

Choosability of the square of a planar graph with maximum degree four

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Abstract

We study squares of planar graphs with the aim to determine their list chromatic number. We present new upper bounds for the square of a planar graph with maximum degree $\Delta \leq 4$. In particular G^2 is 5-, 6-, 7-, 8-choosable if the girth of G is at least 16, 11, 9, 7 respectively. In fact we prove more general results, in terms of maximum average degree, that imply the results above.

1 Introduction

The square of a graph G , denoted by G^2 , is the graph whose vertex set is the same as G , and where two vertices are adjacent in G^2 if they are at distance at most two in G . If Δ is the maximum degree of G , then to colour its square G^2 we will need at least $\Delta + 1$ colours while the upper bound is $\Delta^2 + 1$ using the greedy algorithm. This upper bound is also achieved by a few graphs, for example by the Petersen graph.

Regarding the colouring of the square of planar graphs, Wegner [24] posed the following conjecture in 1977:

Conjecture 1.1 (Wegner). *For a planar graph G of maximum degree Δ*

$$\chi(G^2) \leq \begin{cases} 7, & \Delta = 3; \\ \Delta + 5, & 4 \leq \Delta \leq 7; \\ \lceil \frac{3}{2}\Delta \rceil + 1, & \Delta \geq 8. \end{cases}$$

In [14] Havet, van den Heuvel, McDiarmid, and Reed showed that the following holds: $\chi(G^2) \leq \frac{3}{2}\Delta(1 + o(1))$, which is also true for the choice number (defined below). Dvořák, Král', Nejedlý, and Škrekovski [10] showed that the square of every planar graph of girth at least six with sufficiently large maximum degree Δ is $(\Delta + 2)$ -colorable. Borodin and Ivanova [4] strengthened this result to prove that for every planar graph G of girth at least six with maximum degree $\Delta \geq 24$, the choice number of G^2 is at most $\Delta + 2$. For colouring (rather than list-colouring), the same authors showed [5] that for every planar graph G of girth at least six with maximum degree $\Delta \geq 18$, the chromatic number of G^2 is at most $\Delta + 2$.

Lih, Wang, and Zhu [20] showed that the square of a K_4 -minor free graph with maximum degree Δ has chromatic number at most $\lceil \frac{3}{2}\Delta \rceil + 1$ if $\Delta \geq 4$ and $\Delta + 3$ if $\Delta = 2, 3$. The same bounds were shown to hold for the choice number by Hetherington and Woodall [16].

All graphs in this paper are undirected, simple, and finite. For standard graph definitions see [9]. Denote by $l(f)$ the length of a face f and by $d(v)$ the degree of a vertex v . A k -vertex is a vertex of degree k . A k^- -vertex is a vertex of degree at most k , and a k^+ -vertex is a vertex of degree at least k . If a vertex u is adjacent to a k -vertex v , then v is a k -neighbour of u . A *thread* between two vertices with degree at least three is a path between them consisting of only 2-vertices. A k -thread is a thread with k internal 2-vertices. If vertices u and v lie on a common thread, then u and v are *weak neighbours* of each other. Similarly, we define a weak k -neighbour.

A *colouring* of the vertices of a graph G is a mapping $c : V(G) \rightarrow \mathbb{N}$; we call elements of \mathbb{N} *colours*. A colouring is *proper* if every two adjacent vertices are mapped to different colours. List colouring was first studied by Vizing [23] and is defined as follows. Let G be a simple graph. A *list-assignment* L is an assignment of lists of colours to vertices. A *list-colouring* is then a colouring where each vertex v receives a colour from $L(v)$. The graph G is *L -choosable* if there is a proper L -list-colouring. If G has a list-colouring for every list-assignment with $|L(v)| = k$ for each vertex v , then G is *k -choosable*. We will denote the size of the lists of colours in a specific case simply by χ_l . The minimum k such that G is k -choosable is called the *choice number* of G .

To prove our theorem we will use the discharging method, which was first used by Wernicke [25]; this technique is used to prove statements in structural graph theory, and it is commonly applied in the context of planar graphs. It is most well-known for its central role in the proof of the Four Colour Theorem. Here we apply the discharging method in the more general context of the *maximum average degree*, denoted $\text{mad}(G)$, which is defined as $\text{mad}(G) := \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$, where H ranges

over all subgraphs of G . A straightforward consequence of Euler's Formula is that every planar graph G with girth at least g satisfies $\text{mad}(G) < \frac{2g}{g-2} = 2 + \frac{4}{g-2}$. We call this Fact 1. Most of our results for planar graphs will follow from corresponding results for maximum average degree, via Fact 1.

