GRAPHS WITH A PRESCRIBED ADJACENCY PROPERTY

W. Ananchuen and L. Caccetta

School of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987
Perth 6001
Western Australia.

ABSTRACT:

A graph G is said to have property P(m,n,k) if for any set of m + n distinct vertices of G there are at least k other vertices, each of which is adjacent to the first m vertices of the set but not adjacent to any of the latter n vertices. The problem that arises is that of characterizing graphs having property P(m,n,k). In this paper, we present properties of graphs satisfying the adjacency property. In addition, for small m and n we show that all sufficiently large Paley graphs satisfy P(m,n,k).

1. INTRODUCTION

For our purposes graphs are finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [5]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. However, we denote the complement of G by $\overline{G}$.

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Australasian Journal of Combinatorics 6(1992), pp.155-175

n distinct vertices there are at least k other vertices, each of which is adjacent to the first m vertices but not adjacent to any of the latter n vertices. The class of graphs having property $P(m,n,k)$ is denoted by $\mathcal{G}(m,n,k)$. Observe that if $G \in \mathcal{G}(m,n,k)$, then $\overline{G} \in \mathcal{G}(n,m,k)$. The cycle C_ν of length ν is a member of $\mathcal{G}(1,1,1)$ for every $\nu \geq 5$. The well known Petersen graph is a member of $\mathcal{G}(1,2,1)$ and also of $\mathcal{G}(1,1,2)$. In fact, as observed by Exoo [10], any graph with girth at least 5 and minimum degree at least k is in $\mathcal{G}(1,n,k-n)$ for $1 \leq n \leq k - 1$. Despite these relatively simple examples few members of $\mathcal{G}(m,n,k)$ have been found. The problem that arises is that of characterizing the class $\mathcal{G}(m,n,k)$; this problem is difficult for $m \geq 2$ and $n \geq 2$. One particularly interesting problem that has attracted attention is that of determining the function

$$p(m,n,k) = \min\{\nu(G) : G \in \mathcal{G}(m,n,k)\}.$$

Exoo [10] established bounds on $p(n,n,1)$.

Blass and Harary [3] established, using probabilistic methods, that almost all graphs have property $P(n,n,1)$. From this it is not too difficult to show that almost all graphs have property $P(m,n,k)$. Despite this result few graphs have been constructed which exhibit the property $P(m,n,k)$. Exoo and Harary [9] studied the class $\mathcal{G}(1,n,1)$ and established a number of important properties including the connection with cages. In particular, they established that for $n \leq 6$ the smallest order graphs of this class are the $(n+1,5)$ - cages. They conjectured that if $G \in \mathcal{G}(1,n,1)$ and G has girth at most 4, then $\nu(G) \geq n^2 + 3n + 2$. A particular case (n sufficiently large) of this conjecture was established by Caccetta and Vijayan [7].

An important graph in the study of the class $\mathcal{G}(m,n,k)$ is the so called Paley graph G_p defined as follows. Let $p \equiv 1 (\mathrm{mod}\ 4)$ be a prime. The vertices of G_p are the elements of the Galois field GF(p) and are labelled $0,1,\ldots,p-1$. Two vertices i and j are joined by an edge if and only if their difference is a quadratic residue modulo p, that is $i-j \equiv y^2 (\mathrm{mod}\ p)$ for some $y \in GF(p)$.

Blass, Exoo and Harary [4] showed that $G_p \in \mathcal{G}(n,n,1)$ for $p > n^2 2^{4n}$. Caccetta, Vijayan and Wallis [8] established that $G_p \notin \mathcal{G}(2,2,1)$ for $p < 61$ and $G_p \in \mathcal{G}(2,2,1)$ for $61 \le p \le 173$. They conjectured that $G_p \in \mathcal{G}(2,2,1)$ for every $p \ge 61$. We shall confirm this conjecture in Section 3. In addition, we prove that : $G_p \in \mathcal{G}(2,2,k)$ for every $p > (5 + 2\sqrt{4k + 6})^2$; $G_p \in \mathcal{G}(n,n,1)$ for every $p > ((2n - 3)^{2n-1} + 4)^2$; and $G_p \in \mathcal{G}(1,2,k)$ for every $p > (1 + 2\sqrt{2k})^2$. Computational results are presented which establish the smallest Paley graphs in $\mathcal{G}(2,2,k)$ for small k.

In the next section we present some properties of the class $\mathcal{G}(m,n,k)$. We conclude this section by noting that a variation of this problem has recently been considered by Alspach, Chen and Heinrich [1].

