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Edge growth in graph powers

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Abstract

For a graph G, its rth power G^r is defined as the graph with the same vertex set as G, and an edge between any two vertices whenever they are within distance r of each other in G. Motivated by a result from additive number theory, Hegarty raised the question of how many new edges G^r has when G is a regular, connected graph with diameter at least r. We address this question for $r \neq 3, 6$. We give a lower bound for the number of edges in the rth power of G in terms of the order of G and the minimal degree of G. As a corollary, for $r \neq 3, 6$, we determine how small the ratio $e(G^r)/e(G)$ can be for regular, connected graphs of diameter at least r.

1 Introduction

The rth power of G, denoted G^r, is the graph with vertex set $V(G)$, and xy an edge whenever x and y are within distance r of each other. Consider the following question "How many new edges are added to a graph G by taking its rth power?" If G is a complete graph then G^r doesn't have any new edges. Therefore it is natural to place additional restrictions on G. The diameter of a connected graph G, denoted $\mathrm{diam}(G)$, is defined to be the maximal distance between a pair of vertices in G (alternatively, $\mathrm{diam}(G)$ is the minimal r for which G^r is complete). One would expect that when r is smaller than $\mathrm{diam}(G)$, then G^r has substantially more edges than G. In this paper we study how many new edges are added to a graph G by taking its rth power, for G a connected graph with $\mathrm{diam}(G) \geq r$. In particular, for $r \neq 3, 6$, we determine how small the ratio $e(G^r)/e(G)$ can be for regular, connected graphs G.

The motivation for studying this comes from a corollary of the Cauchy-Davenport Theorem from additive number theory. Before we can state this corollary, we need a definition. The Cayley graph of a subset $A \subseteq \mathbb{Z}_p$ is constructed on the vertex set $\mathbb{Z}_p$.

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For two distinct vertices $x, y \in \mathbb{Z}_p$, we define xy to be an edge whenever $x - y \in A$ or $y - x \in A$. The following is a consequence of the Cauchy-Davenport Theorem (usually stated in the language of additive number theory).

Theorem 1.1 (Cauchy and Davenport, [1, 2]). *Let p be a prime, G the Cayley graph of a set $A \subseteq \mathbb{Z}_p$, and r an integer such that $r < \operatorname{diam}(G)$. Then we have*

$$\frac{e(G^r)}{e(G)} \geq r. \tag{1}$$

One could ask whether inequalities similar to (1) hold for more general families of graphs. Motivated by the fact that Cayley graphs are regular, Hegarty asked this question for regular graphs and proved the following theorem.

Theorem 1.2 (Hegarty, [7]). *Let G be a regular, connected graph, satisfying $\operatorname{diam}(G) \geq 3$. Then we have*

$$\frac{e(G^3)}{e(G)} \geq 1 + \epsilon, \tag{2}$$

where $\epsilon \approx 0.087$.

The constant ϵ has since been improved to 1/6 by the author [8] and to 3/4 by DeVos and Thomassé [4]. The value $\epsilon = 3/4$ is optimal in the sense that there exists a sequence of regular graphs of diameter greater than 3, G_m, satisfying $e(G_m^3)/e(G_m) \to 7/4$ as $m \to \infty$ [4].

Hegarty also asked what happens for other powers of G. For G^2, Hegarty showed that no inequality similar to (2) can hold for regular graphs in general, by exhibiting a sequence of regular, connected graphs of diameter greater than 2, G_m, satisfying $e(G_m^2)/e(G_m) \to 1$ as $m \to \infty$ [7]. Goff [6] studied the 2nd power of regular graphs further. He showed that for any d-regular graph connected graph G such that $\operatorname{diam}(G) > 2$, we have $e(G^2)/e(G) \geq 1 + \frac{3}{2d} - o\left(\frac{1}{d}\right)$. For general d-regular connected graphs G with $\operatorname{diam}(G) > 2$, he showed that the $\frac{3}{2d}$ term in this result cannot be replaced with λ/d for any $\lambda > \frac{3}{2}$. However he showed that with the exception of two families of exceptional graphs, we have $e(G^2)/e(G) \geq 1 + \frac{2}{d} - o\left(\frac{1}{d}\right)$ for all d-regular connected graphs with $\operatorname{diam}(G) > 2$.

The goal of this paper is to deal with the case when $r \geq 4$. In particular, for $r \neq 3, 6$, we find how small the ratio $e(G^r)/e(G)$ can be for a regular, connected graph G of diameter at least r.

