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Reducible and purely heterogeneous decompositions of uniform complete multigraphs into spanning trees

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Abstract

Let $K_n^{(r)}$ be the order n uniform complete multigraph with edge multiplicity r. A spanning tree decomposition of $K_n^{(r)}$ partitions its edge set into a family $\mathcal{T}$ of edge-induced spanning trees. In a purely heterogeneous decomposition $\mathcal{T}$ no trees are isomorphic. Every order n tree occurs in a fully heterogeneous decomposition $\mathcal{T}$. All trees have equal multiplicity in a balanced decomposition $\mathcal{T}$. We say that $\mathcal{T}$ is *reducible* if it has a proper subfamily $\mathcal{T}' \subset \mathcal{T}$ such that $\mathcal{T}'$ is a spanning tree decomposition of $K_n^{(s)}$ for some s with $r > s > 0$. We show that for fixed n and sufficiently large r, every decomposition of $K_n^{(r)}$ is reducible. We also show that when $n \geq 6$, $K_n^{(2)}$ has a purely heterogeneous decomposition $\mathcal{T}$ comprising a path, three trees of maximum degree $\Delta = 3$, and one for each $\Delta > 3$.

1 Heterogeneous Decompositions into Spanning Trees

This paper continues the work done in [1] to examine heterogeneous graph decomposition involving families of spanning trees. Typically the graphs we decompose have at least one edge between each pair of vertices, so they are "over-complete" multigraphs, the multiplicity of each adjacency being at least one. The focus is on decompositions of *uniform* complete multigraphs, that is, complete multigraphs in which all adjacencies have the same multiplicity.

A spanning tree decomposition of a multigraph K is a family $\mathcal{T}$ of edge-induced spanning trees that partitions the edges of K. A given multigraph K need not have any such decomposition. The *purely heterogeneous* case is when $\mathcal{T}$ is a set, so no two members of $\mathcal{T}$ are isomorphic. Typically $\mathcal{T}$ is a multiset; it is a *homogeneous* decomposition of K if all members are isomorphic, or a *heterogeneous* decomposition if at least two members are non-isomorphic. We denote by $\mathcal{T}(n)$ the family of all unlabeled trees of order n. If K has order n, a spanning tree decomposition $\mathcal{T}$ of K is *fully heterogeneous* if $\mathcal{T}(n) \subseteq \mathcal{T}$. Two spanning tree decompositions $\mathcal{T}$ and $\mathcal{T}'$ of K are *similar* if each tree in $\mathcal{T}(n)$ has the same multiplicity in $\mathcal{T}$ and in $\mathcal{T}'$, and they are *equivalent* if $\mathcal{T}$ can be transformed into $\mathcal{T}'$ by some automorphism of K.

Explanation of several of our notational conventions is appropriate. If $\mathcal{T}$ is a set or multiset and k is a positive integer, then $k\mathcal{T}$ is the multiset in which the multiplicity of each member is k times its multiplicity in $\mathcal{T}$. If G and H are two simple graphs or multigraphs, the *union* $G \cup H$ comprises one copy of each, and the copies are vertex-disjoint. In contrast, the *sum* $G + H$ comprises one copy of each, and the copies are edge-disjoint but the vertices of H are identified with distinct vertices of G (assuming the order of G is not less than the order of H). Usually the sum notation is ambiguous without further specification, because the vertex identifications can be achieved in various ways, but there is no ambiguity if G is a complete graph or, more generally, a uniform complete multigraph. In particular, if $G = H$ we write $2G$ for the union and $G^{(2)}$ for the sum with corresponding vertices identified. More generally, if k is any positive integer, then kG is the union of k copies of G, and $G^{(k)}$ is the sum of k copies of G with corresponding vertices identified. (We read $G^{(k)}$ as

"G, k-fold".) Again, the *difference* $G-H$ is defined when G and H are simple graphs or multigraphs, with H a subgraph or submultigraph of G: the vertices of $G-H$ are those of G, and the multiplicity of any adjacency in $G-H$ is its multiplicity in G reduced by the corresponding multiplicity in H.

The complete multigraph $K_n^{(r)}$ is the order n uniform complete multigraph with all adjacencies of multiplicity r. The following two fully heterogeneous decompositions involving particular complete multigraphs are presented in [2], together with some oriented analogs:

Theorem 1.1. *The complete multigraph* $K_6^{(2)}$ *can be decomposed into* $\mathcal{T}(6)$*, one copy of each of the six trees of order* 6.

