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A complete existence theory for Sarvate-Beam triple systems*

Peter Dukes   Jesse Short-Gershman

Mathematics and Statistics
University of Victoria
Victoria, BC V8W 3R4
Canada
dukes@uvic.ca   jesseasg@uvic.ca

Abstract

An ordinary block design has v points, blocks of size k, and each of the $\binom{v}{2}$ unordered pairs of points is contained in a common number λ of blocks. In 2007, Sarvate and Beam introduced a variation where rather than covering all pairs a constant number of times, one wishes to cover all pairs a different number of times. It is of greatest interest when these pair frequencies form an interval of $\binom{v}{2}$ consecutive integers. The case $k = 3$ has received particular attention in the literature, and in this case the object in question is sometimes called a Sarvate-Beam triple system. We prove that the basic (counting) necessary condition $v \equiv 0, 1 \pmod 3$ is sufficient for the existence of Sarvate-Beam triple systems, where any interval of consecutive pair frequencies is possible. Similar results are also obtained, with certain minor restrictions, when $v \equiv 2 \pmod 3$.

1 Introduction

For our purposes, a *block design* is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a multiset of subsets of V, called *blocks*. The *frequency* of a subset $X \subset V$ is the number of blocks in $\mathcal{B}$, counting multiplicity, which contain X.

A block design in which the frequency of every (unordered) pair is one is a *pairwise balanced design*, or (v, K)-PBD, where K includes the set of block sizes in $\mathcal{B}$. More generally, it may be the case that the pairwise frequencies are some constant λ. This is most often considered when the block sizes are also constant, say $K = \{k\}$, and the resulting set system is called a *balanced incomplete block design*, or (v, k, λ)-BIBD.

It is easy to see from a counting argument that a (v, k, λ)-BIBD exists only if $k(k-1)$ divides $\lambda v(v-1)$ and $k-1$ divides $\lambda(v-1)$. It is well-known [2] that these

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conditions are sufficient for the case $k = 3$, when the notation $\mathrm{TS}_\lambda(v)$ is used. The case $\lambda = 1$ corresponds to Steiner triple systems, or $\mathrm{STS}(v)$.

It is sometimes useful to have some pairwise frequencies equal to zero. To this end, define a *group divisible design* $(V, \{V_i\}, \mathcal{B})$ as a set V, equipped with a partition of V into *groups* V_i, and a set of blocks $\mathcal{B}$ such that

- every pair of points in the same group has frequency zero, and
- every pair of points in different groups has frequency one.

This block design is abbreviated as a k-GDD of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ when the block size is a constant k and there are exactly g_i groups of size u_i, $1 \le i \le s$.

Sometimes, higher λ is interesting in this context. One particular case is an *incomplete triple system* $\mathrm{ITS}_\lambda(v; w)$, where $k = 3$ and there is a w-subset W of V such that each pair of distinct points in W has frequency 0 and each remaining pair has frequency λ. See Theorem 2.1 below for necessary and sufficient conditions, and the book [2] for more details on triple systems in general. Apart from this, our use of GDDs is normally for $\lambda = 1$, as defined above.

Both $(v, k, 1)$-BIBDs and k-GDDs are equivalent to edge-decompositions of, respectively, the complete graph K_v and the complete multipartite graph $\overline{\sum u_i K_{g_i}}$ into copies of K_k. With this viewpoint, we may consider even more general block designs as examples of edge-decompositions, and combine them by 'composition'. For example, the existence of both a k-GDD of type g^u and a $(u, k, 1)$-BIBD yields a $(gu, k, 1)$-BIBD. More generally, the existence of a (v, K)-PBD and a $(u, k, 1)$-BIBD for each $u \in K$ yields a $(v, k, 1)$-BIBD, and a similar statement holds for higher λ.

Sarvate and Beam [6] introduced a new condition for set systems in which all unordered pairs of points have a *different* frequency. An *adesign*, abbreviated $\mathrm{AD}(v, k)$, is a block design in which each block has size k and each of the $\binom{v}{2}$ pairs of points has a different frequency. A special case is a *Sarvate-Beam design* $\mathrm{SB}_\mu(v, k)$, where the set of pairwise frequencies is the interval $\{\mu, \mu+1, \dots, \mu + \binom{v}{2} - 1\}$. Since the total of the pairwise frequencies is equal to $\binom{k}{2}$ times the number of blocks, the necessary condition for the existence of such a design is

- $v > k$, and
- $\binom{k}{2}$ divides $\mu\binom{v}{2} + \binom{\binom{v}{2}}{2}$.

