ALL GROUPS OF ODD ORDER HAVE STARTER-TRANSLATE 2-SEQUENCINGS

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ABSTRACT. Bailey defined 2-sequencings (terraces) of groups. She conjectured that all finite groups except elementary Abelian 2-groups (other than the cyclic group Z_2) have 2-sequencings and proved that the direct product of a 2-sequenceable group and a cyclic group of odd order is 2-sequenceable. It is shown here that all groups of odd order have a special type of 2-sequencing called a starter-translate 2-sequencing. It follows that if G is a group of odd order and the finite group H has a symmetric sequencing, then $G \times H$ also has a symmetric sequencing. Bailey used 2-sequencings for the purpose of constructing large numbers of quasi-complete Latin squares of a fixed order. Families of such squares can be useful in certain experiments. With this in mind, an alternative construction for starter-translate 2-sequencings of a large class of groups of odd order is presented. The construction is closely tied to the general method and shows that, with respect to the generation of 2-sequencings, the apparently unimportant step of replacing right cosets of a normal subgroup by left cosets can have major consequences. This method works on a class of groups containing all supersolvable groups of odd order.

§ 1. INTRODUCTION. In [8] Bailey generalized ideas of Gordon [9] to give an algebraic method for constructing quasi-complete Latin squares. One goal of her work was to find "valid randomization sets" of quasi-complete Latin squares of order n. She was able to do this for odd prime powers. Bailey defined the concept of a terraced group. This idea was rediscovered in [1] where it was called a 2-sequencing and has been the subject of several recent investigations [4, 5, 7, 13]. Bailey conjectured that the only finite groups that cannot be terraced are the elementary Abelian 2-groups $Z_2 \times Z_2 \times \cdots \times Z_2$ with at least two factors. She proved that if B is a terraced group and $C = Z_{2n+1}$, then $B \times C$ is terraced. This was generalized in [5] by allowing C to be any group of odd order that carries a special type of terrace called a starter-translate 2-sequencing. All cyclic groups of odd order satisfy this condition. The first, and major, goal of this paper is to show that its title is a true statement.

DEFINITION 1. Suppose G is a group of odd order $2n+1$ and identity e. Then $S = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ is a *left starter* for G if and only if

i) every non-identity element of G occurs in some pair of S,

ii) every non-identity element of G occurs in $\{x_i^{-1}y_i, y_i^{-1}x_i : 1 \le i \le n\}$.

If $h \in G$, then $hS = \{\{hx_i, hy_i\} : 1 \le i \le n\}$ is a *left translate* of S. If G is Abelian the adjective "left" can be omitted when discussing starters and translates.

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The special collection $\{\{x, x^{-1}\} : x \in G\backslash\{e\}\}$, the *patterned starter*, will be denoted PS_G.

LEMMA 1. *If G is a finite group of odd order, then PS_G is a left starter for G.*

PROOF. This is straightforward.

Suppose G is a finite group of order n with identity e. A *sequencing* of G is an ordering

$$\gamma : e, c_2, c_3, \ldots, c_n$$

of all elements of G such that the partial products

$$\delta : e, ec_2, ec_2c_3, \ldots, ec_2 \cdots c_n$$

are distinct and hence also all of G. Sequenceable Abelian groups have been characterized [9] as those Abelian groups with a unique element of order 2. Keedwell [12] has conjectured that all non-Abelian groups of order ≥ 10 are sequenceable. For information on this conjecture, see [2, 3, 11]. A sequencing

$$\gamma : e, c_2, \ldots, c_{n+1}, \ldots, c_{2n}$$

of a group G of order $2n$ with a unique element z of order 2 is said to be *symmetric* if and only if $c_{n+1} = z$ and for $1 \leq i \leq n-1$, $c_{n+1+i} = (c_{n+1-i})^{-1}$. One reason for interest in symmetric sequencings is because they induce 1-factorizations of complete graphs such that the symmetry group of the 1-factorization contains the group sequenced. This paper will deal primarily with the following generalization of the idea of a sequencing.

