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On extending the Bose construction for triple systems to decompositions of complete multipartite graphs into 2-regular graphs of odd order

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Abstract

For integers $r \geq 2$ and $s \geq 1$, let $K_{r\times s}$ denote the complete multipartite graph with r partite sets of order s. Let G be a 2-regular graph of odd order n. If G contains exactly one odd cycle, it is known that there exists a G-decomposition of K_{2kn+1}, of $K_{(2k+1)\times n}$, and of $K_{k'\times 2n}$ for all positive integers k and $k' \geq 3$. If G consists of three vertex-disjoint odd cycles, then the only known general result is a G-decomposition of K_{2n+1}. We use a novel extension of the Bose construction for triple systems to show that in the three odd cycles case, there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k. We also show that there exists a G-decomposition of $K_{k\times 2n}$ as well as of K_{2kn+1} for every integer $k \geq 3$.

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1 Introduction

Let $\mathbb{Z}_n$ be the group of integers modulo n. For integers a and b with $a \leq b$, we denote the set $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$). For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively. We will denote the complete multipartite graph with r partite sets of order s by $K_{r\times s}$. The vertex-disjoint union of r copies of a graph G will be denoted by rG. A non-bipartite graph G is *almost-bipartite* if for some $e \in E(G)$, the graph $G - e$ is bipartite.

A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that the edge sets of the graphs G_i form a partition of the edge set of K. If each G_i is isomorphic to a fixed graph G, such a decomposition is called a *G-decomposition of K* or *(K, G)-design*. In this case, we may say *G decomposes K* or *K is decomposable by G*. A (K_v, G)-design is also known as a *G-design of order v*. For recent surveys on G-designs, we direct to the reader to [2] and [12].

One of the better studied problems in G-designs is the case when G is a cycle. Necessary and sufficient conditions for the existence of C_n-designs of order v were found about a decade ago by Alspach and Gavlas [6] and by Šajna [20]. Necessary and sufficient conditions for the existence of a G-design of order v are found in [3] when G is a 2-regular graph of order at most 10. For an arbitrary 2-regular graph G of order n, the problem of finding necessary and sufficient conditions for the existence of a G-design of order v is far from settled. It is expected however that for such a G, there will exist a G-design of order v for all $v \equiv 1 \pmod{2n}$. This has been confirmed when G is bipartite (see [16] and [8]), when G is almost-bipartite [14], when G is rC_m where m is odd [17], and when G has two components (see [1], [9] and [13]). If in addition n is odd and $(G, v) \notin \{(C_4 \cup C_5, 9), (C_3 \cup C_3 \cup C_5, 11)\}$, then a G-design of order v for all $v \equiv n \pmod{2n}$ is likely to exist. This is confirmed in [15] when G consists of one even and one odd cycle.

A well-known problem on decompositions of complete graphs into 2-regular graphs is the Oberwolfach Problem. Let G be a 2-regular graph of odd order n. The problem of determining whether there exists a G-decomposition of K_n is known as the *Oberwolfach Problem*. This problem was settled in 1989 by Alspach, Schellenberg, Stinson, and Wagner [7] in the case when all the components of G are isomorphic to the same cycle. More recently, Traetta [21] settled the case when G consists of two components. The general problem however is far from settled. For example, very little is known when G consists of three components (see [11] for some known results).

It is easy to see that K_{2kn+n} can be decomposed into $(2k+1)K_n$ and $K_{(2k+1)\times n}$. Thus if there is a G-decomposition of K_n and a G-decomposition of $K_{(2k+1)\times n}$, then there is a G-decomposition of K_{2kn+n}. In [15], an extension of the Bose construction for triple systems is used to show that if G of order n is the vertex-disjoint union of an even cycle and an odd cycle, then G decomposes $K_{(2k+1)\times n}$ for every positive integer k. This is then combined with Traetta's result [21] on the Oberwolfach problem with two components to show that there is a G-decomposition of K_{2kn+n}. In [15], it is also shown that there exists a G-decomposition of $K_{k'\times 2n}$ for every integer $k' \geq 3$. The results on G-decompositions of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$ are extended to all 2-regular almost-bipartite graphs G in [18].

In this article, we use a further extension of the Bose construction for triple systems to show that if G of order n is the vertex-disjoint union of three odd cycles, then there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k. We also show that there exists a G-decomposition of $K_{k\times 2n}$ as well as of K_{2nk+1} for every integer $k \geq 3$. As with the Bose construction, these decompositions make use of commutative quasigroups.

2 Quasigroups and the Bose Construction

A *quasigroup* of *order* q is a pair $(Q, \circ)$ where Q is a set of size q, say $Q = [1, q]$, and $\circ$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. The quasigroup is *idempotent* if $i \circ i = i$ for every $i \in Q$ and it is *commutative* if $i \circ j = j \circ i$ for all $i, j \in Q$. It is known that an idempotent commutative quasigroup of order q exists if and only if q is odd (see [19]).

Let $Q = [1, 2k]$ and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$. In what follows, the two element subsets $\{2i-1, 2i\} \in H$ are called *holes*. A *quasigroup with holes* H is a quasigroup $(Q, \circ)$ of order $2k$ in which for each $h \in H$, we have $(h, \circ)$ is a subquasigroup of $(Q, \circ)$. It is known that for every integer $k \geq 3$, there exists a commutative quasigroup $(Q, \circ)$ of order $2k$ with holes H (see [19]). Commutative quasigroups of order $2k$ with holes H are used to construct C_3-decompositions of $K_{k\times 6}$ for every integer $k \geq 3$.