Theorem 1.2. *The complete multigraph* $K_4^{(2)}$ *can be decomposed into* $2\mathcal{T}(4)$*, two copies of each of the two trees of order* 4.

In [1], we studied the fully heterogeneous problem for K_5^2 and K_5^4. The results are summarized in the following:

Theorem 1.3. *[1] The complete multigraph* $K_5^{(2)}$ *has exactly* 24 *inequivalent fully heterogeneous spanning tree decompositions. There are* 11 *of type* $[2,2,1]$*, and* 13 *of type* $[3,1,1]$*. Those of type* $[2,2,1]$ *comprise two in the similarity class* $(2,1,2)$*, and nine in the similarity class* $(1,2,2)$*. Those of type* $[3,1,1]$ *comprise six in the similarity class* $(1,3,1)$*, and seven in the similarity class* $(1,1,3)$.

Theorem 1.4. *[1] The complete multigraph* $K_5^{(4)}$ *has* 34 *similarity classes of fully heterogeneous spanning tree decompositions; the only potential classes not realized are* $(7,2,1)$ *and* $(8,1,1)$*. With* $(4,5,1)$ *as the sole exception,* 33 *of the similarity classes contain reducible decompositions. Of these,* 30 *contain heterogeneously reducible decompositions, and exactly nine of those contain bi-heterogeneously reducible decompositions. The three similarity classes which contain reducible decompositions, but none that is heterogeneously reducible, are* $(5,2,3)$*,* $(5,3,2)$ *and* $(5,4,1)$.

These are the motivating paradigms for the present paper. Here we investigate the possibility of various decompositions like these two, for the most part involving uniform complete multigraphs of order 5, but subsequently looking at some order n general decomposition results. We are planning a sequel paper to discuss the rich decomposition results in the oriented case.

2 Reducible Spanning Tree Decompositions of $K_n^{(r)}$

We now consider reducibility of spanning tree decompositions in a more general context. Let n and r be any positive integers such that there exists a spanning tree decomposition $\mathcal{T}$ of $K_n^{(r)}$. We say that $\mathcal{T}$ is *reducible* if it has a proper subfamily $\mathcal{T}' \subset \mathcal{T}$ such that $\mathcal{T}'$ is a spanning tree decomposition of $K_n^{(s)}$ for some s with $r > s > 0$; such a subfamily $\mathcal{T}'$ is a *reduction* of $\mathcal{T}$. If $\mathcal{T}'$ is a reduction of $\mathcal{T}$, and $\mathcal{T}(n) \subseteq \mathcal{T}'$, then $\mathcal{T}'$ is a *fully heterogeneous reduction* of $\mathcal{T}(n)$. We shall prove that if n is fixed and

r is sufficiently large, every spanning tree decomposition $\mathcal{T}$ of $K_n^{(r)}$ is reducible, and every fully heterogeneous $\mathcal{T}$ has a fully heterogeneous reduction.

We begin with some notation and terminology required in the proof of the key lemma. For any integer $n \geq 1$, we denote the interval $\{i \in \mathbb{Z} : 1 \leq i \leq n\}$ by $[1, \cdots, n]$. Let $(\mathbb{Z}^+)^n$ be the *dominance poset* on n-tuples of non-negative integers, with the *dominance* partial ordering: if $\alpha = [a_1, \cdots, a_n]$, $\beta = [b_1, \cdots, b_n] \in (\mathbb{Z}^+)^n$ then $\alpha > \beta$ provided $a_i \geq b_i$ for each $i \in [1, \cdots, n]$ and strict inequality holds for at least one i. As usual, a non-empty subset $A \subset (\mathbb{Z}^+)^n$ is an *independent set* (or antichain) if no member of A dominates any other member of A.

Lemma 2.1. *Every independent subset of* $(\mathbb{Z}^+)^n$ *is finite.*

Proof. Let A be any non-empty independent subset of $(\mathbb{Z}^+)^n$. Choose any $\delta = [d_1, \cdots, d_n] \in (\mathbb{Z}^+)^n$. For any non-empty subset $S \subseteq [1, \cdots, n]$, define the subset

$$A_S(\delta) = \{\alpha = [a_1, \cdots, a_n] \in A : a_i = d_i \text{ for each } i \in [1, \cdots, n] \backslash S\} \subseteq A.$$

We call $A_S(\delta)$ the *k-parameter subset* of A determined by δ and S, where $k = |S|$. We shall prove that the k-parameter subsets of A are finite, for each $k \in [1, \cdots, n]$. But A is an n-parameter subset of itself, namely $A_{[1,\cdots,n]}(\delta) = A$ for any $\delta \in (\mathbb{Z}^+)^n$, so it will follow that A is finite.