The key tool in many of our proofs is global discharging, which relies on reducible configurations that may be arbitrarily large. Global discharging was introduced by Borodin [2]. Typically, the vertices in these reducible configurations have degrees only 2 and Δ . Our innovation in this paper is that we consider arbitrarily large reducible configurations consisting entirely of 2-vertices and 3-vertices, even though $\Delta = 4$. For two similar applications of global discharging, see [8] and [4].

Kostochka and Woodall [19] conjectured that, for every square of a graph, the chromatic number and choice number are the same:

Conjecture 1.2 (Kostochka and Woodall). *If G is a simple graph, then*

$$\chi_l(G^2) = \chi(G^2).$$

When G is a planar graph, the upper bound on $\chi(G^2)$ in terms of Δ was successively improved by Jonas [17], Wong [26], Van den Heuvel and McGuinness [15], Agnarsson and Halldorsson [1], Borodin et al. [3] and finally by Molloy and Salavatipour [21] to the best known upper bound so far: $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$.

The choosability of squares of subcubic planar graphs has been extensively studied by Dvořák, Škrekovski, and Tancer [11], Montassier and Raspaud [22], Havet [13], and Cranston and Kim [7]. For the case $\Delta = 4$ there have been few results so far. The only paper we know of is by Bu and Zhu [6]. They proved that if G is planar and $\Delta(G) = 4$, then $\chi(G^2) \leq 13$ (in fact, this proof gives that $\chi_l(G^2) \leq 13$). If G is also triangle-free, then $\chi(G^2) \leq 11$ (we suspect the proof gives $\chi_l(G^2) \leq 11$, but we have not checked all the details). We give some upper bounds on $\chi_l(G^2)$ when $\Delta(G) = 4$ and $\text{mad}(G)$ is bounded. These results imply bounds for $\chi_l(G^2)$ when G is planar with prescribed girth:

Theorem 1.3. *Let G be a graph with maximum degree $\Delta = 4$. The following bounds hold.*

- (a) G^2 is 5-choosable if $\text{mad}(G) < 16/7$, specifically, if G is planar with girth at least 16.
- (b) G^2 is 6-choosable if $\text{mad}(G) < 22/9$, specifically, if G is planar with girth at least 11.
- (c) G^2 is 7-choosable if $\text{mad}(G) < 18/7$, specifically, if G is planar with girth at least 9.
- (d) G^2 is 8-choosable if $\text{mad}(G) < 14/5$, specifically, if G is planar with girth at least 7.

This theorem is summarized in the following table.

$\chi_l \leq$	5	6	7	8
$\text{mad}(G) <$	16/7	22/9	18/7	14/5
planar and $g \geq$	16	11	9	7

Table 1: Upper bounds on the choice number for squares of graphs with $\Delta = 4$ and bounded maximum average degree, including planar graphs with bounded girth.

We will prove each of the claims by contradiction while studying the smallest counterexample to the claim with respect to the number of vertices. If we remove one or more vertices from this graph we know that its square can be properly coloured with the lists provided. We will use this fact in the proofs of the claims.

1.1 Reducible configurations

A *configuration* is an induced subgraph H of a graph G . We call a configuration *reducible* if it cannot appear in a minimal counterexample. To prove that a configuration is reducible, we infer from the minimality of G that subgraph $G - H$ can be properly coloured, and then prove that this colouring can be extended to a proper colouring of the original graph G ; this gives a contradiction. A configuration is *k-reducible* if it is reducible in the setting of k -choosability. Clearly a k -reducible configuration is also $(k + 1)$ -reducible.

We split our proof of the main theorem into four lemmas, one for each part of the theorem. Within each lemma, we prove the reducibility of the configurations used in that lemma. Once we prove a configuration is reducible, we will assume that such a configuration is not present in a minimal counterexample to that lemma.

