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Antimagic and magic labelings in Cayley digraphs

T. Tamizh Chelvam N. Mohamed Rilwan

G. Kalaimurugan

Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli 627 012, Tamil Nadu
India
tamche59@gmail.com rilwan2020@gmail.com
kalaimurugan@gmail.com

Abstract

A Cayley digraph is a digraph constructed from a group Γ and a generating subset S of Γ. It is denoted by $\mathrm{Cay}_D(\Gamma, S)$. In this paper, we prove for any finite group Γ and a generating subset S of Γ, that $\mathrm{Cay}_D(\Gamma, S)$ admits a super vertex (a, d)-antimagic labeling depending on d and $|S|$. We provide algorithms for constructing the labelings.

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. MacDougall et al. [3] introduced the notion of a vertex-magic total labeling. A digraph $G = (V, E)$ with $|V| = p$ and $|E| = q$ is called a (p, q) digraph. For a graph $G = (V, E)$, a one-to-one mapping $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ is a *vertex-magic total labeling* if there is a constant k, called the *magic constant*, such that for every vertex v, $f(v) + \sum_{uv \in E} f(uv) = k$ where the sum is over all vertices u adjacent to v. Thirusangu et al. [4] studied the super vertex (a, d)-antimagic and vertex-magic total labelings for a particular class of Cayley digraphs. For a survey of magic and antimagic labelings one can refer to Gallian [1].

A (p, q)-digraph $G = (V, E)$ is said to have a *super vertex (a, d)-antimagic labeling* if there exists a function $f : V \cup E \to \{1, 2, \ldots, p + q\}$ such that $f(V) = \{1, 2, \ldots, p\}$ and the sum of the labels of the outgoing arcs of v and its own label are distinct for all $v \in V$. Moreover, the set of all such distinct labels associated with vertices in V of G is $\{a, a + d, a + 2d, \ldots, a + (p - 1)d\}$, where a and d are two positive integers. A (p, q)-digraph G is said to be $(0, 1)$-*vertex-magic* with common vertex count k if there exists a bijection $f : E(G) \to \{1, 2, \ldots, q\}$ such that for each $u \in V(G)$,

$\sum_{\vec{e}\in E(u)} f(\vec{e}) = k$ where $E(u)$ is the set of all outgoing arcs from u. It is said to be $(0,1)$-*vertex-antimagic* if, for each $u \in V(G)$, the sums $\sum_{\vec{e}\in E(u)} f(\vec{e})$ are distinct.

Let Γ be a finite nontrivial group and S be a generating subset of Γ. The *Cayley digraph* G, $\mathrm{Cay}_D(\Gamma, S)$, is the digraph whose vertices are the elements of Γ, and there is an arc from α to $\alpha\sigma$ whenever $\alpha \in \Gamma$ and $\sigma \in S$. If $S = S^{-1}$, then there is an arc from α to $\alpha\sigma$ if and only if there is an arc from $\alpha\sigma$ to α. Cayley digraphs are useful designs for many well-known interconnection networks. For example, rings, tori, hypercubes, butterflies, cubic-connected cycle networks, are Cayley digraphs [2].

In this paper, we prove that $G = \mathrm{Cay}_D(\Gamma, S)$ admits a super vertex (a,d)-antimagic labeling for $d = 1$ when $|S|$ is even, and for $d = 2$ when $|S|$ is odd. Also $\mathrm{Cay}_D(\Gamma, S)$ admits a $(0,1)$-vertex-magic labeling when $|S|$ is even, whereas the same admits a $(0,1)$-vertex-antimagic labeling and a vertex-magic total labeling when $|S|$ is odd.

2 Super vertex (a,d)-antimagic labeling of Cayley digraphs

In this section, we give an algorithm to construct a super vertex (a,d)-antimagic labeling of the Cayley digraph $\mathrm{Cay}_D(\Gamma, S)$. The algorithm produces a super vertex $(a,1)$-antimagic labeling if $|S|$ is even and a super vertex $(a,2)$-antimagic labeling if $|S|$ is odd.

Algorithm 1: Super vertex (a,d)-antimagic labeling of $\mathrm{Cay}_D(\Gamma, S)$

Input: A finite group $\Gamma = \{v_1, v_2, \ldots, v_m\}$ and a generating subset $S = \{\sigma_1, \sigma_2, \ldots, \sigma_{|S|}\}$.