Fix $\alpha \in A$. If S is a singleton, say $S = \{r\}$, then α is a member of the 1-parameter subset $A_r(\alpha)$, where we write r to denote the subscript $\{r\}$. If $\beta = [b_1, \cdots, b_n]$ is any other member of $A_r(\alpha)$, then $a_r \neq b_r$ so either $a_r > b_r$ or $b_r > a_r$. Then $\alpha > \beta$ or $\beta > \alpha$, contradicting the independence of A. Hence $A_r(\alpha) = \{\alpha\}$. Now fix some $k \in [1, \cdots, n]$ with $k < n$, and suppose every k-parameter subset of A is finite. Choose any $S \subseteq [1, \cdots, n]$ with $|S| = k + 1$. Clearly $\alpha \in A_S(\alpha)$. If $A_S(\alpha) \backslash \{\alpha\}$ is non-empty, choose any other member $\beta \in A_S(\alpha)$. Since β is independent of α, there is at least one $r \in S$ such that $b_r < a_r$. For any $s \in \mathbb{Z}^+$ let $\alpha(r, s) \in (\mathbb{Z}^+)^n$ be the n-tuple derived from α by replacing its rth entry a_r by s. Then

$$\beta \in A_{S \backslash \{r\}}(\alpha(r, b_r)).$$

Clearly $\alpha \in A_{S \backslash \{r\}}(\alpha(r, b_r))$, since $\alpha(r, a_r) = \alpha$, so it follows without exception that

$$A_S(\alpha) = \bigcup_r \bigcup_s \{A_{S \backslash \{r\}}(\alpha(r, s)) : r \in S, s \in [0, \cdots, a_r]\}.$$

Since $A_{S \backslash \{r\}}(\alpha(r, s))$ is a k-parameter subset of A, our hypothesis ensures that it is finite. Moreover, there are only finitely many choices for r and s, so $A_S(\alpha)$ is also finite. We note that if $\alpha \in A_S(\delta)$, then $A_S(\delta) = A_S(\alpha)$ for any S and δ. It follows inductively that each k-parameter subset of A is finite, for every $k \in [1, \cdots, n]$. In particular, $A_{[1,\cdots,n]}(\alpha) = A$ is finite. □

Note that even though all independent subsets of $(\mathbb{Z}^+)^n$ are finite, when $n \geq 2$ there is no upper bound on their cardinality. For instance, for any $m \geq 0$ the set $A = \{(x, y) \in (\mathbb{Z}^+)^2 : x + y = m\}$ is an independent subset of $(\mathbb{Z}^+)^2$ with cardinality $|A| = m + 1$, which can be arbitrarily large.

Theorem 2.2. *For any given $n \geq 2$, there are integers $r_0(n)$ and $r_1(n)$ such that if $r > r_0(n)$ then every spanning tree decomposition of $K_n^{(r)}$ is reducible, and if $r > r_1(n)$ then every fully heterogeneous spanning tree decomposition of $K_n^{(r)}$ is heterogeneously reducible.*