The requirement of G being regular in Theorem 1.2 is quite restrictive. DeVos and Thomassé noticed that it is possible to remove this assumption, and bound $e(G^3)$ in terms of the order of G and the minimum degree of G. They proved the following theorem.

Theorem 1.3 (DeVos and Thomassé, [4]). *Let G be a connected graph of minimum degree $\delta(G)$ and satisfying $\operatorname{diam}(G) \geq 3$. Then we have*

$$e(G^3) \geq \frac{7}{8}\delta(G)|G|. \tag{3}$$

When G is regular, the above theorem immediately implies Theorem 1.2 with the optimal constant of $\epsilon = 3/4$. The main theorem which we shall prove in this paper is a generalisation of Theorem 1.3 to higher powers of G.

Theorem 1.4. *Suppose that* $r \neq 6$. *Let* G *be a connected graph satisfying* $\operatorname{diam}(G) \geq r$ *and having minimum degree* $\delta(G)$.

- *If* $r \equiv 0 \pmod 3$, *then we have*

$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right)\delta(G)|G|.$$

- *If* $r \not\equiv 0 \pmod 3$, *then we have*

$$e(G^r) \geq \frac{1}{2}\left\lceil \frac{r}{3} \right\rceil \delta(G)|G|.$$

The case $r = 3$ of Theorem 1.4 is due to DeVos and Thomassé [4], and will not be proved here. Applying Theorem 1.4 to regular graphs gives the following corollary.

Corollary 1.5. *Suppose that* $r \neq 6$. *Let* G *be a connected, regular graph, and* r *a positive integer such that* $\operatorname{diam}(G) \geq r$.

- *If* $r \equiv 0 \pmod 3$, *then we have*

$$\frac{e(G^r)}{e(G)} \geq \frac{r+3}{3} - \frac{3}{2(r+3)}.$$

- *If* $r \not\equiv 0 \pmod 3$, *then we have*

$$\frac{e(G^r)}{e(G)} \geq \left\lceil \frac{r}{3} \right\rceil.$$

Corollary 1.5 gives a lower bound on the ratio $e(G^r)/e(G)$ for regular graphs. The bounds on $e(G^r)/e(G)$ in Corollary 1.5 are optimal in the following sense. For each r, there exists a sequence of regular, connected graphs of diameter at least r, G_m, such that $e(G_m^r)/e(G_m)$ tends to the bound given by Corollary 1.5 as m tends to infinity. We will give a construction of such sequences in Section 3.

The structure of this paper is as follows. In Section 2 we define some notation and prove Theorem 1.4. In Section 3, we construct sequences of regular graphs which show that the bounds on $e(G^r)/e(G)$ in Corollary 1.5 are optimal. In Section 4 we make some remarks about the case when $r = 6$, as well as some open problems in this area.

2 Proofs

In this section we prove Theorem 1.4.

Although Theorem 1.4 is a theorem about loopless graphs, in this section we will also consider graphs which may contain loops. This is because the proof of our results is more natural in this setting.

We will denote graphs which may contain loops by curly letters such as "$\mathcal{G}$". Graphs with loops explicitly forbidden are donoted by Roman letters such as "G". For two vertices x and y (possibly $x = y$) we only ever allow one edge between x and y. The neighbourhood of a vertex x, $N(x)$, is defined as the set of vertices adjacent to x. (If there is a loop at x, then $N(x)$ will contain x itself.) The degree of x is $|N(x)|$. Notice that this ensures that a loop is counted only once in the degree of a vertex. The minimal degree of a graph, taken over all vertices in $\mathcal{G}$ is denoted by $\delta(\mathcal{G})$. For graphs with loops allowed, $\mathcal{G}^r$ is again defined to be the graph with vertex set $V(\mathcal{G})$, and xy an edge whenever x and y are within distance r of each other in $\mathcal{G}$. Notice that this definition implies that $\mathcal{G}^r$ always has a loop at each vertex (since for any vertex v we have $d(v, v) = 0$). For two sets of vertices X and Y, let $d(X, Y)$ denote the length of a shortest path between a vertex in X and a vertex in Y. If X is a set of vertices, let $N^r(X)$ be the set of vertices at distance at most r from X. We abbreviate $N^r(\{x\})$ as $N^r(x)$. Notice that since $\mathcal{G}^r$ has a loop at every vertex, we always have $e(\mathcal{G}^r) = \frac{1}{2}\sum_{v\in V(\mathcal{G})}(|N^r(v)| + 1)$. For all other notation, we refer to [5].