DEFINITION 2. Suppose H is a finite group of order n with identity e. A *2-sequencing* of H is an ordering

$$\sigma : e, s_2, s_3, \ldots, s_n$$

of certain elements of H (not necessarily distinct) such that

i) the associated partial products

$$\rho : e, es_2, es_2s_3, \ldots, es_2 \cdots s_n = e, t_2, t_3, \ldots, t_n$$

are distinct and hence all of H,

ii) if $y \in H$ and $y \neq y^{-1}$, then

$$|\{i : 2 \leq i \leq n \text{ and } (s_i = y \text{ or } s_i = y^{-1})\}| = 2,$$

iii) if $y \in H$ and $y = y^{-1}$, then

$$|\{i : 1 \leq i \leq n \text{ and } s_i = y\}| = 1.$$

The statement that the 2-sequencing σ is a *starter-translate* 2-sequencing (st-2-sequencing) means that both

$$S_{\sigma(H)} = \{s_3, s_5, \ldots, s_n\} \quad \text{and} \quad T_{\sigma(H)} = \{s_2, s_4, \ldots, s_{n-1}\}$$

are transversals of PS_H (so $|H|$ must be odd).

Note that if $S_{\sigma(H)}$ is a transversal of PS_H, then

$$A = \{\{t_2, t_3\}, \{t_4, t_5\}, \ldots, \{t_{n-1}, t_n\}\}$$

is a left starter for H and if $T_{\sigma(H)}$ is a transversal of PS_H, then

$$B = \{\{e, t_2\}, \{t_3, t_4\}, \ldots, \{t_{n-2}, t_{n-1}\}\}$$

is a left translate by t_n of a left starter

$$C = \{\{t_n^{-1}, t_n^{-1}t_2\}, \{t_n^{-1}t_3, t_n^{-1}t_4\}, \ldots \{t_n^{-1}t_{n-2}, t_n^{-1}t_{n-1}\}\}$$

for H. All Abelian groups of odd order have st-2-sequencings [5].

Suppose now that G is a group of odd order with normal subgroup K and $G/K \approx Q$ where Q has an st-2-sequencing $\hat{\sigma}(Q)$ and associated $\hat{\rho}(Q)$

$$\hat{\sigma}(Q) : \hat{e}, \hat{s}_2, \hat{s}_3, \ldots, \hat{s}_n$$
$$\hat{\rho}(Q) : \hat{e}, \hat{t}_2, \hat{t}_3, \ldots, \hat{t}_n.$$

Let $\rho : e, t_2, t_3, \ldots, t_n$ be an ordered collection of coset representatives such that for $1 \leq i \leq n$, $t_i \in \hat{t}_i$.

DEFINITION 3. With $\hat{\rho}(Q)$ and ρ as above, the statement that ρ is *compatible* with $\hat{\rho}$ means

i) e is the identity of G,

ii) if $\hat{s}_i = \hat{s}_j$, then $t_{i-1}^{-1}t_i = t_{j-1}^{-1}t_j$, and if $\hat{s}_i = \hat{s}_j^{-1}$, then $t_{i-1}^{-1}t_i = t_j^{-1}t_{j-1}$.

THEOREM 2. *If $G, K, Q, \hat{\rho}(Q)$ and ρ are as above, then it is always possible to choose ρ compatible with $\hat{\rho}$.*

PROOF. Pick ρ step-by-step as follows. Let $t_1 = e$ and choose t_2 arbitrarily in $\hat{t}_2$. From now on be more careful. For each i, $3 \leq i \leq n$, there are two cases. First, if $\hat{s}_i \notin \{\hat{s}_j, \hat{s}_j^{-1} : 2 \leq j < i\}$, then pick t_i arbitrarily in $\hat{t}_i$. If, on the other hand, there is a j, $2 \leq j < i$ such that $\hat{s}_i \in \{\hat{s}_j, \hat{s}_j^{-1}\}$, then the j is unique since $\hat{\sigma}(Q)$ is an st-2-sequencing of Q. If $\hat{s}_i = \hat{s}_j$, then t_i must be picked so that $t_{i-1}^{-1}t_i = t_{j-1}^{-1}t_j$. Since $t_{i-1}^{-1}\hat{t}_i$ is the entire appropriate coset, this can be done. The case $\hat{s}_i = \hat{s}_j^{-1}$ is similar and the proof is complete.