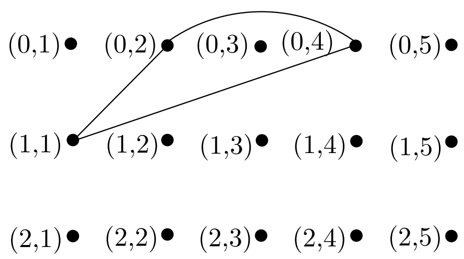
We give a brief description of Bose's construction for Steiner triple triple systems of order $6k + 3$. We direct the reader to the book by Lindner and Rodger [19] for detailed information on quasigroups and triple systems.

We will define a *Steiner triple system* of *order* v to be a C_3-decomposition of K_v. It has long been known that a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$. In 1939, Bose [10] used the existence of an idempotent commutative quasigroup of order $2k + 1$ to construct a C_3-decomposition of K_{6k+3} for every positive integer k. One can view K_{6k+3} as $(2k+1)K_3 \bigcup K_{(2k+1)\times 3}$. Thus to construct a C_3-decomposition of K_{6k+3}, it suffices to construct a C_3-decomposition of $K_{(2k+1)\times 3}$. Let $\langle a, b, c\rangle$ denote the C_3 with vertex set $\{a, b, c\}$.

Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$ where $Q = [1, 2k + 1]$ and let $V(K_{(2k+1)\times 3}) = \mathbb{Z}_3 \times Q$ with the obvious vertex partition. Let $T = \{\langle (0, i), (0, j), (1, i \circ j)\rangle, \langle (1, i), (1, j), (2, i \circ j)\rangle, \langle (2, i), (2, j), (0, i \circ j)\rangle : 1 \leq i < j \leq 2k + 1\}$. Then the C_3's in T form a C_3-decomposition of $K_{(2k+1)\times 3}$.

Figure 1 shows an idempotent commutative quasigroup of order 5 and one triple from the Bose construction of a Steiner triple system of order 15.

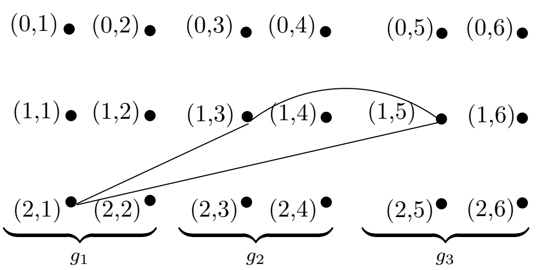
Alternatively, let $k \geq 3$ be an integer and for $i \in [1, k]$, let $h_i = \{2i - 1, 2i\}$ and $g_i = \mathbb{Z}_3 \times h_i$. Let $Q = [1, 2k]$ and $H = \{h_1, h_2, \ldots, h_k\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $2k$ with holes H. Let $V(K_{k\times 6}) = \mathbb{Z}_3 \times Q$ with the vertex-set partition $\{g_1, g_2, \ldots, g_k\}$. Let $T = \{\langle (0, i), (0, j), (1, i \circ j)\rangle, \langle (1, i), (1, j), (2, i \circ$



$\circ$	1	2	3	4	5
1	1	5	2	3	4
2	5	2	4	1	3
3	2	4	3	5	1
4	3	1	5	4	2
5	4	3	1	2	5

Figure 1: An idempotent commutative quasigroup of order 5 and one triple from the Bose construction of a Steiner triple system of order 15.

$j)\rangle, \langle((2,i),(2,j),(0,i\circ j)\rangle\colon 1 \leq i < j \leq 2k, \{i,j\} \notin H\}$. Then the C_3's in T form a C_3-decomposition of $K_{k\times 6}$. This process is part of what is known as the *quasigroups with holes construction* for triple systems (see [19]). Figure 2 shows a commutative quasigroup of order 6 with holes and one triple from the corresponding C_3-decomposition of $K_{3\times 6}$.



$\circ$	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

Figure 2: A commutative quasigroup of order 6 with holes and one triple from the corresponding C_3-decomposition of $K_{3\times 6}$.

3 Some notation

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

For the remainder of this section, we consider only subgraphs of a complete bipartite graph $K_{m,m}$ with vertex set $[0, m-1] \times [1,2]$ and the obvious vertex bipartition. Furthermore, if m, n, and i are integers with $m \leq n$, we denote $\{(m,i),(m+1,i),\ldots,(n,i)\}$ by $[(m,i),(n,i)]$. Define the *length* of an edge $\{(i,1),(j,2)\}$ to be $j-i$ if $j \geq i$; otherwise the edge length is $n+j-i$.

Let $P(k)$ be the path with k edges and $k+1$ vertices given by $((0,1),(k,2),(1,1),(k-1,2),(2,1),(k-2,2),\ldots,(\lceil k/2\rceil, \lceil k/2\rceil - \lfloor k/2\rfloor)+1)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [(0,1),(\lfloor k/2\rfloor,1)]$, $B = [(\lfloor k/2\rfloor+1,2),(k,2)]$, and every edge joins a vertex of A to one of B. Furthermore, the set of lengths of the edges of $P(k)$ is $[1, k]$.