We will prove that the configurations are reducible by using the following method each time. Remove some vertices and colour the square of the remaining graph by minimality. If necessary, uncolour some vertices; for example, if two non-adjacent vertices in the square of the remaining graph are adjacent in G^2 , uncolour them. Finally, extend this colouring to the whole of G^2 .

To simplify the presentation of the reducibility proofs we provide figures using the following notation: a removed vertex is marked with a square around it; an uncoloured vertex is marked with a circle around it; the minimum number of colours left in the list of a removed or uncoloured vertex is written next to it. These figures allow the reader to quickly verify that the configurations pictured are reducible. In the first few reducibility proofs we will provide detailed reasoning but in the remaining ones we will only present the corresponding figure and leave the details to the reader.

We call a graph *degree-choosable* if it can be coloured from any list assignment L such that $|L(v)| = d(v)$ for all $v \in V(G)$. For a few of the reducibility proofs, we will need the following result of Erdős, Rubin, and Taylor [12]:

Lemma 1.4 (Choosability Lemma). *A connected graph fails to be degree-choosable if and only if every block is a complete graph or an odd cycle.*

2 Proof of the Main Theorem

In this section, we prove our main result, Theorem 1.3. The four parts of Theorem 1.3 are completely independent, so we present the proof as four self-contained lemmas, each proving a corresponding part of the theorem. All of our proofs use only maximum average degree (rather than planarity). The proofs of Lemmas 2.1, 2.2, and 2.4 make use of global discharging; the easiest of these proofs is Lemma 2.1, while Lemmas 2.2 and 2.4 require additional details and subtlety. We now prove the four lemmas without further comment.

Lemma 2.1. *If $\Delta(G) \leq 4$ and $\text{mad}(G) < 16/7$, then $\chi_l(G^2) \leq 5$. In particular, for every planar graph G with $\Delta(G) \leq 4$ and girth at least 16, we have $\chi_l(G^2) \leq 5$.*

Proof. The second statement follows from the first by Fact 1. To prove the first, we use discharging. Let G be a minimal counterexample to the lemma, i.e., a minimal graph with $\Delta(G) \leq 4$ and $\text{mad}(G) < 16/7$ such that $\chi_l(G^2) > 5$. For each vertex v , we begin with charge $\mu(v) = d(v)$; we will show that after the discharging phase each vertex finishes with charge at least $16/7$, which gives a contradiction and proves the lemma.

We call a configuration *5-reducible* if it cannot appear in a minimal counterexample to the lemma. We use the following configurations (see Fig. 1):

- (i) *A 4-thread is 5-reducible.* Let v and w be the middle two vertices of the 4-thread. By the minimality of G we can 5-list-colour $(G \setminus \{v, w\})^2$. Now v and w each have at least 2 colours available, so we can extend the colouring to G^2 .
- (ii) *A 3-thread S incident to a 3-vertex u is 5-reducible.* Let v be the 2-vertex on S adjacent to u and let w be the 2-vertex adjacent to v . By the minimality of G , we can 5-list-colour $(G \setminus \{v, w\})^2$. Now v and w have at least one and two available colours, respectively. So we can extend the colouring to G^2 by colouring v then w .
- (iii) *A $3k$ -cycle C_{3k} with $d(v_{3i}) = 3$ for all i and $d(v_{3i+1}) = d(v_{3i+2}) = 2$ for all i (the subscripts are modulo $3k$) is 5-reducible.* Let $S = \{v_{3i} : 1 \leq i \leq k\}$. We delete all vertices on C_{3k} with degree 2. By minimality, the subgraph $(C_{3k})^2 \setminus S$ of G^2 remains to be coloured, and is isomorphic to an even cycle C_{2k} . Each uncoloured vertex has at most 3 restrictions on its colour, so it has a list of at least 2 available colours. Now we can extend the colouring to G^2 since $\chi_l(C_{2k}) = 2$ (this is an easy exercise, and also follows immediately from the Choosability Lemma).

Let H denote the subgraph of G induced by 2-threads with 3-vertices at both ends. Since configuration (iii) is 5-reducible, H must be acyclic. Since every tree has one more vertex than edge, we can recursively assign each 2-thread in H to be sponsored by an incident 3-vertex such that each 3-vertex sponsors at most one 2-thread.

We use the initial charge function $\mu(v) = d(v)$ and the following discharging rules.