We will prove the following theorem, and then deduce Theorem 1.4 as a corollary.

Theorem 2.1. *Let $\mathcal{G}$ be a connected graph, and r a positive integer such that $r \neq 3, 6$ and* $\mathrm{diam}(\mathcal{G}) \geq r$.

- *If $r \equiv 0 \pmod 3$, then we have*

$$e(\mathcal{G}^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right)\delta(\mathcal{G})|\mathcal{G}| + \frac{1}{2}|\mathcal{G}|.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$e(\mathcal{G}^r) \geq \frac{1}{2}\left\lceil\frac{r}{3}\right\rceil\delta(\mathcal{G})|\mathcal{G}| + \frac{1}{2}|\mathcal{G}|.$$

The basic strategy of the proof is simple—for each vertex v, we show that $|N^r(v)|$ is large, thereby showing that $e(\mathcal{G}^r) = \frac{1}{2}\sum_{v\in V(\mathcal{G})}(|N^r(v)| + 1)$ is large as well. When $r \not\equiv 0 \pmod 3$, this is an easy task—it will turn out that in this case each vertex in v satisfies $|N^r(v)| \geq \lceil r/3\rceil\,\delta$. When $r \equiv 0 \pmod 3$, the proof is more complicated. In that case, we will show that a large proportion of the vertices of $\mathcal{G}$ satisfy $|N^r(v)| \geq (r/3 + 1)\delta$, which in turn will imply the bound in Theorem 2.1. This is the same general strategy as the one used by DeVos and Thomassé [4] in the proof of Theorem 1.3. However, many of our intermediate steps are different from their proof.

Proof of Theorem 2.1. For convenience, we will set $\delta = \delta(\mathcal{G})$. If P is a path between two vertices x and y, we say that P is a *geodesic* if the length of P is $d(x, y)$. The notion of a geodesic is useful because the neighbourhood of a geodesic must be quite large. This is quantified in the following claim.

Claim 2.2. *Let P be a length k geodesic. Then $|N(P)| \geq \left(\left\lfloor \frac{k}{3} \right\rfloor + 1\right) \delta$ holds.*

Proof. Let $x_0,\ x_1, \ldots, x_k$ be the vertices of P (in the order in which they occur along the path). Notice that $N(x_0),\ N(x_3), \ldots, N(x_{3\lfloor \frac{k}{3} \rfloor})$ must all be disjoint, since otherwise we could find a shorter path between x_0 and x_k. The sets $N(x_0),\ N(x_3), \ldots, N(x_{3\lfloor \frac{k}{3} \rfloor})$ must also be contained in $N(P)$, and each have order at least δ. This implies the result. $\square$

We now prove the theorem in the case when $r \not\equiv 0 \pmod 3$.

The diameter of $\mathcal{G}$ is at least r, so $\mathcal{G}$ must contain a length r geodesic, P. Claim 2.2 implies that the following holds:

$$|\mathcal{G}| \geq |N(P)| \geq \left(\left\lfloor \frac{r}{3} \right\rfloor + 1\right) \delta = \left\lceil \frac{r}{3} \right\rceil \delta. \tag{4}$$

Since $\mathcal{G}^r$ contains a loop at every vertex, we have $e(\mathcal{G}^r) = \sum_{v \in V(\mathcal{G})} \left(\frac{1}{2}|N^r(v)| + \frac{1}{2}\right)$. Thus to prove Theorem 2.1 it is sufficent to exhibit $\left\lceil \frac{r}{3} \right\rceil \delta$ elements of $N^r(v)$ for each vertex $v \in V(G)$.

Suppose that there exists a length $r-1$ geodesic P_v starting from a vertex v. Then $N(P_v)$ is contained in $N^r(v)$, giving

$$|N^r(v)| \geq |N(P_v)| \geq \left(\left\lfloor \frac{r-1}{3} \right\rfloor + 1\right) \delta = \left\lceil \frac{r}{3} \right\rceil \delta.$$

The second inequality is an application of Claim 2.2.