Now let a be a nonnegative integer and b be an integer such that $|b| \leq \lfloor k/2 \rfloor + 1$, and let us add $(a, 0)$ to all the vertices of A and $(b, 0)$ to all the vertices of B. We denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1 $P(a, b, k)$ is a path with first vertex $(a, 1)$ and second vertex $(b + k, 2)$. Its last vertex is $(a + k/2, 1)$ if k is even and $(b + (k + 1)/2, 2)$ if k is odd.

P2 Each edge of $P(a, b, k)$ joins a vertex of $A' = \big[(a, 1), (\lfloor k/2 \rfloor + a, 1)\big]$ to a vertex of $B' = \big[(\lfloor k/2 \rfloor + 1 + b, 2), (k + b, 2)\big]$.

P3 The set of edge lengths of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ by replacing each vertex (i, j) with $(k - i, 3 - j)$. The new graph is the path $\big((k, 2), (0, 1), (k - 1, 2), (1, 1), \ldots, (\lfloor k/2 \rfloor, \lfloor k/2 \rfloor - \lceil k/2 \rceil + 2)\big)$. The set of vertices of $Q(k)$ is $A \cup B$, where $A = \big[(0, 1), (\lceil k/2 \rceil - 1, 1)\big]$ and $B = \big[(\lceil k/2 \rceil, 2), (k, 2)\big]$, and every edge joins a vertex of A to one of B. The set of edge lengths is still $[1, k]$. We again add $(a, 0)$ to the vertices of A and $(b, 0)$ to vertices of B, where a is nonnegative integer and b is an integer with $|b| \leq \lceil k/2 \rceil$. We denote the resulting graph by $Q(a, b, k)$. Note that this graph has the following properties.

Q1 $Q(a, b, k)$ is a path with first vertex $(k + b, 2)$. Its last vertex is $(b + k/2, 2)$ if k is even and $(a + (k - 1)/2, 1)$ if k is odd.

Q2 Each edge of $Q(a, b, k)$ joins a vertex of $A' = \big[(a, 1), (a + \lceil k/2 \rceil - 1, 1)\big]$ to a vertex of $B' = \big[(b + \lceil k/2 \rceil, 2), (b + k, 2)\big]$.

Q3 The set of edge lengths of $Q(a, b, k)$ is $[b - a + 1, b - a + k]$.

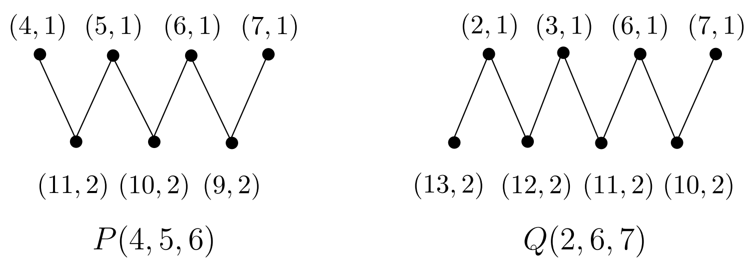


Figure 3: Examples of the $P(a, b, k)$ and $Q(a, b, k)$ notation.

4 G-decompositions of $K_{(2k+1)\times n}$ and of $K_{k\times 2n}$

Let $n \geq 3$ be an odd integer and let k be a positive integer. Let $K_{(2k+1)\times n}$ have vertex set $\mathbb{Z}_n \times [1, 2k + 1]$ with the obvious vertex partition. As before, we define the *length* of an edge $\{(i, r), (j, s)\}$ where $r < s$, to be $j - i$ if $j \geq i$; otherwise the edge length is $n + j - i$. Thus, between any two parts, there are edges of lengths

$0, 1, \ldots, n-1$. We will often write $-j$ for edge length $n-j$ when n is understood. Therefore, between any two parts, there are edges of lengths $0, \pm 1, \pm 2, \ldots, \pm\frac{(n-1)}{2}$. For ease of notation, we henceforth use i_r and i_s to denote the vertices (i, r) and (i, s), respectively.

We first prove a lemma that shows the existence of paths with certain edge lengths in $K_{n,n}$.

Lemma 1. *Let $n \geq 3$ be an odd integer and let $m \leq (n-1)/2$ be a positive integer. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times \{1, 2\}$ with the obvious vertex partition. Let $d_1, d_2, \ldots, d_{m-1}$ be an increasing sequence of consecutive positive integers with $d_{m-1} \leq (n-1)/2$. There exists a path P in $K_{n,n}$ of length $2m-1$ whose edges have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{m-1}$ with endpoints 0_1 and 0_2. Furthermore, $V(P) \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1, 2]$.*

Proof. If $m = 1$, let P be the path consisting of the edge $\{0_1, 0_2\}$. Otherwise, for $k \in [1, m-1]$, define $e_k = \sum_{i=0}^{k-1} (-1)^i d_{m-1-i}$. Note that since $d_{i+1} - d_i = 1$, we have $e_{2j} = j$ and $e_{2j+1} = d_{m-1} - j$. Thus, $e_{m-1} = \lceil \frac{m}{2} \rceil - 1$ if $m-1$ is even and $e_{m-1} = d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1$ if $m-1$ is odd. Similarly, $e_{m-2} = \lceil \frac{m}{2} \rceil - 1$ or $d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1$ if $m-1$ is odd or even, respectively.