The focus of this paper is the case $k = 3$; we call an $\mathrm{SB}_\mu(v, 3)$ a *Sarvate-Beam triple system* and label it $\mathrm{SBTS}_\mu(v)$. Since it is easily seen that 3 divides $\binom{\binom{v}{2}}{2}$ for all positive integers v, the necessary condition for the existence of an $\mathrm{SBTS}_\mu(v)$ reduces to

- $v > 3$, and
- $v \equiv 0, 1 \pmod 3$ or $\mu \equiv 0 \pmod 3$.

Ma, Chang and Feng proved in [5] that the condition $v \equiv 0, 1 \pmod 3$ is sufficient when $\mu = 1$, although they used the term *strictly pairwise distinct triple system* and the notation sPDTS(v) to mean what we call an $\mathrm{SBTS}_1(v)$.

When $v \equiv 2 \pmod 3$, such an interval of pairwise frequencies is possible only if μ is a multiple of 3. We define a *Sarvate-Beam incomplete triple system* $\mathrm{SBITS}_\mu(v; w)$ in analogy with an $\mathrm{ITS}_\lambda(v; w)$; there is a w-subset W of V such that each pair of distinct points in W has frequency 0 and the remaining pair frequencies exhaust the interval $\{\mu, \mu + 1, \ldots, \mu + \binom{v}{2} - \binom{w}{2} - 1\}$. Note that an $\mathrm{SBITS}_\mu(v; w)$ for $w = 0, 1$ reduces to an $\mathrm{SBTS}_\mu(v)$. Following Stanton [8], we are interested in $\mathrm{SBITS}_\mu(v; 2)$ for $v \equiv 2 \pmod 3$. Note that an $\mathrm{SBITS}_1(v; 2)$ is the same object as an $\mathrm{SBTS}_0(v)$.

In general, we find it disappointing that the case $\mu = 0$ has not received more attention in the literature so far. As we shall see, this case admits systems for all $v > 4$ (not just two-thirds of the cases), leads to the only (non-trivial) non-existence result, and is 'basic' in the sense that $\mathrm{SBTS}_0(v)$ sometimes even implies $\mathrm{SBTS}_1(v)$.

Since the definition of an SBTS allows the distribution of the frequencies to be assigned freely among the pairs, it is intuitively clear that the total number of nonisomorphic systems grows incredibly fast with v. Stanton [7] showed that the $\mathrm{SBTS}_1(4)$ is unique up to isomorphism; we mention a generalization of this in Section 3. But Hein and Li [4] showed that there are 20 nonisomorphic $\mathrm{SBITS}_2(5; 2)$ and 16,444,260 nonisomorphic $\mathrm{SBTS}_1(6)$.

The purpose of this paper is to prove the following.

Theorem 1.1.

(a) *There exists an* $SBTS_\mu(v)$ *whenever the necessary conditions are satisfied, except in the case* $(v, \mu) = (4, 0)$.

(b) *There exists an* $SBITS_\mu(v; 2)$ *for all* $\mu \geq 0$ *and all* $v \equiv 2 \pmod 3$, $v \geq 5$.

The outline is as follows. Some elementary constructions to aid in the proof are given next, in Section 2. Certain small examples required are discussed in Section 3, and given fully in the Appendix. A certain type of 'number cube' is defined and investigated in Section 4, and this is used to construct a few more (slightly larger) examples. Sections 5 and 6 outline the recursive proofs of Theorem 1.1, parts (a) and (b), respectively. We conclude with a short list of open directions for this research.

2 Lifting and decomposing

As in [3], we can 'pad' or 'lift' an $\mathrm{SBTS}_\mu(v)$ using a $\mathrm{TS}_\lambda(v)$ to obtain an $\mathrm{SBTS}_{\lambda+\mu}(v)$. This is just multiset union on the block collections. For example, since a Steiner triple system STS(v) exists for $v \equiv 1, 3 \pmod 6$, the existence of $\mathrm{SBTS}_0(v)$ in these cases imply the existence of $\mathrm{SBTS}_\mu(v)$ for all nonnegative integers μ. Similar statements hold in other cases, although the minimum admissible λ for a $\mathrm{TS}_\lambda(v)$ requires λ examples with an interval of starting frequencies.