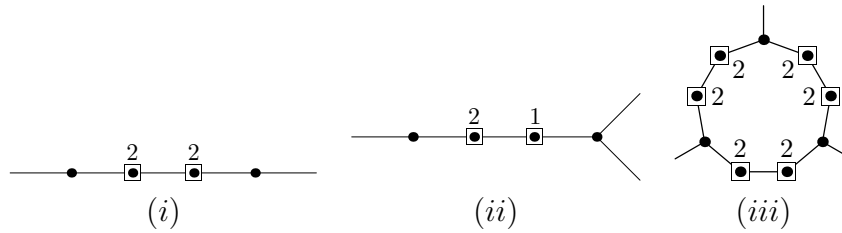


Figure 1: Configurations (i), (ii), and (iii) from Lemma 2.1 are 5-reducible.

- **R1:** Each 3-vertex gives charge $1/7$ to each incident thread.
- **R2:** Each 4-vertex gives charge $3/7$ to each incident thread.¹
- **R3:** Each 3-vertex incident with a 2-thread that it sponsors gives an additional charge of $2/7$ to that 2-thread.

Now we show that each 3^+ -vertex finishes with charge at least $16/7$ and that each k -thread receives charge at least $2k/7$ (so that it finishes with charge at least $16k/7$). Note that a 1-vertex is 5-reducible, so $\delta(G) \geq 2$. First we consider 3^+ -vertices. If $d(v) = 4$, then v gives charge $3/7$ to each incident thread, so the new charge, $\mu^*(v)$, satisfies $\mu^*(v) \geq 4 - 4(3/7) = 16/7$. If $d(v) = 3$, then v sends charge $1/7$ to each incident thread and an additional charge of $2/7$ to at most one incident 2-thread, so $\mu^*(v) \geq 3 - 3(1/7) - 1(2/7) = 16/7$.

Now we consider threads. Each 3-thread receives charge $3/7$ from each endpoint, which are both 4-vertices by (ii). Each 1-thread receives charge at least $1/7$ from each endpoint. Each 2-thread with at least one degree 4 endpoint receives charge $3/7$ from one endpoint and at least $1/7$ from the other. Finally, each 2-thread with two degree 3 endpoints receives charge $1/7$ from each endpoint and an additional charge of $2/7$ from its sponsor, for a total of $4/7$. Thus $\text{mad}(G) \geq 16/7$. This contradiction completes the proof. \square

Lemma 2.2. *If $\Delta(G) \leq 4$ and $\text{mad}(G) < 22/9$, then $\chi_l(G^2) \leq 6$. In particular, for every planar graph G with $\Delta(G) \leq 4$ and girth at least 11, we have $\chi_l(G^2) \leq 6$.*

Proof. The second statement follows from the first by Fact 1. To prove the first, we use discharging. Let G be a minimal counterexample to the lemma. For each vertex v , we begin with charge $\mu(v) = d(v)$, and we will show that after discharging each vertex finishes with charge at least $22/9$, which gives a contradiction and proves the lemma.

We call a configuration *6-reducible* if it cannot appear in a minimal counterexample to the lemma. We use the following configurations (see Fig. 2):

- (i) *A 3-thread S is 6-reducible.* Let v and w be adjacent 2-vertices on S , with v adjacent to an endpoint of S . By the minimality of G we can 6-list-colour $(G \setminus \{v, w\})^2$. Now v and w have at least 1 and 3 colours available, respectively. So we can extend the colouring to G^2 by colouring v then w .

¹If a 4-vertex v is adjacent to two vertices in the same thread, i.e., v serves as both endpoints of the thread, then v sends twice the normal charge to the thread; similarly for Lemma 2.

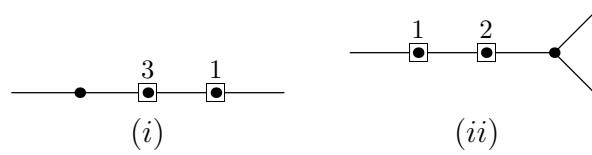


Figure 2: Configurations (i) and (ii) from Lemma 2.2 are 6-reducible.

(ii) A 2-thread T incident to a 3-vertex u is 6-reducible. Let v and w be the two 2-vertices of T , with v adjacent to u . By the minimality of G we can 6-list-colour $(G \setminus \{v, w\})^2$. Now v and w have at least 2 and 1 colours available, respectively. So we can extend the colouring to G^2 by colouring w then v .