Consider the path of length $m-1$ given by $P'\colon 0_1, (e_1)_2, (e_2)_1, (e_3)_2, \ldots$ where P' ends with $(e_{m-1})_2$ if $m-1$ is odd or $(e_{m-1})_1$ if $m-1$ is even. Thus, $V(P') \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1, 2]$. Also, observe that the lengths of the edges of P', in the order encountered, are $d_{m-1}, d_{m-2}, \ldots, d_1$.

Next consider the path $P''\colon 0_2, (e_1)_1, (e_2)_2, (e_3)_1, \ldots$ where P'' ends with $(e_{m-1})_1$ if $m-1$ is odd or $(e_{m-1})_2$ if $m-1$ is even, and observe that the edges of P'', in the order encountered, are $-d_{m-1}, -d_{m-2}, \ldots, -d_1$. Since P'' is constructed in the same way as P' with the corresponding vertices lying in the opposite parts of $V(K_{n,n})$, we have $V(P'') \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1, 2]$, and $V(P') \cap V(P'') = \varnothing$.

Construct the path P from the paths P' and P'' by adding the edge from $(e_{m-1})_1$ to $(e_{m-1})_2$. Note that P has length $2m-1$, the edges of P have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{m-1}$, and $V(P) \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1, 2]$. ∎

Let K be a subgraph of a graph with vertex set $\mathbb{Z}_n \times [1, q]$. For a positive integer ℓ, the graph $K + \ell$ has vertex set $\{(i+\ell)_z \colon i_z \in V(K)\}$ and edge set $\{\{(i+\ell)_r, (j+\ell)_s\} \colon \{i_r, j_s\} \in E(K)\}$.

Theorem 2. *Let G be a 2-regular graph of order n consisting of exactly three odd cycles. For every positive integer k, there exists a G-decomposition of $K_{(2k+1)\times n}$.*

Proof. Let $G = C_{2x+1} \cup C_{2y+1} \cup C_{2z+1}$ where x, y, and z are positive integers and let $n = 2x + 2y + 2z + 3$. Let $k \geq 1$ be an integer. Label the vertex set of $K_{(2k+1)\times n}$ with the elements of the group $\mathbb{Z}_n \times [1, 2k+1]$ with the obvious vertex partition. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$, where $Q = [1, 2k+1]$.

Fix r and s with $1 \leq r < s \leq 2k+1$. We will construct a graph $G_{r,s}$ consisting of the vertex disjoint union of the following three cycles: $C_{r,s}$ of length $2x+1$, $C'_{r,s}$ of length $2y+1$, and $C''_{r,s}$ of length $2z+1$. We will consider two cases.

Case 1: G has at least two cycles of length 3. Without loss of generality, we may assume that $x = y = 1$. Then the vertex sets of $C_{r,s}$ and $C'_{r,s}$ can be given by $\{0_r, 1_s, 3_{r\circ s}\}$ and $\{3_r, 2_s, 5_{r\circ s}\}$, respectively. If $z = 1$, then the vertex set of $C''_{r,s}$ can be given by $\{4_r, 4_s, 8_{r\circ s}\}$. Suppose that $z \geq 2$. By Lemma 1, there exists a path $P^*_{r,s}$ of length $2z-1$, between parts r and s, whose edges have lengths $\{0\}\cup\pm[5, z+3]$. In the lemma, we would use $d_1 = 5$, $d_2 = 6$, $\ldots$, $d_{z-1} = z+3$, so $V(P^*_{r,s}) \subseteq [0, z+3] \times \{r, s\}$ with endpoints 0_r and 0_s. Let $P''_{r,s} = P^*_{r,s} + 4$. Thus $P''_{r,s}$ has endpoints 4_r and 4_s. Then $V(P''_{r,s}) \subseteq [4, z + 7] \times \{r, s\}$. Thus, $P''_{r,s}$ is vertex disjoint from $C_{r,s}$ and $C'_{r,s}$. Construct the cycle $C''_{r,s}$ of length $2z + 1$ from the path $P''_{r,s}$ by adding the edges $\{4_r, 8_{r\circ s}\}$ and $\{4_s, 8_{r\circ s}\}$. Note that in the induced subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$, $G_{r,s}$ contains one edge of each length $i \in [-1, 1] \cup \pm[5, z + 3]$ (if $z = 1$, then $G_{r,s}$ contains one edge of each length $i \in [-1, 1]$). Moreover, the three edges of $G_{r,s}$ that are incident only with vertices in $\mathbb{Z}_n \times \{r, r\circ s\}$ are all of different lengths. In fact, the edges $\{0_r, 3_{r\circ s}\}$ in $C_{r,s}$, $\{3_r, 5_{r\circ s}\}$ in $C'_{r,s}$, and $\{4_r, 8_{r\circ s}\}$ in $C''_{r,s}$, have lengths 3, 2, and 4, respectively, if $r < r \circ s$, and lengths -3, -2, and -4, respectively, otherwise. Similarly, the three edges of $G_{r,s}$ that are incident only with vertices in $\mathbb{Z}_n \times \{s, r \circ s\}$ are all of different lengths. In fact, the edges $\{1_s, 3_{r\circ s}\}$ in $C_{r,s}$, $\{2_s, 5_{r\circ s}\}$ in $C'_{r,s}$, and $\{4_s, 8_{r\circ s}\}$ in $C''_{r,s}$, have lengths 2, 3, and 4, respectively, if $s < r \circ s$, and lengths -2, -3, and -4, respectively, otherwise. Figure 4 shows an example of $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = y = 1$ and $z = 4$.