Even more generally, we can lift an $\mathrm{SBITS}_\mu(v;w)$ with an $\mathrm{ITS}_\lambda(v;w)$ to obtain an $\mathrm{SBITS}_{\lambda+\mu}(v;w)$. The relevant existence result for ITS is below.

Theorem 2.1. (Stern; see [2]) *An $ITS_\lambda(v;w)$ exists if and only if*

1. $w = 0$, $\lambda(v-1) \equiv 0 \pmod 2$, *and* $\lambda v(v-1) \equiv 0 \pmod 6$, *or*
2. $w = v$, *or*
3. $0 < w < v$ *and*
   - $v \geq 2w+1$,
   - $\lambda(\binom{v}{2} - \binom{w}{2}) \equiv 0 \pmod 3$,
   - $\lambda(v-1) \equiv \lambda(v-w) \equiv 0 \pmod 2$.

Setting $w = \lambda = 2$, we see that there exists an $\mathrm{ITS}_2(v;2)$ for all $v \equiv 2 \pmod 3$, $v \geq 5$.

These lifting constructions are now summarized according to congruence classes for later use.

Lemma 2.2. *Let $v > 3$ be an integer.*

(a) *If $v \equiv 1, 3 \pmod 6$ and there exists an $SBTS_0(v)$, then there exists an $SBTS_\mu(v)$ for all nonnegative integers μ.*

(b) *If $v \equiv 0, 4 \pmod 6$ and there exists both an $SBTS_0(v)$ and an $SBTS_1(v)$, then there exists an $SBTS_\mu(v)$ for all nonnegative integers μ.*

(c) *If $v \equiv 5 \pmod 6$ and there exists an $SBTS_0(v)$, then there exists an $SBTS_{3\mu}(v)$ for all nonnegative integers μ.*

(d) *If $v \equiv 2 \pmod 6$ and there exists both an $SBTS_0(v)$ and an $SBTS_3(v)$, then there exists an $SBTS_{3\mu}(v)$ for all nonnegative integers μ.*

(e) *If $v \equiv 2 \pmod 3$ and there exists both an $SBITS_0(v;2)$ and an $SBITS_1(v;2)$, then there exists an $SBITS_\mu(v;2)$ for all nonnegative integers μ.*

Now, in order to form larger Sarvate-Beam systems from smaller ones, we describe a general construction. First, however, an important definition is needed.

For a simple graph G with e edges, define an $\mathrm{SBTS}_\mu(G)$ to be a collection of triples of vertices in G such that

- the frequency of all pairs which are non-edges of G is zero, and
- the frequencies of pairs which are edges of G exhaust the interval from μ to $\mu + e - 1$.

In practice, we are usually interested in some complete balanced multipartite graph $\overline{mK_n}$. The following construction is a routine extension of the technique in [3] and [5], where PBD-closure was used to construct SBTS.

Construction 2.3. *Let μ be a nonnegative integer and let G be a graph with an edge-decomposition into subgraphs $G_1, G_2, \ldots, G_m$, having $e_1, e_2, \ldots, e_m$ edges, respectively. Define $\mu_1 = \mu$ and $\mu_i = \mu_{i-1} + e_{i-1}$ for $i > 1$. Align an SBTS$_{\mu_i}(G_i)$ on G_i for each i to form an SBTS$_\mu(G)$.*

Remark 2.4. *In the case $G = K_v$, Construction* 2.3 *produces an SBTS$_\mu(v)$, and in the case $G = K_v - K_2$, it yields an SBITS$_\mu(v; 2)$.*

3 Examples for $4 \leq v \leq 9$

When a block design $(V, \mathcal{B})$ is clear from context, write $F(X)$ for the frequency of $X \subset V$. For our purposes, F acts on both triples and pairs.

We begin with a structural result for $v = 4$. Let $F(123) = a, F(124) = b, F(134) = c, F(234) = d$, and call an SBTS$_\mu(4)$ in *standard form* if $a \leq b \leq c \leq d$.

