_n with $m \geq n \geq 2$. Then the other upper bound $\max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2$ is attained because $G_1 \vee G_2 \cong K_{m,n}$ and $\chi_i(G_1) = \chi_i(G_2) = 0$. $\square$

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