Let H be the subgraph induced by 2-threads; recall that the endpoints of each 2-thread must be 4-vertices, by (ii). As in the proof of Lemma 1, H must be acyclic. Thus, we can assign each 2-thread of H to be sponsored by an incident 4-vertex such that each 4-vertex sponsors at most one 2-thread.

If a 2-vertex v has two 3-neighbours, call the 1-thread containing v *light*. Let J be the subgraph induced by light 1-threads. We will show that each component of J must be a tree or a cycle. Suppose instead that J contains a cycle with an incident edge. We denote the cycle by $u_1v_1u_2v_2 \dots u_kv_k$ where $d(u_i) = 2$ and $d(v_i) = 3$ for all i and v_1 is adjacent to a 2-vertex z not on the cycle (which is adjacent to a second 3-vertex). By minimality, we can 6-list-colour $(G \setminus \{u_1, v_1, u_2, z\})^2$. Now only three neighbours of v_1 in G^2 are coloured, so we can colour v_1 . Finally, we uncolour each vertex u_i . Now the uncoloured vertices induce in G^2 a subgraph K consisting of a cycle with a single vertex z adjacent to two successive vertices on the cycle. For each vertex $x \in V(K)$, let $L(x)$ denote the colours available for x . Note that we have $|L(x)| \geq d_K(x)$ for all $x \in V(K)$. Thus, by the Choosability Lemma, we can extend the list-colouring to all of $V(G)$. So each component of J must be a tree or a cycle; hence we can assign each 1-thread of J to be sponsored by an incident 3-vertex such that each 3-vertex sponsors at most one 1-thread.

We use the initial charge function $\mu(v) = d(v)$ and the following discharging rules.

- **R1:** Each 3-vertex gives charge $1/9$ to each incident thread.
- **R2:** Each 4-vertex gives charge $3/9$ to each incident thread.
- **R3:** Each 3^+ -vertex incident with a sponsored thread gives an additional charge of $2/9$ to that thread.

Now we show that each 3^+ -vertex finishes with charge at least $22/9$ and that each k -thread receives charge at least $4k/9$ (so it finishes with charge at least $22k/9$). As in Lemma 2.1, note that $\delta(G) \geq 2$. If $d(v) = 4$, then v gives charge $3/9$ to each incident thread and an additional $2/9$ to at most one sponsored thread, so $\mu^*(v) \geq 4 - 4(3/9) - 1(2/9) = 22/9$. If $d(v) = 3$, then v sends charge $1/9$ to each incident thread and an additional $2/9$ to at most one incident thread, so $\mu^*(v) \geq 3 - 3(1/9) - 1(2/9) = 22/9$.

Now we consider threads. Each 2-thread receives charge $3/9$ from each endpoint and charge $2/9$ from its sponsor, for a total charge of $8/9$. Consider a 1-thread with interior 2-vertex v . If v has at least one 4-neighbour, then the 1-thread receives charge at least $3/9 + 1/9 = 4/9$. Each 1-thread with both endpoints of degree 3 receives charge $1/9$ from each endpoint and charge $2/9$ from its sponsor for a total charge of $4/9$. Thus $\text{mad}(G) \geq 22/9$. This contradiction completes the proof. \square

Lemma 2.3. *If $\Delta(G) \leq 4$ and $\text{mad}(G) < 18/7$, then $\chi_l(G^2) \leq 7$. In particular, for every planar graph G with $\Delta(G) \leq 4$ and girth at least 9, we have $\chi_l(G^2) \leq 7$.*

Proof. The second statement follows from the first by Fact 1. To prove the first, we use discharging. Let G be a minimal counterexample to the lemma. For each vertex v , we begin with charge $\mu(v) = d(v)$, and we will show that after discharging each vertex finishes with charge at least $18/7$, which gives a contradiction and proves the lemma. We leave to the reader the details of verifying that each of the three following configurations is 7-reducible (see Fig. 3):

- (i) *A thread of two 2-vertices;*
- (ii) *A 3-vertex adjacent to three 2-vertices;*
- (iii) *A 3-vertex, adjacent to two 2-vertices, one of which is adjacent to a second 3-vertex.*

We use the initial charge function $\mu(v) = d(v)$ and the following discharging rules.