Proof. By Cayley's Theorem, there are $N = n^{n-2}$ labeled trees of order n. Let $\mathcal{T}^*(n)$ be the set of these labeled trees, and let $f : \mathcal{T}^*(n) \to [1, \cdots, N]$ be a bijective numbering of them. Let $\mathcal{T}$ be any spanning tree decomposition of a labeled complete multigraph $K_n^{(k)}$, for some $k \geq 1$. With $\mathcal{T}$ we associate the N-tuple $\alpha = [a_1, \cdots, a_N] \in (\mathbb{Z}^+)^N$ such that $a_{f(T)}$ is the multiplicity of $T \in \mathcal{T}$, for each $T \in \mathcal{T}^*(n)$. Let $D \subset (\mathbb{Z}^+)^N$ be the set of all N-tuples corresponding to spanning tree decompositions of labeled complete multigraphs of order n. If $\alpha, \beta \in D$ correspond to decompositions $\mathcal{T}$ and $\mathcal{T}'$, then $\alpha > \beta$ if and only if $\mathcal{T} \supset \mathcal{T}'$, so $\mathcal{T}$ is reducible if and only if α dominates some nonzero member corresponding to an irreducible decomposition of D. Let $D^* \subset D$ be the subset corresponding to all irreducible decompositions. Then D^* is an independent subset of $(\mathbb{Z}^+)^N$, so D^* is finite, by Lemma 2.1. Hence there are only finitely many irreducible spanning tree decompositions of labeled complete multigraphs of order n. Again, let $H \subset D$ be the set of all N-tuples corresponding to fully heterogeneous spanning tree decompositions of labeled complete multigraphs of order n, and let $H^* \subset H$ be the subset corresponding to all such decompositions which are heterogeneously irreducible. Then H^* is an independent subset of $(\mathbb{Z}^+)^N$, so is finite, by Lemma 2.1. The theorem follows. ☐

In view of Theorem 2.2 we define $r_0(n)$ to be the largest integer r such that $K_n^{(r)}$ has an irreducible spanning tree decomposition, and $r_1(n)$ to be the largest integer r such that $K_n^{(r)}$ has a fully heterogeneous spanning tree decomposition that is heterogeneously irreducible. Evidently $r_0(2) = r_1(2) = 1$ and $r_0(3) = r_1(3) = 2$. What is the case for $n \geq 4$? Note that Theorem **??** implies $r_0(5) \geq 6$ and $r_1(5) \geq 6$.

3 Purely Heterogeneous Spanning Tree Decompositions of $K_n^{(2)}$

To conclude, let us consider decompositions which use any tree at most once. There are fewer than n distinct trees of order n when $2 \leq n \leq 5$, so in those cases $K_n^{(2)}$ has no purely heterogeneous spanning tree decomposition. However $|\mathcal{T}(6)| = 6$, and $K_6^{(2)}$ does have a purely heterogeneous decomposition into $\mathcal{T}(6)$, guaranteed by Theorem 1.1. In what follows we shall show this is the starting point for purely heterogeneous spanning tree decompositions of $K_n^{(2)}$ for every $n \geq 6$.

As detailed in the proof of the following lemma, Fig. 6 shows a purely heterogeneous spanning tree decomposition of $K_8^{(2)}$, with the special property that pruning vertex 7 yields a purely heterogeneous decomposition of $K_7^{(2)}$, then pruning vertex 6 leaves our initial purely heterogeneous decomposition of $K_6^{(2)}$.

Lemma 3.1. *For $6 \leq n \leq 8$, there is a purely heterogeneous spanning tree decomposition of $K_n^{(2)}$.*

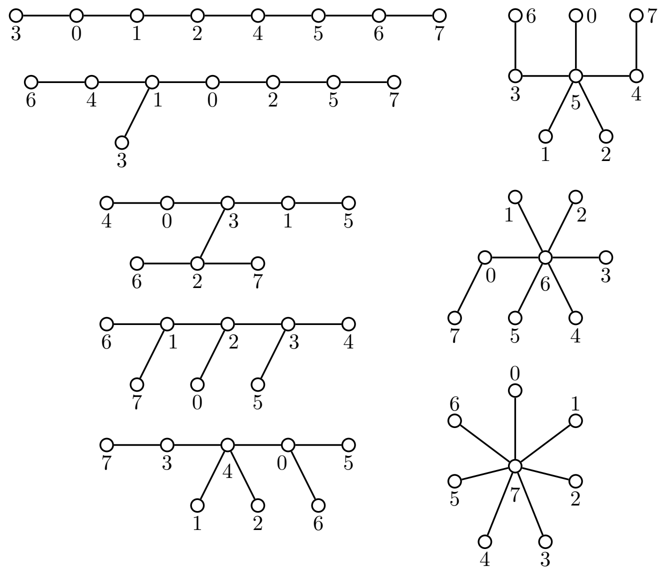

Figure 1: A special purely heterogeneous decomposition of $K_8^{(2)}$.