Suppose that all the vertices in $\mathcal{G}$ are within distance $r-1$ of v. In this case we have $N^r(v) = V(\mathcal{G})$, which is of order at least $\left\lceil \frac{r}{3} \right\rceil \delta$ by (4). This completes the proof of the case "$r \not\equiv 0 \pmod 3$" of the theorem.

For the rest of the proof fix r such that $r \equiv 0 \pmod 3$ and $r \geq 9$.

If v is a vertex of $\mathcal{G}$, we say that v is *sufficient* if $|N^r(v)| \geq \left(\frac{r}{3} + 1\right) \delta$. Otherwise we say that v is *insufficient*.

The following is a useful property of insufficient vertices.

Claim 2.3. *Let v be an insufficient vertex. Then there is some vertex at distance $r+1$ from v.*

Proof. Since $\operatorname{diam}(\mathcal{G}) \geq r$, Claim 2.2 implies that $|\mathcal{G}| \geq \left(\frac{r}{3} + 1\right) \delta$. Since v is insufficient, we have $|N^r(v)| < \left(\frac{r}{3} + 1\right) \delta$, and so v cannot be within distance r from all the vertices in the graph. $\square$

The following three claims will allow us to bound the number of insufficient vertices in $\mathcal{G}$.

Claim 2.4. *If $2 < d(x, y) < r$ holds for $x, y \in V(\mathcal{G})$, then either x or y is sufficient.*

Proof. Suppose that x is insufficient. By Claim 2.3, we can find a length r geodesic starting from x with vertex sequence x, x_1, $x_2, \dots, x_r$.

Suppose that $N(y) \cap N(x_i) \neq \emptyset$ for some i with $3 \leq i \leq r-3$. In this case $N(x)$, $N(x_3)$, $N(x_6), \dots, N(x_r)$ are all contained in $N^r(y)$. There are $\frac{r}{3}+1$ of these, they are all disjoint (since x, x_1, $x_2, \dots, x_r$ form a geodesic), and are of order at least δ. Hence y is sufficient.

Otherwise $N(y) \cap N(x_i) = \emptyset$ for all $r \leq i \leq r-3$. In this case $N(x)$, $N(y)$, $N(x_3)$, $N(x_6), \dots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. This contradicts our initial assumption that x is insufficient. □

Claim 2.5. *Let x and y be two vertices in $\mathcal{G}$ such that $r \leq d(x, y) \leq r+2$. If there exists a vertex $z \in \mathcal{G}$ such that $d(z, x) \geq r-1$ and $d(z, y) \geq r-1$, then either x or y is sufficient.*

Proof. Choose any z in $N^{r-1}(\{x, y\}) \setminus N^{r-2}(\{x, y\})$. This set is nonempty by the second assumption of the claim. We will have $d(z, x), d(z, y) \geq r-1$ and either $d(z, x)$ or $d(z, y) = r-1$. Without loss of generality assume that $d(z, x) = r-1$ and $d(z, y) \geq r-1$.

We will show that x is sufficient. Let x, $x_1, \dots, x_{d(x,y)-1}$, y be a geodesic between x and y. For $i = 1, \dots, d(x, y)-1$, the triangle inequality implies that

$$d(x, z) - i = d(x, z) - d(x, x_i) \leq d(x_i, z), \tag{5}$$

$$d(y, z) - d(x, y) + i = d(y, z) - d(y, x_i) \leq d(x_i, z). \tag{6}$$

Averaging (5) and (6), and using the inequalities $d(z, x), d(z, y) \geq r-1$ and $d(x, y) \leq r+2$ gives

$$\frac{r-4}{2} \leq d(x_i, z). \tag{7}$$

Using $r \geq 9$ and the fact that $d(x_i, z)$ is an integer, (7) implies that $d(z, x_i) \geq 3$ for all i. Hence $N(x)$, $N(z)$, $N(x_3)$, $N(x_6), \dots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. Hence x is sufficient. □

Claim 2.6. *If $d(x, y) = r$ holds for $x, y \in V(\mathcal{G})$, then either x or y is sufficient.*

Proof. Suppose that x and y are insufficient. By Claim 2.3 there exists $z \in V(\mathcal{G})$ such that $d(x, z) = r+1$. Let x, $x_1, \dots, x_{r-1}, y$ be a geodesic between x and y. Since x and y are insufficient, Claim 2.5 implies that we have $d(z, y) < r-1$. Note that $d(x, z) = r+1$ implies that $N(z) \cap N(x_i) = \emptyset$ for all $i \leq r-2$. Thus $N(z)$, $N(x_1)$, $N(x_4), \dots, N(x_{r-2})$ are all disjoint and contained in $N^r(y)$. This contradicts our assumption that y is insufficient. □

Let X be the set of insufficient vertices in $\mathcal{G}$. We define an equivalence relation "$\sim$" on X by letting $x \sim y$ if $d(x, y) \leq 2$. For $r \geq 9$, Claim 2.4 implies that this is an equivalence relation. Let $X_1, \dots, X_l$ be the equivalence classes of "$\sim$".