Lemma 3.1. *If there exists an SBTS$_\mu(4)$ in standard form, then its triple frequencies are*

$$(a,b,c,d) = \begin{cases} (s-1, s+1, s+2, s+3) & \text{if } \mu \text{ is even and } \mu = 2s; \\ (s, s+1, s+2, s+4) & \text{if } \mu \text{ is odd and } \mu = 2s+1. \end{cases}$$

Proof: First, if $F(\{x,y,z\}) = F(\{x,y,w\}) = f$, then $F(\{x,z\}) = F(\{x,w\}) = f + F(\{x,z,w\})$, a contradiction. Therefore $a < b < c < d$ and necessarily $F(12) = a + b = \mu$.
CASE 1: $\mu = 2s$. Then $a + b = 2s$ and $c + d = 2s + 5$. So $2s + 5 \geq (2s - a + 1) + (2s - a + 2) = 4s + 3 - 2a$, which implies that $a \geq s - 1$. Since $a < b < c < d$, $(a,b,c,d) = (s-1, s+1, s+2, s+3)$ is the only possibility.
CASE 2: $\mu = 2s + 1$. Then $a + b = 2s + 1$ and $c + d = 2s + 6$. So $2s + 6 \geq (2s - a + 2) + (2s - a + 3) = 4s + 5 - 2a$, which implies that $a \geq s - 1/2$. Since a is an integer and $a < b < c < d$, $(a,b,c,d) = (s, s+1, s+2, s+4)$ is the only possibility. □

From this, we obtain our only nonexistence result as well as a general uniqueness result.

Corollary 3.2. *There does not exist an SBTS$_0(4)$.*

Corollary 3.3. *For each $\mu > 0$, there exists a unique SBTS$_\mu(4)$ up to isomorphism.*

In [3], an SBTS$_0(6)$ was constructed. Additional examples of SBTS$_0(v)$ for $v = 5, 7, 8, 9$ are given in Appendix A.

Examples of $\mathrm{SBTS}_1(v)$ are given in [7] for $v = 6, 7$ and in [5] for $v = 9$. An $\mathrm{SBTS}_3(8)$ is also given in Appendix B.

Examples of $\mathrm{SBITS}_0(v; 2)$ for $v = 5, 8$ are given in Appendix C. The triple systems that are called $\mathrm{SB}(v, 3)$ in [4] and [8] correspond to $\mathrm{SBITS}_2(v; 2)$ under our notation.

Together with Lemma 2.2, we are able to prove the main theorem for $v \leq 9$.

Lemma 3.4. *Theorem* 1.1 *holds for each* $v \leq 9$ *and all admissible* μ.

Before closing this section, we mention one more uniqueness result.

Lemma 3.5. *The* $SBITS_0(5; 2)$ *given in Appendix C is unique up to isomorphism.*

Proof: Let $F(12) = 0$. The other pairs must cover the frequencies $0, 1, \ldots, 8$. Note that $F(13) = 0$ implies $F(14) = F(15) = F(145)$, a contradiction. Similarly, none of $F(14)$, $F(15)$, $F(23)$, $F(24)$, $F(25)$ can be 0. So there is no loss of generality in taking $F(34) = 0$. Letting $F(135) = a$, $F(145) = b$, $F(235) = c$, and $F(245) = d$, we have $a+b+c+d = 12$, none of them are 0 and they are all distinct. Without loss of generality $a = 1$, and also $b < c$ because the permutation $(13)(24)(5)$ has the effect of switching b and c. $b = 2$ leads to the contradiction $F(25) = c + d = 9$. Also $b \geq 3$ forces d to be 2; then $(b, d) = (4, 2)$ leads to $F(35) = F(45) = 6$ and $b \geq 5$ leads to $b > c$. So therefore $(b, c, d) = (3, 6, 2)$ is the only possibility. □

4 Sarvate-Beam squares and cubes

A Latin square is a $n \times n$ array, whose entries are from 1 to n, such that every row and every column is a permutation of the entries. These are useful objects in design theory, particularly for constructing triple systems since they are equivalent to 3-GDDs of type n^3. Their analog for Sarvate-Beam systems is essentially the case $d = 3$ in the definition that follows.

A *Sarvate-Beam hypercube* d-$\mathrm{SBHC}_\mu(n)$ is a d-cube of side length n so that each cell contains a nonnegative integer and such that the dn^{d-1} line sums exhaust the interval of integers $\{\mu, \mu + 1, \ldots, \mu + dn^{d-1} - 1\}$.

A 2-SBHC is a *Sarvate-Beam square*, denoted by SBS, and a 3-SBHC is a *Sarvate-Beam cube*, denoted by SBC. The subscript μ is understood to be 0 if it is absent.

By counting the sum of the entries in a d-SBHC in two ways, it is easily seen that a necessary condition for the existence of a d-$\mathrm{SBHC}_\mu(n)$ is that $n > 1$ and d divides $dn^{d-1}(dn^{d-1} - 1)/2$. The following theorem shows that this condition is sufficient in the case $d = 2$.