Before stating and proving the main result, it will be helpful to build a picture of the proposed construction as in [5]. Suppose G is a group of odd order with proper normal subgroup K and suppose K has an st-2-sequencing $\sigma(K)$ with associated partial product sequence

$$\rho(K) : e, v_2, v_3, \ldots, v_m.$$

Let G/K be denoted by Q and suppose Q has an st-2-sequencing $\hat{\sigma}(Q)$ with associated partial product sequence $\hat{\rho}(Q)$. By Theorem 2, one can choose

$$\rho : e, t_2, t_3, \ldots, t_n$$

compatible with $\hat{\rho}$. Now construct a $|Q| \cdot |K|$ array of points. Label the rows from top to bottom by the elements of ρ in order. Label the columns left to right by the elements of $\rho(K)$ in order. Build a Hamiltonian path through the array as shown in Figure 1.

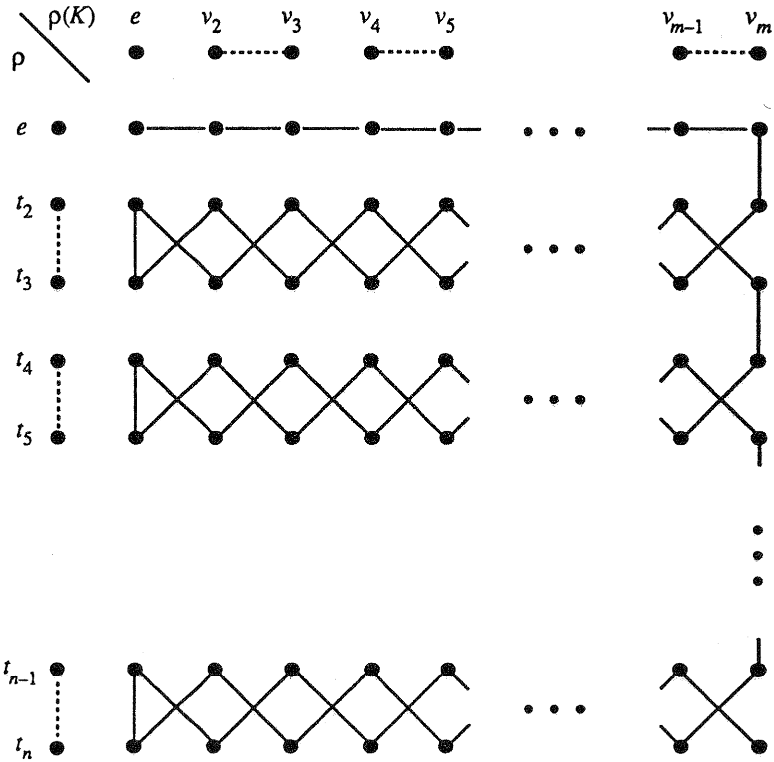


FIGURE 1

The starter pairs of $\rho(K)$ are designated by dotted lines. The pairs of ρ that correspond to starter pairs in $\hat{\rho}(Q)$ are also shown by dotted lines. The goal is to show that this path is the partial product sequence $\rho(G)$ of an st-2-sequencing of G if appropriate ideas are implemented. Orient the path by starting at e and proceeding as indicated. Make the obvious definitions with respect to the terms "horizontal edge," "vertical edge" and "diagonal edge" of the path in Figure 1. Every vertical edge may be thought of as being "projected from" an edge directly left of it in ρ. Similarly, every horizontal edge and diagonal edge is projected from the edge in $\rho(K)$ directly above it. Define edges $1, 3, 5, \ldots$ of the oriented path in Figure 1 to be *translate edges* for the proposed $\rho(G)$ and edges $2, 4, 6, \ldots$, to be the *starter edges*. It is easy to see that

i) starter edges in $\rho(K)$ and ρ project only to starter edges in $\rho(G)$,

ii) translate edges in $\rho(K)$ and ρ project only to translate edges in $\rho(G)$.