Denote the vertices and arcs of the Cayley digraph $G = \mathrm{Cay}_D(\Gamma, S)$ as $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $E(G) = \{\vec{e_{ij}} : \vec{e_{ij}} = v_i.\sigma_j, 1 \le i \le m, 1 \le j \le |S|\}$.

Procedure: Define $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ by $f(v_i) = i$, for $1 \le i \le m$, and

$$f(\vec{e_{ij}}) = \begin{cases} (j+1)m + 1 - i & \text{if } j \text{ is even,} \\ jm + i & \text{if } j \text{ is odd,} \end{cases}$$

for $1 \le i \le m$ and $1 \le j \le |S|$.

Output: The sum corresponding to the vertex v_i is given by

$$f(v_i) + \sum_{j=1}^{|S|} f(\vec{e_{ij}}) = \begin{cases} \frac{|S|}{2}[(|S|+2)m+1] + i & \text{if } |S| \text{ is even,} \\ \left(\frac{|S|-1}{2}\right)[(|S|+1)m+1] + |S|m + 2i & \text{if } |S| \text{ is odd.} \end{cases}$$

Example 2.1 We illustrate Algorithm 1 for $\mathrm{Cay}_D(\mathbb{Z}_6, S)$, where $S = \{1, 2, 4, 5\} \subset \mathbb{Z}_6$. Actually we give a super vertex $(75,1)$-antimagic labeling of $\mathrm{Cay}_D(\mathbb{Z}_6, \{1,2,4,5\})$ in Figure 1.

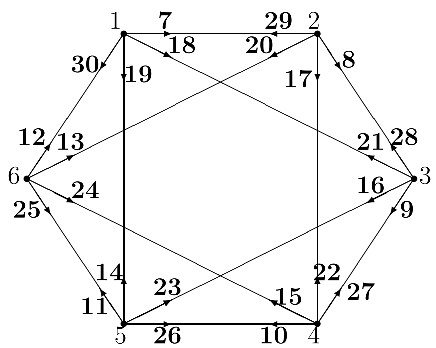

Figure 1

Theorem 2.2 *Let Γ be a finite group of order m and let S be a generating subset of Γ. Then the Cayley digraph $Cay_D(\Gamma, S)$ admits a super vertex $(a,1)$-antimagic labeling if $|S|$ is even and a super vertex $(a,2)$-antimagic labeling if $|S|$ is odd.*

Proof: Let $\Gamma = \{v_1, v_2, \ldots, v_m\}$ and $S = \{\sigma_1, \sigma_2, \ldots, \sigma_{|S|}\}$. By the construction of $G =$ Cay$_D(\Gamma, S)$, $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $E(G) = \{\vec{e_{ij}} : \vec{e_{ij}} = v_i.\sigma_j,\ 1 \leq i \leq m,\ 1 \leq j \leq |S|\}$. Let us consider the assignment of labels as per Algorithm 1 and realize that $f(v_i) + \sum_{j=1}^{|S|} f(\vec{e_{ij}})$ are distinct for every i, $1 \leq i \leq m$. Note that the following are true:

Case 1: When $|S|$ is even, the set of numbers corresponding to various sums are $\{a, a+1, a+2, \ldots, a+(m-1)\}$ where $a = \frac{|S|}{2}[(|S|+2)m+1]+1$. Hence Cay$_D(\Gamma, S)$ admits a super vertex $(a,1)$-antimagic labeling.

Case 2: When $|S|$ is odd, the set of numbers corresponding to various sums are $\{a, a+2, a+4, \ldots, a+2(m-1)\}$ where $a = \left(\frac{|S|-1}{2}\right)[(|S|+1)m+1] + |S|m + 2$. Hence Cay$_D(\Gamma, S)$ admits a super vertex $(a,2)$-antimagic labeling. □

By taking Γ as the Dihedral group $D_n = \langle r, s \mid r^n = s^2 = e,\ rsr = s\rangle$ and $S = \{r, s\}$, we get the following:

Corollary 2.3 [4, Theorem 3.1] *The Cayley digraph associated with the group D_n, with the generating set $\{r, s\}$, admits a super vertex (a,d)-antimagic labeling.*

3 Vertex-magic total labeling of Cayley digraphs

In this section, we give an algorithm to construct a vertex-magic total labeling of the Cayley digraph Cay$_D(\Gamma, S)$ when $|S|$ is odd. Also we give an algorithm to construct a $(0,1)$-vertex-magic and a $(0,1)$-vertex-antimagic labeling of Cay$_D(\Gamma, S)$.