Theorem 4.1. *There exists an* $SBS_\mu(n)$ *if and only if* n *is even.*

Proof: Necessity follows by substituting $d = 2$ in the condition above. For sufficiency, note that $[[\mu, 0], [1, 2 + \mu]]$ is an $\mathrm{SBS}_\mu(2)$ for any μ. For n even and greater than 2, divide an n by n array into 2 by 2 subsquares and place an $\mathrm{SBS}_{\mu+4i}(2)$ in the i-th subsquare along the block diagonal, $0 \le i < n/2$. Place zeros off the block diagonal to give an $\mathrm{SBS}_\mu(n)$. □

We are most interested here in Sarvate-Beam cubes. Note there is no numerical restriction on the order n, because 3 clearly divides $(9n^4 - 3n^2)/2$ for all n. An SBC(2) and an SBC(3) are given in Appendix F.

The following two lemmas provide means to 'lift' and 'inflate' an SBC.

Lemma 4.2. *If there exists an $SBC_\mu(n)$, then there exists an $SBC_{\mu+\nu}(n)$ for all nonnegative integers ν.*

Proof: Let L be a Latin square of order n with entries in $\{0, 1, \dots, n-1\}$. Adding ν to the (i, j, k)-entry of the $\mathrm{SBC}_\mu(n)$ whenever $L_{ij} = k$ gives an $\mathrm{SBC}_{\mu+\nu}(n)$. □

Theorem 4.3. *If there exists an SBC(n), then there exists an SBC(mn) for every positive integer m.*

Proof: Let L be a Latin square of order m with entries in $\{0, 1, \dots, m-1\}$ and, by Lemma 4.2, let C_t be an $\mathrm{SBC}_{3tn^2}(n)$ for $0 \le t < m^2$. Arrange these cubes in an $m \times m \times m$ array by putting C_{km+i} in position (i, j, k) whenever $L_{ij} = k$. □

Inflating an SBC(2) and an SBC(3) yields many orders, although it appears difficult to completely settle the existence question for SBCs, particularly for orders whose smallest prime factor is large.

Corollary 4.4. *There exists an SBC(n) for all $n \equiv 0, 2, 3, 4 \pmod 6$, $n \ge 2$.*

We now show that an $\mathrm{SBC}_\mu(n)$ is equivalent to an $\mathrm{SBTS}_\mu(\overline{3K_n})$, which is actually an extremal example of a Sarvate-Beam 3-GDD; see [5].

Lemma 4.5. *If there exists an $SBC_\mu(n)$, then there exists an $SBTS_\mu(\overline{3K_n})$.*

Proof: Let $V = \{x_\alpha \mid 1 \le x \le n, 0 \le \alpha \le 2\}$ and include the block $\{i_0, j_1, k_2\}$ with multiplicity equal to the (i, j, k)-entry of the $\mathrm{SBC}_\mu(n)$. The frequencies of pairs $\{x_a, y_b\}$, $a \ne b$, are the line sums and thus cover the desired interval. □

5 Existence results for $v \equiv 0, 1 \pmod 3$, $v \ge 10$

As mentioned earlier, Ma, Chang, and Feng have already settled this case when $\mu = 1$. By Lemma 2.2, it suffices to settle the case $\mu = 0$. For this, we (temporarily) extend the definition of an $\mathrm{SBTS}_\mu(v)$ to allow negative block frequencies. This is a convenient way to sidestep the nonexistence of $\mathrm{SBTS}_0(4)$ in the recursive construction. Otherwise, the following lemma is similar to corresponding results in [3] and [5].

Lemma 5.1. *There exists an SBTS$_0(v)$ for all $v \equiv 0,1 \pmod 3$, $v > 9$, $v \neq 10,12,15,18,19,24,27$.*

Proof: By the PBD-closure table in [1], the given conditions imply v admits a $(v,\{4,6,7,9\})$-PBD. Let its blocks be $B_1, B_2, \dots,$ ordered by decreasing size. If not all blocks are of size 4, then $|B_1| > 4$ and we may simply apply Construction 2.3 with $\mu = 0$ and $G_i = K_{B_i}$ for each i. On the other hand, if all blocks are of size 4, we instead apply Construction 2.3 with $\mu = -2$ and $G_i = K_{B_i}$, beginning with an SBTS$_{-2}(4)$ on B_1. By Lemma 3.1, there is a unique triple $B \subset B_1$ with frequency -2, and the other three triple frequencies are $0,1,2$. Finally, lift the resulting SBTS$_{-2}(v)$ using a TS$_2(v)$ having B as a repeated block, the latter existing by Theorem 2.1 with $w = 3$ and $\lambda = 2$. The resulting SBTS$_0(v)$ has nonnegative frequencies on triples, as required. □

The existence of SBCs of sides 4, 6, and 9 easily handle 3 of the above outstanding cases for SBTS$_0(v)$.