Proof. We begin with the decomposition of $K_6^{(2)}$ into $\mathcal{T}(6)$ from [2], and apply the permutation (05314) to its vertex labels to obtain a labeled decomposition $\mathcal{T}$ better adapted to our present needs. Fig. 6 shows how $\mathcal{T}$ can be extended to a purely heterogeneous spanning tree decomposition $\mathcal{T}'$ of $K_7^{(2)}$ by adding a suitable adjacency from each tree in $\mathcal{T}$ to the new vertex 6, and adjoining a star centered on 6. Fig. 6 also shows how $\mathcal{T}'$ extends to a purely heterogeneous decomposition $\mathcal{T}''$ of $K_8^{(2)}$ by adding a suitable adjacency from each tree in $\mathcal{T}'$ to the new vertex 7, and adjoining a star centered on 7. The lemma follows. $\square$

For the next lemma it is convenient to adapt the standard notation $\Delta(G)$ for the maximum degree in any graph G, by using $n_\Delta(G)$ to denote the number of vertices of maximum degree in G.

Lemma 3.2. *For $n \geq 8$, a purely heterogeneous spanning tree decomposition $\mathcal{T}^* = \{T_i : 0 \leq i < n\}$ exists for the complete multigraph $K_n^{(2)}$, in which the trees are distinguished by their vertices of maximum degree: $\Delta(T_0) = 2$ with $n_\Delta(T_0) = n-2$; $\Delta(T_1) = \Delta(T_2) = \Delta(T_3) = 3$ with $n_\Delta(T_1) = 1$, $n_\Delta(T_2) = 2$ and $n_\Delta(T_3) = 3$; and $\Delta(T_i) = i$ with $n_\Delta(T_i) = 1$, for $4 \leq i < n$.*

Proof. For some $n \geq 8$, label the vertices of $K_n^{(2)}$ with the elements of $\mathbb{Z}_n$, and let $\mathcal{T}^*(n) = \{T_i(n) : 0 \leq i < n\}$ be a purely heterogeneous decomposition of $K_n^{(2)}$ into n spanning trees with the following six properties:

(a) the tree $T_0(n)$ is a path;

(b) the trees $T_1(n)$, $T_2(n)$ and $T_3(n)$ have maximum degree 3, and the sets of vertices at which they attain degree 3 are $\{1\}$, $\{2,3\}$ and $\{1,2,3\}$ respectively;

(c) for $4 \le i < n$, the maximum degree of $T_i(n)$ is i;

(d) each $T_i(n)$ attains its maximum degree at the vertex i, and this is its unique vertex of maximum degree if $4 \le i < n$;

(e) for $0 \le i < 4$, each vertex in the set $\{n-j-1 : 0 \le j \le i\}$ is a leaf of $T_i(n)$;

(f) for $4 \le i < n$, the set $A_i(n) = \{j : 0 \le j < n-4, \deg(j) < i-1 \text{ in } T_i(n)\}$ has cardinality $a_i(n) \ge i-3$.

When $n = 8$, the purely heterogeneous decomposition of $K_8^{(2)}$ in Fig. 6 satisfies all these requirements. Indeed, it has $a_4(8) = 3$ and $a_5(8) = a_6(8) = a_7(8) = 4$.

From $\mathcal{T}^*(n)$ we shall grow a family of trees $\mathcal{T}^*(n+1) = \{T_i(n+1) : 0 \le i \le n\}$, each of order $n+1$. For $0 \le i < n$, the tree $T_i(n+1) \in \mathcal{T}^*(n+1)$ will be grown from the tree $T_i(n) \in \mathcal{T}^*(n)$ by adjoining a new vertex n and an edge between n and a suitably chosen vertex of $T_i(n)$. To specify how to choose the vertex of $T_i(n)$ to be adjacent to n, recursively define a bijection $f_n : \mathbb{Z}_n \to \mathbb{Z}_n$, by setting $f_n(i) = n-i-1$ for $0 \le i < 4$, and for $4 \le i < n$ requiring $f_n(i) = \max A_i(n) \backslash B_i(n)$, where $B_i(n) = \{f_n(j) : 0 \le j < i\}$. We claim that f_n is well-defined. The first four values of f_n are explicitly specified. Suppose $B_k(n)$ is specified for some $k \ge 4$; it contains only $k-4$ values less than $n-4$, whereas property (f) ensures that there are $a_k(n) \ge k-3$ potential values for $f_n(k)$ available in $A_k(n)$, so at least one choice is available in $A_k(n) \backslash B_k(n)$ to be assigned as the value for $f_n(k)$. It follows inductively that f_n is well-defined. Now we produce $\mathcal{T}^*(n+1)$, as follows. The tree $T_n(n+1)$ is the star on $\mathbb{Z}_{n+1}$, centered on n. For $0 \le i < n$, the tree $T_i(n+1)$ results from copying $T_i(n)$, reinterpreting the copied vertex labels as coming from $\mathbb{Z}_{n+1}$, and adding vertex n and an edge between $f_n(i)$ and n. Thus n is a leaf of every $T_i(n+1)$ with $0 \le i < n$. In particular, $T_0(n+1)$ is a path, and n is not adjacent to any maximum degree vertex in $T_i(n+1)$ when $1 \le i < n$, so it is routine to verify that $\mathcal{T}^*(n+1)$ has properties (a)-(e).