§ 2. THE CONSTRUCTIONS. The most general result will follow from Figure 1 by using the right coset Kt_i to label the t_i-row of Figure 1.

THEOREM 3. *Suppose the odd order group G is an extension of the normal subgroup K by Q. If K and Q have* st-2-*sequencings, then so does G.*

PROOF. Let $\rho(K)$ be the partial product sequence associated with an st-2-sequencing of K. Let $\hat{\rho}(Q)$ perform the corresponding function for Q and choose ρ compatible with $\hat{\rho}(Q)$. Construct the Figure 1 array and oriented path. The point at the intersection of the row labelled by t_i and the column labelled by v_j is associated with the group element $v_j t_i$. Thus, *right* cosets of K are used to label the rows of Figure 1.

If an edge of the oriented path is travelled from vertex c to vertex d, define

$$y_{cd} \in G \text{ such that } c y_{cd} = d. \tag{1}$$

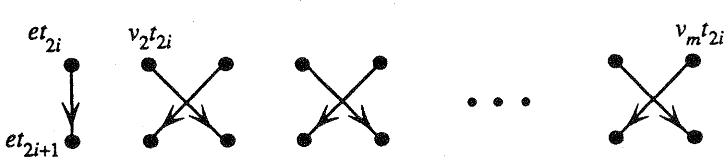
If the given path is denoted by $\rho(G)$, the idea is to compute the associated proposed st-2-sequencing $\sigma(G)$ via (1). It will suffice to show that applying (1) to the oriented starter (translate) edges of the path gives a transversal of PS_G.

Consider the starter edges first. The path begins by traversing K and clearly, if (1) is applied to these starter edges, the result is a transversal of PS_K. The right cosets of K in G that are not K itself come in inverse pairs. The next step is to show that when (1) is applied to a "band" of oriented starter edges, the result is a complete right coset of K and that different "bands" are associated with different pairs of cosets.

Consider the "band" in the path to the right of the Q starter edge (t_{2i}, t_{2i+1}). The starter edges in this band are all diagonal edges except for one vertical edge.

$$\begin{cases} (v_{2j}t_{2i}, v_{2j+1}t_{2i+1}), & 1 \le j \le (|K|-1)/2 = (m-1)/2 \\ (v_{2j+1}t_{2i}, v_{2j}t_{2i+1}), & 1 \le j \le (|K|-1)/2 \\ (et_{2i}, et_{2i+1}) & \end{cases} \tag{2}$$

Pictorially, this looks like



Define $t_{2i}^{-1}t_{2i+1} = w_i$ and recall that $\{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}$ are exactly the pairs of a starter S for K. Then applying (1) to (2) yields y_{cd} values like

$$t_{2i}^{-1}(v_{2j}^{-1}v_{2j+1})t_{2i+1} \quad \text{and} \quad t_{2i}^{-1}(v_{2j+1}^{-1}v_{2j})t_{2i+1}.$$

Let j vary, add the vertical edge and one gets

$$\begin{aligned} t_{2i}^{-1}Kt_{2i+1} &= (t_{2i}^{-1}Kt_{2i})(t_{2i}^{-1}t_{2i+1}) \\ &= Kw_i \end{aligned}$$

since K is normal in G. But $\hat{\rho}(Q)$ is an st-2-sequencing of Q and it is therefore clear that (1) applied to the oriented starter edges gives a transversal of PS_G.