The following claim gives a lower bound on the order of $\mathcal{G}$.

Claim 2.7. $|\mathcal{G}| \geq \left(\frac{r+3}{6}\right) \delta l$.

Proof. First suppose that $l \leq 2$. Let P be a geodesic in G of length $\mathrm{diam}(\mathcal{G}) \geq r$. Then Claim 2.2 implied that $|\mathcal{G}| \geq |N(P)| \geq (r+3)\delta/3$, proving the result.

Thus we can suppose that $l \geq 3$. Claims 2.4 and 2.6 imply that $d(X_i, X_j) \geq r+1$ for all $i \neq j$.

If we had $d(X_i, X_j) \leq r+2$ for some i and j, then Claim 2.5 would imply that we have $d(X_i, z) < r-1$ or $d(X_j, z) < r-1$ for all $z \in V(\mathcal{G})$. Then, Claim 2.4 would imply that all the vertices outside of X_i and X_j are sufficient, contradicting the assumption that $l \geq 3$.

Therefore we can suppose that $d(X_i, X_j) \geq r+3$ for all $i \neq j$. For each i, choose x_i to be any vertex in X_i. Note that $N^{\lfloor \frac{r}{2} \rfloor}(x_i)$ contains a length $\lfloor \frac{r}{2} \rfloor$ geodesic, P. Using Claim 2.2 gives

$$\left| N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \right| \geq |N(P)| \geq \left(\left\lfloor \frac{1}{3} \left\lfloor \frac{r}{2} \right\rfloor \right\rfloor + 1 \right) \delta \geq \left(\frac{r+3}{6} \right) \delta.$$

For the last inequality we are using the fact that $r \equiv 0 \pmod 3$. Note that $d(X_i, X_j) \geq r+3$ implies that $N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \cap N^{\lfloor \frac{r}{2} \rfloor + 1}(X_j) = \emptyset$ for all i, j. This implies that the following holds:

$$|V(\mathcal{G})| \geq \sum_{i=1}^{l} \left| N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \right| \geq \left(\frac{r+3}{6} \right) \delta l.$$

□

When x is insufficient, the following claim gives a lower bound on the order of $N^r(x)$.

Claim 2.8. *Suppose that x is an insufficient vertex in the equivalence class X_i. Then, $|N^r(x)| \geq |X_i| + \frac{r}{3}\delta$ holds.*

Proof. By Claim 2.3, we can choose a length r geodesic from x. Let $x, x_1, \dots, x_r$ be the vertices of this geodesic. Suppose that $X_i \cap N(x_j)$ is nonempty for some x_j. Choose $y \in X_i \cap N(x_j)$. Clearly $j \leq 1$ must hold, since otherwise $N(x)$, $N(x_3)$, $N(x_6)$, $\dots$, $N(x_r)$ would all be contained in $N^r(y)$, contradicting that y is insufficient (since $y \in X_i$).

Hence X_i, $N(x_2)$, $N(x_5), \dots, N(x_{r-1})$ are all disjoint and contained in $N^r(x)$ proving the claim. □

Combining Claims 2.7 and 2.8 we prove the theorem.

$$
\begin{aligned}
2e(\mathcal{G}^r) - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathcal{G}| - |\mathcal{G}| &= \sum_{x\in V(\mathcal{G})} |N^r(x)| - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathcal{G}| \\
&\geq \sum_{x \text{ sufficient}} \left(\frac{r}{3}+1\right)\delta + \sum_{i=1}^{l}\left(|X_i| + \frac{r}{3}\delta\right)|X_i| \\
&\qquad - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathcal{G}| \\
&= \frac{3}{2(r+3)}\delta|\mathcal{G}| + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&\geq \frac{1}{4}\delta^2 l + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&= \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta + \frac{1}{4}\delta^2\right) \\
&= \sum_{i=1}^{l}\left(|X_i| - \frac{1}{2}\delta\right)^2 \\
&\geq 0.
\end{aligned}
$$