Lemma 5.2. *There exists an SBTS$_0(v)$ for $v = 12,18,27$.*

Proof: Let $n = v/3$. Put $V_\alpha = \{x_\alpha \mid 1 \leq x \leq n\}$ for $\alpha = 0,1,2$, and let $V = V_0 \cup V_1 \cup V_2$. Apply Construction 2.3 with ingredient SBTS(G_i) on the following graphs: the complete tripartite graph $\overline{\sum K_{V_\alpha}}$ (from SBC(n)), and the three complete graphs K_{V_α}, $0 \leq \alpha \leq 2$ (from SBTS$_\mu(n)$). Note that for $n = 4$ we must sequence the SBTS$(\overline{3K_4})$ first (for the low frequencies), due to the nonexistence of SBTS$_0(4)$. □

Two more cases follow from SBCs upon appending an extra point.

Lemma 5.3. *There exists an SBTS$_0(v)$ for $v = 10,19$.*

Proof: Let $n = (v-1)/3$, $V_\alpha = \{x_\alpha \mid 1 \leq x \leq n\}$ for $\alpha = 0,1,2$, and $V = V_0 \cup V_1 \cup V_2 \cup \{\infty\}$. Apply Construction 2.3 with G_1 as in Lemma 5.2, and G_2, G_3, G_4 as $K_{V_\alpha \cup \{\infty\}}$, $0 \leq \alpha \leq 2$. □

Similarly, it is possible to append 3 extra points, taking care to use SBITS with 'holes' on those points. Here, and in what follows, we compactly describe a valid sequencing order with shifted subscripts on the graphs G_i.

Lemma 5.4. *There exists an SBTS$_0(15)$.*

Proof: Let $V_\alpha = \{x_\alpha \mid 1 \leq x \leq 4\}$, for $\alpha = 0,1,2$, $V_\infty = \{\infty_1, \infty_2, \infty_3\}$, and $V = V_0 \cup V_1 \cup V_2 \cup V_\infty$. Apply Construction 2.3 with G_1 as in Lemma 5.2, $G_2 = K_{V_0 \cup V_\infty}$, and $G_{\alpha+2} = K_{V_\alpha \cup V_\infty} - K_{V_\infty}$, $\alpha = 1,2$, by lifting the SBITS$_1(7;3)$ from Appendix D with suitably many copies of an ITS$_1(7;3)$. □

Our final special construction uses the existence of orthogonal incomplete Latin squares of order six having a 2×2 hole. More details on this well-known object can be found in [1]. Additionally, we must find a Sarvate-Beam system on a graph isomorphic to $K_2 \times K_4$ to 'fill' the hole.

Lemma 5.5. *There exists an* $SBTS_0(24)$.

Proof: Let $V_\alpha = \{x_\alpha \mid 1 \le x \le 6\}$ and $W_\alpha = \{x_\alpha \mid x = 5, 6\}$ for $\alpha = 0, 1, 2, 3$. Let $G_1 = \overline{\sum K_{W_\alpha}}$, which is isomorphic to $K_2 \times K_4$. Now let B_k, $1 \le k \le 32$, denote the blocks $\{i_0, j_1, (L_{ij})_2, (M_{ij})_3\}$ where L and M are orthogonal incomplete Latin squares of order 6 missing (just) the entries 5 and 6 in their 5th and 6th rows and columns. Let $G_{k+1} = K_{B_k}$, $1 \le k \le 32$. Finally, let $G_{\alpha+33} = K_{V_\alpha}$, $0 \le \alpha \le 3$. Apply Construction 2.3, using the SBTS(G_1) given in Appendix E, along with SBTS$_\mu(4)$ and SBTS$_\mu(6)$ for the rest of the sequencing. □

Incidentally, an alternate construction for this case is possible following [5], where a Sarvate-Beam GDD of type 4^6 is used. We have chosen to avoid the extra definition and construction.