The argument for the translate edges is slightly more complicated. As before, when (1) is applied to the translate edges of K, the result is a transversal of PS_K. Recall that $\rho(K) = hT \cup S$, where

$$T = \{\{c_j, d_j\} : 1 \leq j \leq (|K|-1)/2\}$$

is a starter for K. Then

$$hT = \{\{hc_j, hd_j\} : \{c_j, d_j\} \in T\}$$

and since hT has no pair containing h, $h = v_m$, the last element of $\rho(K)$. The remaining translate edges of the Figure 1 path consist of the band edges in band i

(3) $$\begin{cases} (hc_jt_{2i+1}, hd_jt_{2i}), & 1 \leq j \leq (|K|-1)/2 \\ (hd_jt_{2i+1}, hc_jt_{2i}), & 1 \leq j \leq (|K|-1)/2 \end{cases}$$

and the right vertical edges

(4) $$(ht_{2i-1}, ht_{2i}), \qquad 1 \leq i \leq (|Q|-1)/2.$$

Applying (1) to (3) yields

$$\begin{aligned} t_{2i+1}^{-1}[K\backslash\{e\}]t_{2i} &= t_{2i+1}^{-1}[K\backslash\{e\}]t_{2i+1}(t_{2i+1}^{-1}t_{2i}) \\ &\subset K(t_{2i+1}^{-1}t_{2i}). \end{aligned}$$

Now $e \notin t_{2i+1}^{-1}[K\backslash\{e\}]t_{2i+1}$, and $t_{2i+1}^{-1}t_{2i} = w_i^{-1}$ so

$$t_{2i+1}^{-1}[K\backslash\{e\}]t_{2i} = Kw_i^{-1}\backslash\{w_i^{-1}\}.$$

Each starter edge of ρ will lead to a unique w_i^{-1} and so the problem reduces to showing that if W^{-1} is the collection of all w_i^{-1} that arise, then applying (1) to (4) gives a transversal of

$$\{(w_i, w_i^{-1}) : w_i^{-1} \in W^{-1}\}.$$

Here is where use is made of the compatibility of ρ. Each translate pair (t_{2i-1}, t_{2i}) of ρ is associated with a unique starter pair (t_{2j}, t_{2j+1}) of ρ that gives left-starter differences in the same pair of cosets of K. By compatibility

(5) $$t_{2i-1}^{-1}t_{2i} \in \{t_{2j}^{-1}t_{2j+1}, t_{2j+1}^{-1}t_{2j}\}.$$

The result of applying (1) to the right vertical edge associated with (t_{2i-1}, t_{2i}) is

$$t_{2i-1}^{-1}h^{-1}ht_{2i} = t_{2i-1}^{-1}t_{2i}$$

and thus is either w_j or w_j^{-1}, depending on the element in the set described in (5) that equals $t_{2i-1}^{-1}t_{2i}$. This completes the proof of Theorem 3.

COROLLARY 4. *All groups of odd order have* st-2-*sequencings.*

PROOF. The argument is by induction. Since all Abelian groups of odd order have st-2-sequencings [5], it is possible to get started (the group of order 1 satisfies the starter-translate part of Definition 2 vacuously). Suppose all groups of odd order $\leq 2n-1$ have st-2-sequencings and let G be a group of order $2n+1$. By the Feit-Thompson Theorem, G is solvable and so it has a normal Abelian subgroup K of order more than one [14, p. 85]. Since $|K| \leq |G|$ and $|G/K| < |G|$ and all orders are odd, the result follows.

COROLLARY 5. *Suppose the odd order group G is an extension of K by Q. If both K and Q have* st-*sequencings (so both K and Q are non-Abelian), then so does G.*

PROOF. This follows from the argument for Theorem 3 mutatis mutandis (st-sequencings are known to exist [5]).

COROLLARY 6. *If G is a group of odd order and H is a group with a symmetric sequencing, then $G \times H$ has a symmetric sequencing.*

PROOF. Since H has a symmetric sequencing, H has a unique element of order 2 which generates a normal subgroup isomorphic to Z_2. One has

$$(G \times H)/Z_2 \approx G \times H/Z_2.$$

Now G has an st-2-sequencing and H/Z_2 has a 2-sequencing [1, Theorem 2]. Thus, $G \times H/Z_2$ has a 2-sequencing [5, Theorem 6] and the result follows [1, Theorem 4].