Assume that $\mathcal{T}^*(n+1)$ has been built iteratively from the decomposition in Fig. 6. Let us check that it also has property (f). For $4 \le i < n$ the vertex n is adjacent in $T_i(n+1)$ to at most one vertex in $A_i(n)$, so $A_i(n+1)$ contains all but at most one of the vertices in $A_i(n)$. Moreover, $n-4$ is an end vertex in $T_i(n+1)$ when $i \ne n-4$, and then $A_i(n+1) \backslash A_i(n) = \{n-4\}$. Hence $a_i(n+1) \ge a_i(n)$ when $i \ne n-4$, and $a_{n-4}(n+1) \ge a_{n-4}(n) - 1$. Then $a_4(8) = 3$ ensures $a_4(n) \ge 2$ if $n \ge 9$, so we certainly have $a_4(n) \ge 1$ for $n \ge 8$. Again, $a_5(8) = 4$ ensures $a_5(9) \ge 4$ and $a_5(n) \ge 3$ if $n \ge 10$, so we certainly have $a_5(n) \ge 2$ for $n \ge 8$. Similarly $a_6(8) = 4$ ensures $a_6(n) \ge 3$ if $n \ge 8$. Now the star $T_{n-1}(n)$ has $A_{n-1}(n) = \{j : 0 \le j < n-4\}$ so $a_{n-1}(n) = n-4$, and adding a new leaf gives $a_{n-1}(n+1) = n-3$, for $n \ge 8$. Hence we certainly have $a_i(n) \ge i-3$ for $7 \le i < n$ and $n \ge 8$. So (f) holds.

Note that $\mathcal{T}^*(n+1)$ is a purely heterogeneous spanning tree decomposition of $K_{n+1}^{(2)}$. Every member of $\mathcal{T}^*(n+1)$ has order $n+1$, and is a star or else is formed by adding a new leaf to a tree of order n. It is a decomposition of $K_{n+1}^{(2)}$, since every adjacency jk with $0 \leq j < k < n$ occurs twice in $\mathcal{T}^*(n)$, and therefore twice in $\mathcal{T}^*(n+1)$, and every adjacency jn with $0 \leq j < n$ occurs once in $T_n(n+1)$ and once in $T_i(n+1)$ with $f_n(i) = j$. Properties (a)–(c) ensure that no two trees in $\mathcal{T}^*(n+1)$ are isomorphic.

Induction on n, beginning with the decomposition in Fig. 1, now completes the proof. □

Combining Lemmas 3.1 and 3.2, we have

Theorem 3.3. *There is a purely heterogeneous spanning tree decomposition of* $K_n^{(2)}$ *if and only if* $n \geq 6$.

It seems natural to conjecture that for every even multiplicity $r \geq 2$ there is a smallest integer $n_0(r)$ such that $K_n^{(r)}$ has a purely heterogeneous spanning tree decomposition for every $n \geq n_0(r)$. Theorem 3.3 shows that $n_0(2) = 6$. Suppose the conjecture is true: is it true that $n_0(r)$ is the smallest n such that $K_n^{(r)}$ has a purely heterogeneous spanning tree decomposition? If so, Theorem 3.3 would generalize to assert that there is a purely heterogeneous spanning tree decomposition of $K_n^{(r)}$ if and only if $n \geq n_0(r)$. Corresponding conjectures apply for every odd multiplicity $r \geq 1$ and sufficiently large even integers n. It is known [5] that the conjecture holds for simple graphs ($r = 1$), with $n_0(1) = 6$. Indeed, a decomposition akin to that in Lemma 3.2 is known [3].

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