Algorithm 2: Vertex-magic total labeling of $\mathbf{Cay}_D(\Gamma, S)$

Input: A finite group $\Gamma = \{v_1, v_2, \dots, v_m\}$ and a generating subset $S = \{\sigma_1, \sigma_2, \dots, \sigma_{|S|}\}$.

Denote the vertices and arcs of the Cayley digraph $G = \mathrm{Cay}_D(\Gamma, S)$, by $V(G) = \{v_1, v_2, \dots, v_m\}$ and $E(G) = \{\vec{e_{ij}} : \vec{e_{ij}} = v_i.\sigma_j, 1 \leq i \leq m, 1 \leq j \leq |S|\}$.

Procedure: Define $f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ by $f(v_i) = i$, for each i, $1 \leq i \leq m$. Further,

$$f(\vec{e_{ij}}) = \begin{cases} jm + i & \text{if } j \text{ is even,} \\ (j+1)m + 1 - i & \text{if } j \text{ is odd,} \end{cases}$$

for $1 \leq i \leq m$ and $1 \leq j \leq |S|$.

Output: The sum corresponding to the vertex v_i is given by

$$f(v_i) + \sum_{j=1}^{|S|} f(\vec{e_{ij}}) = \begin{cases} \frac{|S|}{2}[(|S|+2)m + 1] + i & \text{if } |S| \text{ is even,} \\ \frac{m(|S|+1)^2}{2} + \frac{|S|+1}{2} & \text{if } |S| \text{ is odd.} \end{cases}$$

Theorem 3.1 *Let Γ be a finite group of order m and S be a generating subset of Γ such that $|S|$ is odd. Then the Cayley digraph $Cay_D(\Gamma, S)$ admits a vertex-magic total labeling.*

Proof: Let us take $\Gamma = \{v_1, v_2, \dots, v_m\}$ and $S = \{\sigma_1, \sigma_2, \dots, \sigma_{|S|}\}$, where $|S|$ is odd. By the construction of $G = \mathrm{Cay}_D(\Gamma, S)$, $E(G) = \{\vec{e_{ij}} : \vec{e_{ij}} = v_i.\sigma_j, 1 \leq i \leq m, 1 \leq j \leq |S|\}$. Consider the assignment of labels by Algorithm 2. Since $|S|$ is odd, for each vertex v_i, the sum $f(v_i) + \sum_{j=1}^{|S|} f(\vec{e_{ij}}) = \frac{m(|S|+1)^2}{2} + \frac{|S|+1}{2}$ is the same. □

Remark 3.2 If $|S|$ is even, then Algorithm 2 produces a super vertex $(a, 1)$-antimagic labeling of $\mathrm{Cay}_D(\Gamma, S)$.

Example 3.3 We illustrate Algorithm 2 for $\mathrm{Cay}_D(D_4, S)$, where

$$D_4 = \langle r, s \mid r^4 = s^2 = e,\ rsr = s \rangle \quad \text{and } S = \{r, r^3, s\}.$$

Actually we exhibit a vertex-magic total labeling of $\mathrm{Cay}_D(D_4, \{r, r^3, s\})$ with total sum 66 in Figure 2.

Let $\Gamma = \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and $S = \{a, b, b+k\} \subseteq \mathbb{Z}_n$ such that $\gcd(a, b, k) = 1$ and any one of the following holds:

(i) $\gcd(a - b, k) \neq 1$,

(ii) $\gcd(a, 2k) = 1$,

(iii) $\gcd(b, k) = 1$,

(iv) both a and k are even,

(v) a is odd and either b or k is odd.