- **R1:** Each 4-vertex gives charge $5/14$ to each 2-neighbour.
- **R2:** Each 3-vertex with a single 2-neighbour gives charge $4/14$ to that 2-neighbour.
- **R3:** Each 3-vertex with two 2-neighbours gives charge $3/14$ to each 2-neighbour.

Now we show that each vertex finishes with charge at least $18/7$. Note that $\delta(G) \geq 2$. If $d(v) = 2$ and v has a 4-neighbour, then $\mu^*(v) \geq 2 + 5/14 + 3/14 = 2 + 4/7$. If $d(v) = 2$ and v has no 4-neighbour, then v receives charge $4/14$ from each of its 3-neighbours, since otherwise we have configuration (iii) in Figure 3. Now $\mu^*(v) \geq 2 + 2(4/14) = 18/7$. If $d(v) = 3$, then by (ii) v has at most two 2-neighbours, so $\mu^*(v) \geq 3 - 2(3/14) = 18/7$. Finally, if $d(v) = 4$, then v has at most four 2-neighbours, so $\mu^*(v) \geq 4 - 4(5/14) = 18/7$. Thus, $\text{mad}(G) \geq 18/7$. This contradiction completes the proof. \square

Lemma 2.4. *If $\Delta(G) \leq 4$ and $\text{mad}(G) < 14/5$, then $\chi_l(G^2) \leq 8$. In particular, for every planar graph G with $\Delta(G) \leq 4$ and girth at least 7, we have $\chi_l(G^2) \leq 8$.*

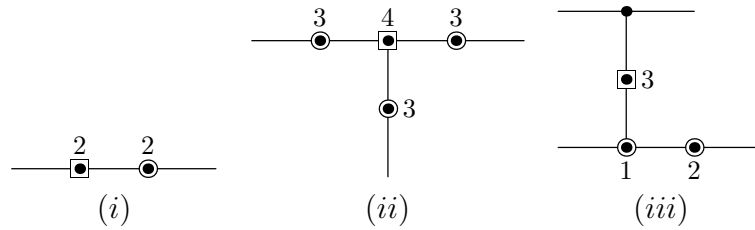


Figure 3: Configurations (i), (ii), and (iii) from Lemma 2.3 are 7-reducible.

Proof. The second statement follows from the first by Fact 1. To prove the first, we use discharging. Let G be a minimal counterexample to the lemma. For each vertex v , we begin with charge $\mu(v) = d(v)$, and we will show that after discharging each vertex finishes with charge at least $14/5$, which gives a contradiction and proves the lemma. We call a 2-vertex with two 3-neighbours a *light 2-vertex*. We call a 2-vertex with a 3-neighbour and a 4-neighbour a *medium 2-vertex*. We call a 2-vertex with two 4-neighbours a *heavy 2-vertex*. We call a 3-vertex adjacent to a light 2-vertex a *needy 3-vertex*. Adjacent 2-vertices are 8-reducible (we showed above that they are 7-reducible), so this implies that every 2-vertex is heavy, medium, or light.

We leave to the reader the details of verifying the following 8-reducible configurations, which are depicted in Fig. 4:

- (i) a 3-vertex with two 2-neighbours;
- (ii) a 3-vertex with two 3-neighbours and a light 2-neighbour;
- (iii) a 4-vertex with three 2-neighbours, one of which is medium;
- (iv) A 4-vertex with a needy 3-neighbour and two 2-neighbours, one of which is medium.

If a 1-thread S contains a heavy 2-vertex v then we call S *heavy*. Let J be the subgraph induced by heavy 1-threads. Each component of J must be a tree or a cycle. Since the proof is identical to that given in Lemma 2.2, here we do not repeat the details. Since each component of J is a tree or a cycle, we can assign each 2-vertex on a heavy 1-thread to be sponsored by an adjacent 4-vertex, so that each 4-vertex sponsors at most one such 2-vertex.

We use the initial charge function $\mu(v) = d(v)$ and the following discharging rules.