The proof of Theorem 1.1(a) for $v \equiv 0, 1 \pmod 3$ now follows from lifting these various constructed systems using Lemma 2.2.

6 Existence results for $v \equiv 2 \pmod 3$, $v \ge 11$

For asymptotic existence when $v \equiv 2 \pmod 3$, we write $v = 3u + m + 2$ where $u \equiv 0, 1 \pmod 4$ and $m = 0$ or 6.

Theorem 6.1. (See [1]) *A* 4-*GDD of type* $3^u m^1$ *exists if and only if*
1. $u \equiv 0 \pmod 4$ *and* $m \equiv 0 \pmod 3$, $u \ge (2m + 6)/3$, *or*
2. $u \equiv 1 \pmod 4$ *and* $m \equiv 0 \pmod 6$, $u \ge (2m + 3)/3$, *or*
3. $u \equiv 3 \pmod 4$ *and* $m \equiv 3 \pmod 6$, $u \ge (2m + 3)/3$.

We summarize our usage of Theorem 6.1 in the following table, where $v_{\min}$ is the minimum value of $v = 3u + m + 2$ for which a 4-GDD of type $3^u m^1$ exists.

$v \pmod{12}$	$u \pmod 4$	m	$v_{\min}$
2	0	0	14
5	1	0	17
8	0	6	32
11	1	6	23

Lemma 6.2. *There exist* $SBTS_0(v)$, $SBTS_3(v)$, $SBITS_0(v; 2)$, *and* $SBITS_1(v; 2)$ *for all* $v \equiv 2 \pmod 3$, $v \ge 14$, $v \ne 20$.

Proof: Write $v = 3u+m+2$ using the cases in the table above. Put $V_\alpha = \{1_\alpha, 2_\alpha, 3_\alpha\}$ for $1 \le \alpha \le u$, $W = \emptyset$ or $\{1, \ldots, 6\}$ according as $m = 0$ or 6, and $V_\infty = \{\infty_1, \infty_2\}$. Let $V = (\bigcup_{\alpha=1}^{u} V_\alpha) \cup W \cup V_\infty$ and take a 4-GDD of type $3^u m^1$ on $V \setminus V_\infty$. Let this 4-GDD have blocks $B_1, B_2, \ldots, B_N$. Apply Construction 2.3 with the following ingredients:

- either $\mathrm{SBTS}_0(5)$, $\mathrm{SBTS}_3(5)$, or $\mathrm{SBITS}_0(5;2)$ on $K_{V_1 \cup V_\infty}$
- suitably lifted $\mathrm{SBITS}_\mu(5;2)$ on $K_{V_\alpha \cup V_\infty} - K_{V_\infty}$, $2 \le \alpha \le u$
- suitably lifted $\mathrm{SBTS}_\mu(4)$ on each GDD block B_i for $1 \le i \le N$
- suitably lifted $\mathrm{SBITS}_\mu(8;2)$ on $K_{W \cup V_\infty} - K_{V_\infty}$ when $m = 6$

□

Note that the construction of Lemma 6.2 was not available in the previous section for $v \equiv 0, 1 \pmod 3$, since $\mathrm{SBTS}_\mu(3)$ does not exist.

Lemma 6.3. *There exist $SBTS_0(11)$, $SBITS_0(11;2)$, and $SBITS_1(11;2)$.*

Proof: Let $V_\alpha = \{1_\alpha, 2_\alpha, 3_\alpha\}$ for $\alpha = 0,1,2$, $V_\infty = \{\infty_1, \infty_2\}$, and $V = V_0 \cup V_1 \cup V_2 \cup V_\infty$. Apply Construction 2.3 with G_1 as in Lemma 5.2 and $G_{\alpha+2} = K_{V_\alpha \cup V_\infty} - K_{V_\infty}$, $\alpha = 0,1,2$, with suitable $\mathrm{SBITS}_\mu(5;2)$ used on these latter graphs. □

Lemma 6.4. *There exist $SBTS_0(20)$, $SBTS_3(20)$, $SBITS_0(20;2)$, and $SBITS_1(20;2)$.*