The first equality uses the fact that $\mathcal{G}^r$ contains a loop at every vertex, hence $2e(\mathcal{G}^r) = \sum_{x\in V(\mathcal{G})} |N^r(x)| + |\mathcal{G}|$. The first inequality follows from the definition of "sufficient vertex" and Claim 2.8. The second equality follows from the fact that there are $|\mathcal{G}| - \sum_{i=1}^{l} |X_i|$ sufficient vertices in $\mathcal{G}$. The second inequality follows from Claim 2.7. This completes the proof. $\square$

We now deduce Theorem 1.4 from Theorem 2.1.

Proof of Theorem 1.4. Let $\mathcal{G}$ be a copy of G with a loop added at every vertex. Then $\mathcal{G}^r$ will be isomorphic to G^r with a loop added at every vertex. Note that we have $e(\mathcal{G}^r) = e(G^r) + |G|$, and $\delta(\mathcal{G}) = \delta(G) + 1$. Substitute these into Theorem 2.1 to obtain the following.

- If $r \equiv 0 \pmod 3$, then we have
$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right)\delta(G)|G| + \left(\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}\right)|G|.$$

- If $r \not\equiv 0 \pmod 3$, then we have
$$e(G^r) \geq \frac{1}{2}\left\lceil\frac{r}{3}\right\rceil \delta(G)|G| + \left(\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}\right)|G|.$$

Note that for $r \geq 3$, both $\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}$ and $\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}$ are non-negative, so Theorem 1.4 follows. □

3 Extremal constructions

In this section we construct graphs which demonstrate the optimality of Theorem 1.4 and Corollary 1.5. Specifically, for each r, we will construct a sequence of regular, connected graphs of diameter at least r, G_m, such that $e(G_m^r)/e(G_m)$ tends to the bound given by Corollary 1.5 as m tends to infinity. See Figure 1 for a diagram of the sequences that we will construct. We prove the following.

Proposition 3.1. *Let r be an integer greater than 3. There exists a sequence of regular, connected graphs of diameter $\geq r$, G_m, which satisfy the following.*

- *If $r \equiv 0 \pmod 3$, then we have*

$$\lim_{m\to\infty} \frac{e(G_m^r)}{e(G_m)} = \frac{r+3}{3} - \frac{3}{2(r+3)}.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$\lim_{m\to\infty} \frac{e(G_m^r)}{e(G_m)} = \left\lceil\frac{r}{3}\right\rceil.$$

Proof. For $r \not\equiv 0 \pmod 3$, we construct the following sequence of graphs G_m. Take disjoint sets of vertices $N_0, \dots, N_r$, with $|N_i| = m-1$ if $i \equiv 1 \pmod 3$ and $|N_i| = 2$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \dots, r-1$. Add all the edges within N_i for all i. Remove a cycle passing through all the vertices in $N_1 \cup \dots \cup N_{r-1}$. It is easy to see that G_m is m-regular and of diameter r. If $r \equiv 1 \pmod 3$ then $|G_m| = \frac{1}{3}(rm + 2m + 3r - 6)$ will hold. Since G_m is m-regular, we have $e(G_m) = \frac{1}{6}(rm + 2m + 3r - 6)m$. Since G_m^r is complete, we have $e(G_m^r) = \frac{1}{18}(rm + 2m + 3r - 6)(rm + 2m + 3r - 7)$. This implies that $e(G_m^r)/e(G_m) \to \left\lceil\frac{r}{3}\right\rceil$ as $m \to \infty$. A similar calculation can be used to show that the same limit holds when $r \equiv 2 \pmod 3$.

For $r \equiv 0 \pmod 3$, we construct the following sequence of graphs G_m to show that Corollary 1.5 is optimal. Take disjoint sets of vertices $N_0, \dots, N_{r+1}$. Let $|N_0| = |N_{r+1}| = 2m+1$, $|N_i| = 1$ if $i \equiv 2 \pmod 3$, and $|N_i| = 2m$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \dots, r$. Add all the edges within N_i for all i. Delete a perfect matching from each of the sets N_1 and N_r. This will ensure that G_m is $4m$-regular and has diameter $r+1$. Note that $|G_m| = \frac{1}{3}(4rm + r + 12m + 6)$, and so we have $e(G_m) = \frac{1}{6}(4rm + r + 12m + 6)4m$. The only edges missing from G_m^r will be between N_0 and N_{r+1}, so we have $e(G_m^r) = \frac{1}{18}(4rm + r + 12m + 6)(4rm + r + 12m + 5) - (2m+1)^2$. This implies that $e(G_m^r)/e(G_m) \to \frac{r+3}{3} - \frac{3}{2(r+3)}$ as $m \to \infty$. This construction is a generalization of one from [4]. □