It is well-known that the Abelian groups with symmetric sequencings are exactly the Abelian groups with sequencings. As stated previously, this is precisely the class of Abelian groups with a unique element of order 2. It is also known [4] that all dicyclic groups of order at least 12 have symmetric sequencings and [6] that all finite Hamiltonian groups with a unique element of order 2, with the single exception of Q_4, have symmetric sequencings.

Here is another result on groups that carry symmetric sequencings.

COROLLARY 7. *Suppose G is a group such that $|G| \equiv 2 \pmod 4$. Then G has a symmetric sequencing if and only if G has a unique element of order* 2.

PROOF. If G has a symmetric sequencing, then G has a unique element of order 2 by definition. Conversely, if G has a unique element of order two, let Z_2 denote the subgroup it generates. Then Z_2 is a central Sylow 2-subgroup of G so it is in the center of its normalizer. Thus Z_2 has a normal complement Q [10, p. 203] which has odd order. Therefore $G \approx Q \times Z_2$ and Corollary 6 applies.

The use of right cosets to label the rows of Figure 1 meant that the application of (1) to (2) led to elements like

$$t_{2i}^{-1}(v_{2j}^{-1}v_{2j+1})t_{2i+1}.$$

As j varied, this became (adding the vertical edge)

$$t_{2i}^{-1}Kt_{2i+1}.$$

Suppose left cosets are used to label the rows of Figure 1. Now the application of (1) to the adjusted (2) leads to elements like

$$v_{2j}^{-1}(t_{2i}^{-1}t_{2i+1})v_{2j+1}.$$

As j varies the pairs $\{v_{2j}, v_{2j+1}\}$ run through the pairs of a starter $S = \{\{a_i, b_i\} : 1 \leq i \leq (|K|-1)/2\}$ for K. If $t_{2i}^{-1}t_{2i+1} = w$, the set that arises is

$$\{a_i^{-1}wb_i, b_i^{-1}wa_i : \{a_i, b_i\} \in S\}$$

and a quite different situation develops. It will be useful to agree to the following

HYPOTHESIS (*). *Let G be a group of odd order with proper normal Abelian subgroup K and starter S for K.*

DEFINITION 4. Suppose G, K and S satisfy hypothesis (*). If $w \in G$, define

$$Sep(w, S) = \{a_i^{-1}wb_i, b_i^{-1}wa_i : \{a_i, b_i\} \in S\}.$$

i) S is *w-separating* if and only if $|Sep(w,S)| = |K| - 1$,

ii) S is *G-separating* if and only if for any $w \in G$, S is w-separating,

iii) S is *absolutely separating* if and only if for any G such that G, K and S satisfy hypothesis (*), S is G-separating,

iv) S is *G-aconjugate* if and only if no pair of S contains elements that are conjugate in G,

v) S is *absolutely aconjugate* if and only if for any G such that G, K and S satisfy hypothesis (*), S is G-conjugate.

It is not hard to see that (i) does not imply (ii), (ii) does not imply (iii) and (iv) does not imply (v). Note that $Sep(w, S) \subset wK$.

LEMMA 8. *Suppose G is a group of odd order and $v, w \in G$. Then $vw = wv^{-1}$ if and only if $v = e$.*

PROOF. If $v = e$, the result is trivial. Conversely, if $w = e$, the result is obvious since $|G|$ is odd. Suppose now that $w \neq e$. Then

$$vw = wv^{-1} \text{ iff } w^{-1}vw = v^{-1}.$$

This means that the inner automorphism defined by conjugation by w takes v to v^{-1} and v^{-1} to v. Now w has odd order $2n+1$. Thus

$$\begin{aligned} v &= w^{-(2n+1)}vw^{2n+1} \\ &= w^{-1}(w^{-2n}vw^{2n})w \\ &= w^{-1}vw \end{aligned}$$

so $v = v^{-1}$ and the result follows.

THEOREM 9. *Suppose G, K and PS_K satisfy hypothesis (*). The following statements hold.*

i) PS_K *is absolutely separating,*

ii) PS_K *is absolutely aconjugate,*

iii) *If* $w \in G$, $Sep(w, PS_K) = wK \backslash \{w\}$.