Next, let $G^*_{r,s} = \{G_{r,s} + \ell : 0 \leq \ell < n - 1\}$. Thus $G^*_{r,s}$ contains n distinct copies of G. Moreover, in the induced subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$, G^* contains all edges of length i for all $i \in [-(n - 1)/2, (n - 1)/2] \setminus \pm[2, 4]$. Let $\mathcal{C} = \{G_{r,s} + \ell : 1 \leq r < s \leq 2k + 1, 0 \leq \ell \leq n - 1\}$ and note that $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G. We will show that every edge of $K_{(2k+1)\times n}$ appears in some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ with $r < s$ be an arbitrary edge of $K_{(2k+1)\times n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$ and $\beta'' = \max\{s, t''\}$. If $j-i \in [-(n-1)/2, (n-2)/2] \setminus \pm[2, 4]$ then e belongs to $G_{r,s} + \ell$ where $0 \leq \ell \leq n-1$.

Note that if $j - i = 2$, then e belongs to the triple $\{(i, r), (i - 1, t'), (j, s)\}$ which is a copy of $C_{t',r}$ if $t' < r$, or a copy of $C'_{r,t'}$ if $r < t'$. If $j - i = 3$, then e belongs to the triple $\{(i, r), (i + 1, t'), (j, s)\}$ which is a copy of $C'_{t',r}$ if $t' < r$, and a copy of $C_{r,t'}$ if $r < t'$. Also, if $j - i = 4$, then e belongs to some copy of $C''_{\alpha',\beta'}$. Thus, if $j - i \in [2, 4]$, then e belongs to $G_{\alpha',\beta'} + \ell$ where $0 \leq \ell \leq n - 1$.

Observe that if $j - i = -2$, then e belongs to the cycle $\langle(j, s), (j - 1, t''), (i, r)\rangle$ which is a copy of $C_{t'',s}$ if $t'' < s$, or a copy of $C'_{s,t''}$ if $s < t''$. If $j - i = -3$, then e belongs to the cycle $\langle(j, s), (j + 1, t''), (i, r)\rangle$ which is a copy of $C'_{t'',s}$ if $t'' < s$, or a copy of $C_{s,t''}$ if $s < t''$. Also, if $j - i = -4$, then e belongs to some copy of $C''_{\alpha'',\beta''}$. Thus, if $j - i \in [-4, -2]$, then e belongs to $G_{\alpha'',\beta''} + \ell$ where $0 \leq \ell \leq n - 1$. Since every edge of $K_{(2k+1)\times n}$ appears in some copy of G in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{(2k+1)\times n}$ into copies of G.

Case 2: G has at most one cycle of length 3. Suppose $y \geq 2$ and $z \geq 2$. By Lemma 1, there exists a path $P_{r,s}$ of length $2x - 1$ using the edge lengths in $\{0\} \cup$

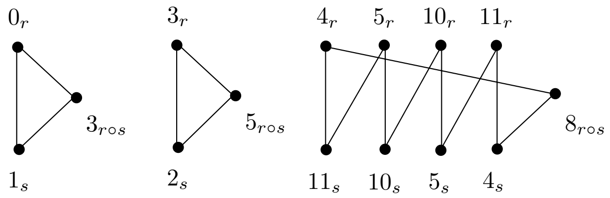


Figure 4: $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = y = 1$ and $z = 4$.

$\pm[y + z + 3, x + y + z + 1]$ with endpoints 0_r and 0_s. In the lemma, we would use $d_1 = y + z + 3$, $d_2 = y + z + 4$, $\ldots$, $d_{x-1} = x + y + z + 1$, so $V(P_{r,s}) \subseteq \big([0, \lceil \frac{x}{2} \rceil - 1] \cup [\lceil \frac{x}{2} \rceil + y + z + 2, x + y + z + 1]\big) \times \{r, s\}$. We construct the cycle $C_{r,s}$ of length $2x + 1$ from $P_{r,s}$ by adding the edges $\{0_r, (y + z)_{ros}\}$ and $\{0_s, (y + z)_{ros}\}$.

Next, we will construct the cycle $C'_{r,s}$ of length $2y + 1$. Let $P'_{r,s} = G'_1 + G'_2 + G'_3$ where

$$G'_1 = P(\lceil \tfrac{x}{2} \rceil, \lceil \tfrac{x}{2} \rceil + 3, y - 2)$$

$$G'_2 = \begin{cases} \big((\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s, (\lceil \frac{x}{2} \rceil + \frac{y+1}{2})_r, (\lceil \frac{x}{2} \rceil + \frac{y-1}{2})_s, \lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r\big), & \text{if } y-2 \text{ odd;} \\ \big((\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r, (\lceil \frac{x}{2} \rceil + \frac{y+2}{2})_s, (\lceil \frac{x}{2} \rceil + \frac{y+4}{2})_r, \lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s\big), & \text{if } y-2 \text{ even,} \end{cases}$$

$$G'_3 = \begin{cases} P\big(\lceil \frac{x}{2} \rceil + \frac{y+5}{2}, \lceil \frac{x}{2} \rceil - \frac{y-1}{2}, y-2\big), & \text{if } y-2 \text{ odd;} \\ Q\big(\lceil \frac{x}{2} \rceil + \frac{y+6}{2}, \lceil \frac{x}{2} \rceil - \frac{y-2}{2}, y-2\big), & \text{if } y-2 \text{ even.} \end{cases}$$

If $y = 2$, then $P'_{r,s} = G'_2 = \big(\lceil \frac{x}{2} \rceil_r, (\lceil \frac{x}{2} \rceil + 2)_s, (\lceil \frac{x}{2} \rceil + 3)_r, \lceil \frac{x}{2} \rceil_s\big)$.