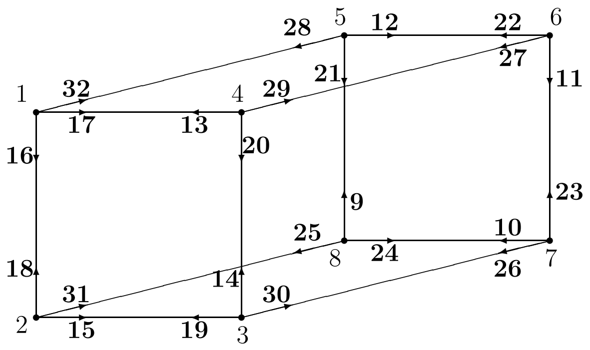

Figure 2

Note that $|S|$ is odd and S is a generating set for $\mathbb{Z}_n$. By Theorem 3.1, we have the following:

Corollary 3.4 *[4, Theorem 4.1] The Cayley digraph associated with the group $\mathbb{Z}_n$, with the generating set $\{a, b, b+k\}$, admits a magic labeling.*

Now let us present an algorithm for a $(0,1)$-vertex-magic labeling and a $(0,1)$-vertex-antimagic labeling of the Cayley digraph $\text{Cay}_D(\Gamma, S)$.

Algorithm 3: $(0,1)$**-vertex-magic labeling and** $(0,1)$**-vertex-antimagic labeling of** $\mathbf{Cay}_D(\Gamma, S)$

Input: A finite group $\Gamma = \{v_1, v_2, \ldots, v_m\}$ and a generating subset $S = \{\sigma_1, \sigma_2, \ldots, \sigma_{|S|}\}$. Denote the vertices and arcs of the Cayley digraph $G = \text{Cay}_D(\Gamma, S)$ by $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $E(G) = \{\vec{e_{ij}} : \vec{e_{ij}} = v_i.\sigma_j, 1 \leq i \leq m, 1 \leq j \leq |S|\}$.

Procedure: Define $f : E(G) \longrightarrow \{1, 2, \ldots, |E(G)|\}$ by

$$f(\vec{e_{ij}}) = \begin{cases} jm + 1 - i & \text{if } j \text{ is even,} \\ (j-1)m + i & \text{if } j \text{ is odd,} \end{cases}$$

for i, j with $1 \leq i \leq m$ and $1 \leq j \leq |S|$.

Output: The sum corresponding to the vertex v_i is given by

$$\sum_{j=1}^{|S|} f(\vec{e_{ij}}) = \begin{cases} \frac{m|S|^2}{2} + \frac{|S|}{2} & \text{if } |S| \text{ is even,} \\ \frac{m(|S-1|)^2}{2} + \frac{|S|-1}{2} + (|S|-1)m + i & \text{if } |S| \text{ is odd.} \end{cases}$$

Example 3.5 We illustrate Algorithm 3 for $\mathrm{Cay}_D(D_6, \{r, r^2, s\})$. Actually we exhibit a $(0, 1)$-vertex-antimagic labeling of $\mathrm{Cay}_D(D_6, \{r, r^2, s\})$ in Figure 3.

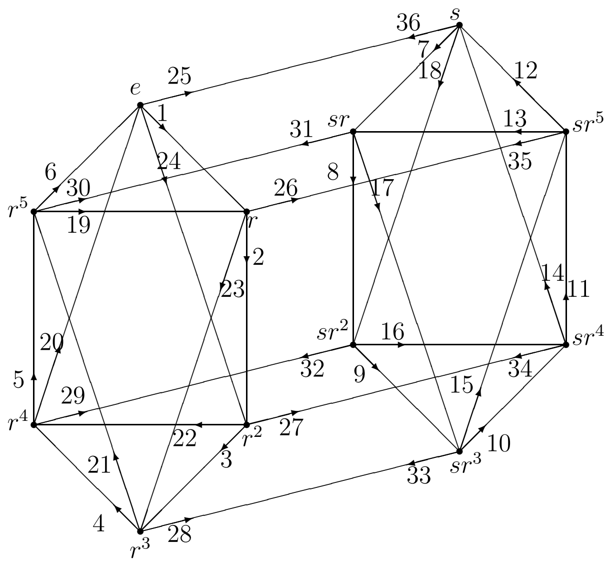

Figure 3

In view of Algorithm 3, we have the following theorem.

Theorem 3.6 *Let S be a generating subset of a finite group Γ and let $G = Cay_D(\Gamma, S)$. Then the following hold:*

(i) If $|S|$ is even, G admits a $(0, 1)$-vertex-magic labeling;

(ii) If $|S|$ is odd, G admits a $(0, 1)$-vertex-antimagic labeling.

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