PROOF. By definition, if $w \in G$,

$$Sep(w, PS_K) = \{awa : a \in K \backslash \{e\}\}.$$

Then

$$awa = bwb \text{ iff } b^{-1}awab^{-1} = w$$
$$\text{iff } (ab^{-1})w(ab^{-1}) = w \text{ iff } vw = wv^{-1}, \quad v = ab^{-1}$$

and (i) follows from Lemma 8. Part (iii) is clear since the argument just given works just as well if $a = e$. Finally, a is conjugate to a^{-1} in G if and only if there is a $w \in G$ such that $w^{-1}aw = a^{-1}$. But then $aw = wa^{-1}$ and again Lemma 8 yields (ii).

If G, K and S satisfy hypothesis $(*)$, $w \in G$ and $h \in K$, then

$$Sep(w, hS) = \{(ha_i)^{-1}w(hb_i), (hb_i)^{-1}w(ha_i) : \{a_i, b_i\} \in S\}$$

and hS is *w-separating* if and only if $|Sep(w, hS)| = |K| - 1$.

THEOREM 10. *Suppose* G, K *and* S *satisfy hypothesis* $(*)$, $w \in G$ *and* $h \in K$. *The following results hold.*

i) S *is w-separating if and only if* hS *is w-separating,*

ii) S *is G-separating if and only if* hS *is G-separating,*

iii) S *is absolutely separating if and only if* hS *is absolutely separating,*

iv) S *is w-separating if and only if* S *is hw-separating,*

v) S *is w-separating if and only if* S *is* w^{-1}*-separating.*

vi) $Sep(w, hS) = Sep(h^{-1}wh, S) = h^{-1}Sep(w, S)h$.

If S *is w-separating and G-aconjugate, then*

vii) $Sep(w, S) = wK \backslash \{w\}$,

PROOF. The arguments are all straightforward.

The first three parts above show that translates have the same separating properties as the starter. Parts (iv) and (v) state that S is w-separating if and only if S is v-separating for all $v \in wK \cup w^{-1}K$. The last two parts give useful information on the element of vK missing from $Sep(v, S)$.

The next simple lemma is a barrier that prevents the following argument from working on all groups of odd order.

LEMMA 11. *Suppose* G *is a group of odd order and* $h \in G \backslash \{e\}$. *If the pairs of* $PS_G \cup h \cdot PS_G$ *form the partial product sequence of a starter-translate 2-sequencing of* G, *then* G *is cyclic.*

PROOF. If G is not cyclic, then h generates a cyclic proper subgroup and the pairs of $PS_G \cup h \cdot PS_G$ cannot form a Hamiltonian path through all of G.

THEOREM 12. *Suppose the odd order group G is an extension of an Abelian group K by Q and Q has an* st-2-*sequencing. If there is an* st-2-*sequencing for K with partial product sequence $\rho(K) : hT \cup S$ where S and T are starters for K such that*

i) *S and T are G-separating, and*

ii) *S and T are G-aconjugate,*

then G has an st-2-*sequencing.*

PROOF. As stated, $\rho(K)$ is the partial product sequence associated with an st-2-sequencing of K. Let $\hat{\rho}(Q)$ perform the corresponding function for Q and choose ρ compatible with $\hat{\rho}(Q)$. Construct the Figure 1 array and oriented path. The point at the intersection of the row labelled by t_i and the column labelled by v_j is associated with the group element $t_i v_j$. Thus *left* cosets of K are used to label the rows of Figure 1. Define y_{cd} as in (1). Analogous to the proof of Theorem 3, it will suffice to show the oriented starter (translate) edges of the path are associated with a transversal of PS_G in $\sigma(G)$.

Consider the starter edges first. As before, the starter edges of the first part of the path give a transversal of PS_K. The starter edges in the (t_{2i}, t_{2i+1})-band are as in (2) but with all products reversed (i.e., $v_{2j}t_{2i}$ is reversed to $t_{2i}v_{2j}$, etc.).