Proof: Let $V_\alpha = \{x_\alpha \mid 1 \le x \le 4\}$ for $0 \le \alpha \le 4$. Consider a 5-GDD of type 4^5, which can be obtained from 3 MOLS of order 4. Enumerate its blocks as $B_1, B_2, \ldots, B_{16}$. Let A_k denote the intersection of block B_k with the two groups $V_3 \cup V_4$. Observe that $|A_k| = 2$, since all blocks intersect all groups in this GDD. Now apply Construction 2.3 with $G_k = K_{B_k} - K_{A_k}$ for $1 \le k \le 16$, $G_{17+\alpha} = K_{V_\alpha}$ for $\alpha = 0,1,2$, and $G_{20} = K_{V_3 \cup V_4} - K_2$. This latter graph requires some suitable $\mathrm{SBITS}_\mu(8;2)$. □

The proofs of Theorem 1.1(a), $v \equiv 2 \pmod 3$, and also part (b), now follow from Lemma 2.2 applied to the systems constructed above in Lemmas 6.2, 6.3 and 6.4.

7 Concluding remarks

Although our main result closes the basic question for triple systems, the general problem of constructing pairwise distinct block designs is still in early stages.

Some of the next steps are:

1. consideration of $\mathrm{SBTS}_\mu(v;w)$ for $w > 2$;
2. imposing extra structure for triples, such as ordering or resolvability;
3. an asymptotic (or complete) solution for block size 4;
4. better upper bounds on the maximum frequency in a general adesign.

Appendix: small examples required for the proofs

In the tables which follow, frequencies of triples are listed according to the colexicographic ordering, as indicated in the first row below.

A SBTS$_0(v)$ for $v = 5, 7, 8, 9$

v	1 1 1 2 1 1 2 1 2 3 ... 2 2 3 3 2 3 3 4 4 4 ... 3 4 4 4 5 5 5 5 5 5 ...	frequencies of triples
5	0 0 0 2 0 1 3 2 5 2	
7	0 0 0 0 0 0 1 0 2 3 0 0 2 1 2 3 1 2 4 5 0 1 2 1 3 4 2 3 4 4 2 3 4 5 6	
8	0 0 0 0 0 0 1 0 2 3 0 0 1 0 2 3 1 2 4 4 0 0 2 1 2 3 1 2 3 4 1 2 4 5 5 ... 0 1 2 1 2 4 1 2 3 5 2 3 3 5 8 2 3 4 5 8 4	
9	0 0 0 1 0 0 1 0 1 3 0 0 1 0 1 2 1 2 3 3 0 0 1 1 2 3 0 2 3 5 1 2 3 4 5 ... 0 0 2 0 2 2 1 2 3 5 1 2 4 5 5 2 2 4 4 6 7 ... 0 1 2 1 2 3 1 2 3 5 1 3 4 5 6 1 3 4 5 6 8 2 3 4 5 6 8 6	

B SBTS$_3(8)$

	0 0 1 2 0 1 2 1 2 3 1 0 2 1 2 3 1 2 4 5 1 1 2 1 2 4 1 3 4 5 2 3 3 5 8 ... 1 1 2 1 3 4 2 3 4 5 2 3 4 6 8 2 3 5 6 8 3

C SBITS$_0(v; 2)$ for $v = 5, 8$

5	0 0 0 0 0 1 6 3 2 0
8	0 0 0 0 0 0 1 0 1 3 0 0 1 0 2 3 0 2 2 4 0 0 1 0 2 3 1 2 3 4 1 2 4 4 6 ... 0 0 2 1 2 3 1 2 4 4 2 2 4 5 8 2 3 4 6 7 3

D SBITS$_1(7; 3)$

	0 0 0 0 0 0 0 1 0 1 0 0 0 0 1 2 5 1 3 5 0 0 0 0 1 3 6 2 3 6 5 2 3 6 1

Note: $F(12) = F(13) = F(23) = 0$ and the other pair frequencies are $1, 2, \ldots, 18$.

E SBTS$_0(K_2 \times K_4)$

	0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 2 1 0 3 1 6 7 4 2 6 10 0 ... 0 0 1 2 1 0 4 2 8 7 5 3 7 8 0 0 0 0 0 0 0

Note: $F(12) = F(34) = F(56) = F(78) = 0$ and the other pair frequencies are $0, 1, \ldots, 23$.

F SBC(2) and SBC(3)

$$\text{SBC}(2) \qquad \begin{bmatrix} (0,4) & (0,6) \\ (1,4) & (2,5) \end{bmatrix} \qquad \text{SBC}(3) \qquad \begin{bmatrix} (0,0,6) & (0,0,13) & (0,12,3) \\ (0,0,3) & (0,0,2) & (9,10,0) \\ (4,1,12) & (7,11,8) & (9,2,5) \end{bmatrix}$$

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