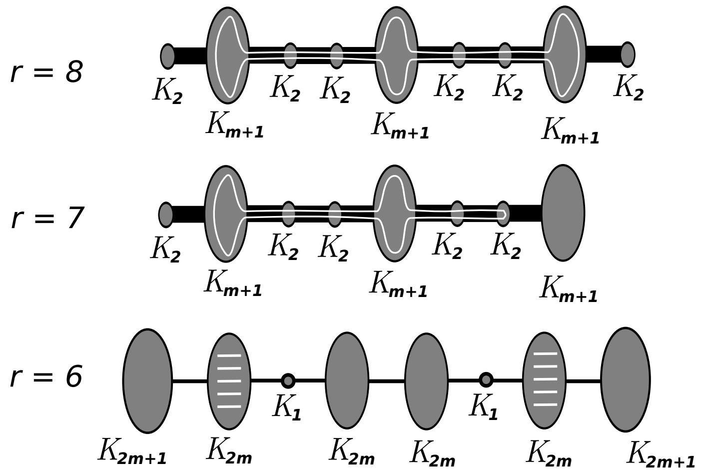

Figure 1: Graphs showing the optimality of the cases "$r = 6$," "$r = 7$," and "$r = 8$" of Corollary 1.5. The grey circles represent complete graphs of specified order. The black lines between the sets represent all the edges being present between them. The white cycle in the "$r = 7$" and "$r = 7$" cases represents a single cycle passing through all the vertices in the specified sets being removed. The white matchings in the "$r = 6$" case represent a perfect matching being removed from the specified sets.

4 Remarks

In this section we discuss some problems which are left open in this paper.

- One natural open problem is to extend the results of this paper to the case when $r = 6$. In particular it would be interesting to know if Theorem 1.4 holds for $r = 6$. It seems to be difficult to extend our proof of this theorem to the case when $r = 6$. One reason for this is that there are examples showing that Claim 2.7 does not always hold when $r = 6$. We sketch one such construction here.

  For fixed m, and $i = 1, 2, 3$, we define a set of vertices B_i of order m as well as three vertices a_i, c_i, d_i. For each i, all the edges inside B_i are present as well as the edges between B_i and $\{a_i, c_i\}$, and the edge $c_i d_i$. We add a set of vertices X of order m and add all the edges inside X and between X and $\{d_1, d_2, d_3\}$. This produces a graph G_m, with minimum degree $m+2$ and order $4m + 9$. However, it is easy to check that for $m \geq 2$ the insufficient vertices in this graph are a_1, a_2, and a_3. Since $d(a_i, a_j) = 8$ for $i \neq j$, we obtain that there are three equivalence classes of insufficient vertices in $\mathcal{G}$. But then we have $|G_m| = 4m + 9 \leq 9\delta(G_m)$, showing that the conclusion of Claim 2.7 does not hold for this graph.

  Therefore it seems that some new ideas would be needed in order to extend the results of this paper to the case when $r = 6$.

- Notice that Theorems 1.4 and Corollary 1.5 have the condition "$\text{diam}(G) \geq$

r" whereas Theorem 1.1 has the condition "diam(G) $> r$". Both bounds are natural to study. Although Corollary 1.5 is a statement about graphs satisfying "diam(G) $\geq r$", we can use it to obtain a lower bound on the quantity $e(G^r)/e(G)$ for graphs satisfying "diam(G) $> r$" as well. When $r \equiv 0$ or 1 (mod 3), it is easy to see that the lower bound Corollary 1.5 gives cannot be increased even when restricted to graphs satisfying "diam(G) $> r$" (using the examples we constructed in Section 3).

In more generality one could ask for bounds on $e(G^r)$ among all graphs satisfying diam(G) $> D$ for some fixed D. When D is larger than r, then it is likely that that the bounds in Theorem 1.4 could be improved. Some results in this direction have already been obtained by DeVos, McDonald, and Scheide. We refer the reader to [3] for details.

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