Define $t_{2i}^{-1}t_{2i+1} = w$ and recall that $\{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}$ are exactly the pairs of the starter

$$S = \{\{a_j, b_j\} : 1 \leq j \leq (m-1)/2\}$$

for K. Then applying (1) to the modified (2) yields

$$Sep(w, S) \cup \{w\}.$$

Since S is G-separating and G-aconjugate,

$$S(w, S) \cup \{w\} = wK$$

by Theorem 10. But $\hat{\rho}(Q)$ is an st-2-sequencing of Q and it is therefore clear that (1) applied to the oriented starter edges gives a transversal of PS_G.

The argument for the translate edges is again slightly more complicated. As before, when (1) is applied to the translate edges of K, the result is a transversal of PS_K. Let T, hT and $h = v_m$ be as in the proof of Theorem 3. The remaining translate edges are as in (3) and (4) but with all products reversed (i.e., $(hc_j)t_{2i+1}$ is reversed to $t_{2i+1}(hc_j)$, etc.).

Define $h^{-1}t_{2i+1}^{-1}t_{2i}h = w_i$ and note that w_i belongs to the same coset of K as $t_{2i+1}^{-1}t_{2i}$. Then applying (1) to the modified (3) yields

$$Sep(w_i, T) = w_i K \backslash \{w_i\}$$

since T is G-separating and G-aconjugate.

Each starter edge of ρ will lead to a unique w_i and so the problem reduces to showing that if W is the collection of all w_i that arise, then applying (1) to the modified (4) gives a transversal of

$$\{(w_i, w_i^{-1}) : w_i \in W\}.$$

Again, the compatibility of ρ allows the argument here to finish as in the proof of Theorem 3.

Here are some consequences of Theorem 12.

COROLLARY 13. *Any extension of a finite cyclic group of odd order by an* st-2-*sequenceable group is* st-2-*sequenceable by the method of Theorem* 12.

PROOF. Every cyclic group of odd order has an st-2-sequencing consisting of the patterned starter and a translate of the patterned starter. This was, perhaps, first proved in [15] although not expressed in those terms. The result follows from Theorems 9, 10 and 12.

COROLLARY 14. *Every supersolvable group of odd order is* st-2-*sequenceable by the method of Theorem* 12.

PROOF. The argument is by induction. Since all Abelian groups of odd order have st-2-sequencings, it is possible to get started. Suppose all supersolvable groups of odd order $\leq 2n-1$ have st-2-sequencings and let G be supersolvable of order $2n+1$. Then G has a normal cyclic subgroup K (see [10] for information on supersolvable groups) and G/K is supersolvable so the result follows from Corollary 13.

An analogue of Corollary 5 could be stated using the construction of Theorem 12.

As things stand currently, the left coset labelling of Figure 1 does not give as general a result as the right coset labelling. A natural way to attempt to extend Corollary 14 to all (solvable) groups of odd order is to note [14, p. 85] that if K is a minimal normal subgroup of a finite solvable group, then there is a prime p such that either K is cyclic of order p or the direct product of at least two cyclic groups of order p. This may lead to some interesting questions about starters in elementary Abelian groups.

How general is Corollary 14? It is known [16, p. 105] that there is a non-supersolvable group of odd order $n < 1000$ if and only if

i) $n < 1000$ is an odd multiple of 75, or

ii) $n \in \{351, 363, 405, 867\}$.

The example of order 75 (there is only one) is a semi-direct product. The authors' first results on the problem of constructing st-2-sequencings were through Theorem 12 (hence Corollary 14) and a special case of Theorem 3 dealing with semi-direct products that was proved in a somewhat different fashion. These methods are sufficient (notes available from the authors on request) to handle all groups of odd order < 1000.

QUESTION. Is there a group of odd order that is not supersolvable and not expressible as a non-trivial semi-direct product?

Finally, if the goal is to use Theorems 3 and 12 to construct large numbers of 2-sequencings on a suitable group G, then one should note the following possibilities.

i) use different st-2-sequencings on K and Q (Theorem 3),

ii) use the full power of [15] on K with Corollary 13,

iii) use the "t-weave" of [5] as well as the "s-weave" that was used here.

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