Note that by **P1**, the first vertex of G'_1 is $\lceil \frac{x}{2} \rceil_r$, and the last vertex is $(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s$ if $y - 2$ is odd and $(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r$ if $y - 2$ is even; the first vertex of G'_3 is $(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r$ and the last vertex is $\lceil \frac{x}{2} \rceil_s$ if $y - 2$ is odd. By **Q1**, the first vertex of G'_3 is $(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s$ and the last vertex is $\lceil \frac{x}{2} \rceil_s$ if $y - 2$ is even.

For $i = 1$ or 3, let A'_i and B'_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the graph G_i. Then using **P2** and **Q2**, we compute

$$A'_1 = [\lceil \tfrac{x}{2} \rceil_r, (\lceil \tfrac{x}{2} \rceil + \lfloor \tfrac{y-2}{2} \rfloor)_r],$$

$$B'_1 = [(\lceil \tfrac{x}{2} \rceil + \lceil \tfrac{y+5}{2} \rceil)_s, (\lceil \tfrac{x}{2} \rceil + y + 1)_s],$$

$$A'_3 = [(\lceil \tfrac{x}{2} \rceil + \lceil \tfrac{y+5}{2} \rceil)_r, (\lceil \tfrac{x}{2} \rceil + y + 1)_r],$$

$$B'_3 = [\lceil \tfrac{x}{2} \rceil_s, (\lceil \tfrac{x}{2} \rceil + \lfloor \tfrac{y-2}{2} \rfloor)_s].$$

Note that $V(G'_1) \cap V(G'_2) = \{(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s\}$ if $y - 2$ is odd and $V(G'_1) \cap V(G'_2) = \{(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r\}$ if $y - 2$ is even and, $V(G'_2) \cap V(G'_3) = \{(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r\}$ if $y - 2$ is odd and $V(G'_2) \cap V(G'_3) = \{(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s\}$ if $y - 2$ is even; otherwise, G'_1, G'_2 and G'_3 are vertex disjoint. Therefore, $G'_1 + G'_2 + G'_3$ is a path of length $2y - 1$ with the endpoints $\lceil \frac{x}{2} \rceil_r$ and $\lceil \frac{x}{2} \rceil_s$. Since $V(P'_{r,s}) \subseteq [\lceil \frac{x}{2} \rceil, \lceil \frac{x}{2} \rceil + y + 1] \times \{r, s\}$, $P'_{r,s}$ is vertex-disjoint from

$P_{r,s}$.

Next, let E_i' denote the set of edge lengths in G_i' for $i = 1$ or 3. By **P3** and **Q3**, we have edge lengths

$$E_1' = [4, y+1],$$
$$E_3' = [-(y+1), -4].$$

Notice that the set of edge lengths in G_2' is $\{2, -1, -3\}$. Then construct the cycle $C_{r,s}'$ of length $2y+1$ from the path $P_{r,s}'$ by adding the edges $\{\lceil \frac{x}{2} \rceil_r, (\lceil \frac{x}{2} \rceil + y + z + 1)_{r\circ s}\}$ and $\{\lceil \frac{x}{2} \rceil_s, (\lceil \frac{x}{2} \rceil + y + z + 1)_{r\circ s}\}$.

Finally we will construct the cycle $C_{r,s}''$ of length $2z+1$. Let $P_{r,s}'' = G_1'' + G_2'' + G_3''$ where

$$G_1'' = P(x+y+z+2, x+2y+z+3, z-2),$$

$$G_2'' = \begin{cases} \big((\frac{2x+4y+3z+5}{2})_s, (\frac{2x+4y+3z-1}{2})_r, (\frac{2x+4y+3z+1}{2})_s, (\frac{2x+4y+3z+5}{2})_r\big), & \text{if } z-2 \text{ odd;} \\ \big((\frac{2x+2y+3z+2}{2})_r, (\frac{2x+2y+3z+8}{2})_s, (\frac{2x+2y+3z+6}{2})_r, (\frac{2x+2y+3z+2}{2})_s\big), & \text{if } z-2 \text{ even,} \end{cases}$$

$$G_3'' = \begin{cases} P\big(\frac{2x+4y+3z+5}{2}, \frac{2x+2y+z+5}{2}, z-2\big), & \text{if } z-2 \text{ odd;} \\ Q\big(\frac{2x+4y+3z+6}{2}, \frac{2x+2y+z+6}{2}, z-2\big), & \text{if } z-2 \text{ even.} \end{cases}$$

If $z = 2$, then $P_{r,s}'' = G_2'' = \big((x+y+4)_r, (x+y+7)_s, (x+y+6)_r, (x+y+4)_s\big)$.