- **R1:** Each 3^+ -vertex gives charge $1/5$ to each adjacent 2-vertex.
- **R2:** Each 4-vertex gives charge $1/5$ to each adjacent needy 3-vertex.
- **R3:** Each needy 3-vertex gives an additional $1/5$ to each adjacent light 2-vertex.
- **R4:** Each 4-vertex gives an additional $2/5$ to each adjacent medium 2-vertex and each adjacent sponsored 2-vertex.

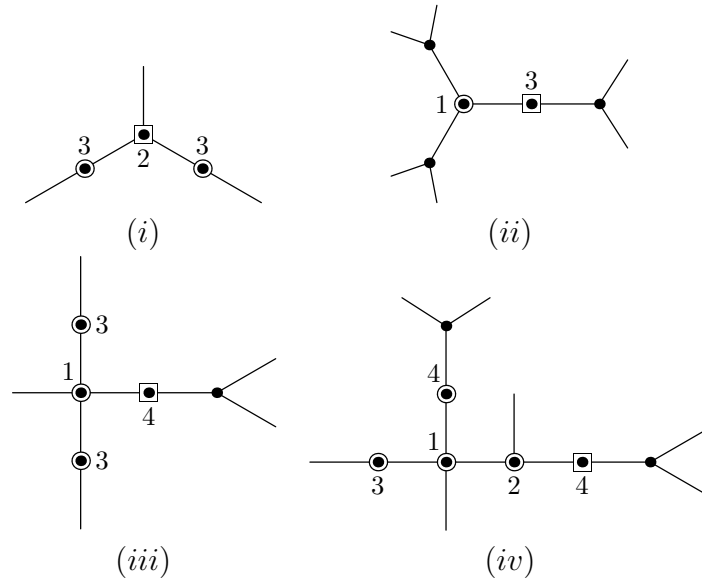


Figure 4: Configurations (i)–(iv) from Lemma 2.4 are 8-reducible.

Now we show that each vertex finishes with charge at least $14/5$. Note that $\delta(G) \geq 2$.

Suppose $d(v) = 2$. If v is heavy, then v receives charge $1/5$ from each neighbour and an additional charge $2/5$ from its sponsor, so $\mu^*(v) = 2 + 2(1/5) + 2/5 = 14/5$. If v is medium, then v receives charge $1/5$ from its 3-neighbour and charge $1/5 + 2/5$ from its 4-neighbour, so $\mu^*(v) = 2 + 1/5 + 1/5 + 2/5 = 14/5$. If v is light, then v receives charge $1/5$ from each neighbour and an additional charge $1/5$ from each neighbour, so $\mu^*(v) = 2 + 2(2/5) = 14/5$.

Suppose $d(v) = 3$. By (i), v has at most one 2-neighbour. If v has a light 2-neighbour, then v gives it charge $1/5 + 1/5$ and v receives charge $1/5$ from some 4-neighbour, since otherwise we have configuration (ii). So $\mu^*(v) \geq 3 - 2/5 + 1/5 = 14/5$. If v has a medium 2-neighbour, then v gives it only charge $1/5$, so $\mu^*(v) \geq 3 - 1/5 = 14/5$.

Suppose $d(v) = 4$. If v has no medium neighbours, then v gives charge at most $1/5$ to each neighbour and an additional charge of $2/5$ to at most one sponsored 2-vertex, so $\mu^*(v) \geq 4 - 4(1/5) - 2/5 = 14/5$. So suppose that v has a medium 2-neighbour. If v has only one 2-neighbour, then v gives charge at most $1/5$ to each other neighbour and charge $1/5 + 2/5$ to its medium 2-neighbour, so $\mu^*(v) \geq 4 - 3(1/5) - 1/5 - 2/5 = 14/5$. If v has at least two 2-neighbours, at least one of which is medium, then by configurations (iii) and (iv), v gives charge to no neighbours besides these two 2-neighbours. Since v gives total charge at most $3/5$ to each of these 2-neighbours, $\mu^*(v) \geq 4 - 2(3/5) = 14/5$.

Thus, each vertex finishes with charge at least $14/5$, so $\text{mad}(G) \geq 14/5$. This contradiction completes the proof. \square

Acknowledgments

We thank an anonymous referee for a very detailed report, which improved the clarity of our exposition. After this paper was written, Kim and Park [18] disproved the conjecture of Kostochka and Woodall [19] that all squares of graphs have list chromatic number equal to chromatic number.

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