Note that by **P1**, the first vertex of G_1'' is $(x+y+z+2)_r$, and the last vertex is $(\frac{2x+4y+3z+5}{2})_s$ if $z-2$ is odd and $(\frac{2x+2y+3z+2}{2})_r$ if $z-2$ is even; the first vertex of G_3'' is $(\frac{2x+4y+3z+5}{2})_r$ and the last vertex is $(x+y+z+2)_s$ if $z-2$ is odd. By **Q1**, the first vertex of G_3'' is $(\frac{2x+2y+3z+2}{2})_s$ and the last vertex is $(x+y+z+2)_s$ if $z-2$ is even.

For $i = 1$ or 3, let A_i'' and B_i'' denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the graph G_i''. Then using **P2** and **Q2**, we compute

$$A_1'' = [(x+y+z+2)_r, (x+y+\lfloor \tfrac{3z}{2} \rfloor + 1)_r],$$
$$B_1'' = [(x+2y+\lceil \tfrac{3z+5}{2} \rceil)_s, (x+2y+2z+1)_s],$$
$$A_3'' = [(x+2y+\lceil \tfrac{3z+5}{2} \rceil)_r, (x+2y+2z+1)_r],$$
$$B_3'' = [(x+y+z+2)_s, (x+y+\lfloor \tfrac{3z}{2} \rfloor + 1)_s].$$

Note that $V(G_1'') \cap V(G_2'') = \{(x+2y+\lceil \frac{3z+5}{2} \rceil)_s\}$ if $z-2$ is odd and $V(G_1'') \cap V(G_2'') = \{(x+y+\lfloor \frac{3z}{2} \rfloor + 1)_r\}$ if $z-2$ is even and, $V(G_2'') \cap V(G_3'') = \{(x+2y+\lceil \frac{3z+5}{2} \rceil)_r\}$ if $z-2$ is odd and $V(G_2'') \cap V(G_3'') = \{(x+y+\lfloor \frac{3z}{2} \rfloor + 1)_s\}$ if $z-2$ is even; otherwise, G_1'', G_2'' and G_3'' are vertex disjoint. Therefore, $G_1'' + G_2'' + G_3''$ is a path of length $2z-1$ with the endpoints $(x+y+z+2)_r$ and $(x+y+z+2)_s$. Since $V(P_{r,s}'') \subseteq [x+y+z+2, x+2y+2z+1] \times \{r, s\}$, $P_{r,s}''$ is vertex disjoint from $P_{r,s}$ and $P_{r,s}'$.

Next, let E_i'' denote the set of edge lengths in G_i'' for $i = 1$ or 3. By **P3** and **Q3**, we have edge lengths

$$E_1'' = [y+2, y+z-1]$$
$$E_3'' = [-(y+z-1), -(y+2)].$$

Notice that the set of edge lengths in G_2'' is $\{3, 1, -2\}$. Then, construct the cycle $C_{r,s}^{''}$ of length $2z+1$ from the path $P_{r,s}^{''}$ by adding the edges $\{(x+y+z+2)_r, (x+2y+2z+4)_{ros}\}$ and $\{(x+y+z+2)_s, (x+2y+2z+4)_{ros}\}$.

Since $(y+z)_{ros}$,$(\lceil \frac{x}{2} \rceil + y+z+1)_{ros}$ and $(x+2y+2z+4)_{ros}$ are different vertices, and $P_{r,s}$, $P'_{r,s}$ and $P''_{r,s}$ are vertex disjoint, we have $C_{r,s}$, $C_{r,s}^{'}$ and $C_{r,s}^{''}$ are also vertex disjoint. Figure 5 shows an example of $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = 4, y = 2$ and $z = 5$.

Let $G^*_{r,s} = \{G_{r,s} + \ell : 0 \le \ell \le n-1\}$. Then $G^*_{r,s}$ contains n distinct copies of G and all the edges of each length $i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y+z, y+z+2]$ in the induced subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$. Let $\mathcal{C} = \{G_{r,s} + \ell : 1 \le r < s \le 2k+1, 0 \le \ell \le n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G. We will show that every edge of $K_{(2k+1)\times n}$ appears in some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ with $r < s$ be an arbitrary edge of $K_{(2k+1)\times n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$ and $\beta'' = \max\{s, t''\}$. If $j - i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y+z, y+z+2]$, then e belongs to $G_{r,s} + \ell$ for some ℓ with $0 \le \ell \le n-1$. If $j - i \in [y+z, y+z+2]$, then e belongs to $G_{\alpha',\beta'} + \ell$ where $0 \le \ell \le n-1$. If $j - i \in [-(y+z+2), -(y+z)]$, then e belongs to $G_{\alpha'',\beta''} + \ell$ where $0 \le \ell \le n-1$. Since every edge of $K_{(2k+1)\times n}$ appears in some copy of G in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{(2k+1)\times n}$ into copies of G. ∎

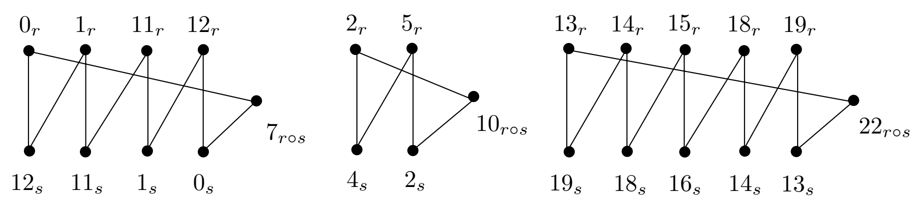


Figure 5: $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = 4$, $y = 2$ and $z = 5$.

In the proof of Theorem 2, if we replace idempotent symmetric quasigroups with symmetric quasigroups with holes, then we obtain a G-decomposition of $K_{k\times 2n}$ for every integer $k \ge 3$.

Theorem 3. *Let G be a 2-regular graph of order n consisting of exactly three odd cycles. For every integer $k \ge 3$, there exists a G-decomposition of $K_{k\times 2n}$.*

Proof. Let $G = C_{2x+1} \cup C_{2y+1} \cup C_{2z+1}$, where $x, y, z \ge 1$. Let $k \ge 3$ be an integer and let $Q = [1, 2k]$. For $i \in [1, k]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_n \times h_i$. Let

$n = 2x + 2y + 2z + 3$ and let $V(K_{k\times 2n}) = \mathbb{Z}_n \times [1, 2k]$ with the vertex-set partition $\{g_1, g_2, \ldots, g_k\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, \cdots, h_k\}$.

Fix r and s with $1 \le r < s \le 2k$ and $\{r, s\} \notin H$. We proceed in the same fashion as in the proof of Theorem 2 producing the graph $G_{r,s}$ consisting of a cycle $C_{r,s}$ of length $2x + 1$, a cycle $C'_{r,s}$ of length $2y + 1$, and a cycle $C''_{r,s}$ of length $2z + 1$ such that $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ are vertex disjoint.

We treat first the case where G contains at most one cycle of length 3 (thus we assume $y \ge 3$ and $z \ge 3$ as in Case 2 in Theorem 2). Note that for fixed r and s with $1 \le r < s \le 2k$ and with $\{r, s\} \notin H$, the set $\{G_{r,s} + \ell \colon 0 \le \ell \le n - 1\}$ contains n distinct copies of G and all the edges of lengths $i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y+z, y+z+2]$ in the induced subgraph of $K_{k\times 2n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$. Let $\mathcal{C} = \{G_{r,s} + \ell \colon 1 \le r < s \le 2k, \{r, s\} \notin H, 0 \le \ell \le n - 1\}$ and note that $\mathcal{C}$ contains $2k(k-1)n$ distinct copies of G. We wish to show that every edge of $K_{k\times 2n}$ appears in some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ where $r < s$ be an arbitrary edge of $K_{k\times 2n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$ and $\beta'' = \max\{s, t''\}$. If $j - i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y + z, y + z + 2]$, then e belongs to $G_{r,s} + \ell$ for some ℓ with $0 \le \ell \le n - 1$. If $j - i = [y + z, y + z + 2]$, then e belongs to $G_{\alpha',\beta'} + \ell$ where $0 \le \ell \le n - 1$. If $j - i = [-(y + z + 2), -(y + z)]$, then e belongs to $G_{\alpha'',\beta''} + \ell$ where $0 \le \ell \le n - 1$. Since every edge of $K_{k\times 2n}$ appears in some copy of G in $\mathcal{C}$ and since $\mathcal{C}$ contains $2k(k-1)n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{k\times 2n}$ into copies of G.

An argument similar to the one above can be used to treat the case where G contains at least two cycles of length 3 (corresponding to Case 1 in Theorem 2). ∎

5 G-decompositions of K_{2kn+1}

Let G of order n be the vertex-disjoint union of three odd cycles. It is shown in [5] and [4] that there exists a G-decomposition of K_{2n+1}. It was not known whether a G-decomposition of K_{2kn+1} exists for every positive integer k. Using the G-decomposition of K_{2n+1} and the result from Theorem 3, we can answer this question in the affirmative for $k \ge 3$.

Theorem 4. *Let G of order n be the vertex-disjoint union of three odd cycles. There exists a G-decomposition of K_{2kn+1} for every positive integer $k \ne 2$.*

Proof. Since there exists a G-decomposition of K_{2n+1}, we can assume that $k \ge 3$. For $i \in [1, k]$, let S_i be a set with $2n$ elements and let H_i be a complete graph of order $2n + 1$ with vertex set $S_i \cup \{\infty\}$. Let $V(K_{2kn+1}) = S_1 \cup S_2 \cup \ldots \cup S_k \cup \{\infty\}$. Thus, $K_{2kn+1} = H_1 \cup H_2 \cup \ldots \cup H_k \cup K_{k\times 2n}$. Since there is a G-decomposition of H_i for $i \in [1, k]$ and there is a G-decomposition of $K_{k\times 2n}$, the result follows. ∎

If a G-decomposition of K_n exists (i.e., if the Oberwolfach problem has a solution in this case), then a G-decomposition of K_{2kn+n} will also exist.

Theorem 5. *Let G of order n be the vertex-disjoint union of three odd cycles. If a G-decomposition of K_n exists, then there exists a G-decomposition of K_{2kn+n} for every positive integer k.*

Proof. Observe that $K_{2kn+n} = (2k+1)K_n \cup K_{(2k+1)\times n}$. Since a G-decomposition of K_n exists, a G-decomposition of $(2k+1)K_n$ will also exist. By Theorem 2, there exists a G-decomposition of $K_{(2k+1)\times n}$. The result follows. ∎

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