2. PROPERTIES OF THE CLASS $\mathcal{G}(m,n,k)$

For disjoint subsets A and B of V(G) we denote by N(A/B) the set of vertices of G not in $A \cup B$ which are adjacent to each vertex of A and not adjacent to any vertex of B. When $A = \{a_1,a_2,\ldots,a_m\}$ and $B = \{b_1,b_2,\ldots,b_n\}$ we sometimes write for convenience N(A/B) as $N(a_1,a_2,\ldots,a_m/b_1,b_2,\ldots,b_n)$. Further, we extend our notation so that for $X \subseteq V(G)$, N(X/) (N(/X)) denotes the set of vertices of G - X which are adjacent (non-adjacent) to every vertex of X. Note that X can be a single element. Where appropriate, lower case letters will denote the cardinality of the set defined by the corresponding upper case letters. Thus, for example, $n(a/b) = |N(a/b)|$.

In the following lemmas we establish a number of properties of the class $\mathcal{G}(m,n,k)$. We often make use of the following simple fact. If $G \in \mathcal{G}(m,n,k)$, then $n(X/Y) \geq k$ for any disjoint set of vertices X and Y with $|X| \leq m$ and $|Y| \leq n$.

Lemma 2.1: If $G \in \mathcal{G}(m,n,k)$, then $\delta(G) \geq m + n + k-1$.

Proof: Suppose to the contrary that $d_G(u) = d \leq m + n + k-2$. Let $v_1,v_2,\ldots,v_d$ denote the neighbours of u. Observe that $d - (m + n-1) \leq k-1$ and hence $n(u,v_1,\ldots,v_{m-1}/v_m,v_{m+1},\ldots,v_{m+n-1}) \leq k-1$, a contradiction. This proves the lemma. □

Lemma 2.2: Let $\{u_1,u_2,\ldots,u_m, v_1,v_2,\ldots,v_n\}$ be a set of m + n vertices in a graph $G \in \mathcal{G}(m,n,k)$. Then

(a) $n(u_1,u_2,\ldots,u_t/\ ) \geq m + n + k-t$, for $t \leq m$,

and

(b) $n(u_1,u_2,\ldots,u_m/v_1,v_2,\ldots,v_\ell) \geq n + k-\ell$, for $\ell \leq n$.

Proof: We prove only (a) as the proof of (b) is similar. Suppose to the contrary that

$$n(u_1,u_2,\ldots,u_t/\ ) = d \leq m + n + k-t-1.$$

Let $x_1,x_2,\ldots,x_d$ denote the vertices of $N(u_1,u_2,\ldots,u_t/\ )$. We have

$$n(u_1,u_2,\ldots,u_t, x_1,x_2,\ldots,x_{m-t}/x_{m-t+1},x_{m-t+2},\ldots,x_{m-t+n})$$
$$\leq d - (m + n-t) \leq k - 1,$$

a contradiction. This proves (a). □

An immediate Corollary of Lemma 2.2(b) is the following.

Corollary: For $1 \le \ell \le n$, $\mathcal{G}(m,n,k) \subseteq \mathcal{G}(m,n-\ell,k+\ell)$. □

The next few lemmas establish the properties of $\mathcal{G}(m,n,k)$ in terms of vertex degrees.

Lemma 2.3: Let G_0 be a graph in $\mathcal{G}(m,n,k)$ having minimum order. Then for any $G \in \mathcal{G}(m,n+1,k)$

$$\nu(G) \ge \nu(G_0) + \Delta(G) + 1.$$

Proof: Let w be any vertex of G. Clearly

$$G_w = G - w - N(w/\ ) \in \mathcal{G}(m,n,k)$$

and hence

$$\nu(G_w) = \nu(G) - 1 - d_G(w)$$
$$\ge \nu(G_0)\ .$$

This proves the lemma. □

Observe that for $m \ge 2$ every vertex of a graph $G \in \mathcal{G}(m,n,k)$ is contained in a triangle. In fact, every edge of G is in some triangle. For $m = 1$ we have the following result.

Lemma 2.4: Let $G \in \mathcal{G}(1,n,k)$. If $d_G(u) = n + k$, then u is on no cycle of length less than 5.

Proof: Let $v_1, v_2, \ldots, v_{n+k}$ be the neighbours of u. Suppose C is the smallest cycle of G containing u. We may suppose without any loss of generality that v_1, $v_2 \in C$. If C has length 3 then $v_1v_2 \in E(G)$. Since $d_G(u) = n + k$, we have

$$n(u/v_2,v_3,\ldots,v_{n+1}) \le k - 1$$

This contradicts the fact that $G \in \mathcal{G}(1,n,k)$. So C cannot have length 3. Suppose it has length 4 and let u, v_1,v_2 and w be the vertices of C. Then, since $d_G(u) = n + k$, we have

$$n(u/w,v_3,v_4,\ldots,v_{n+1}) \le k - 1,$$

again a contradiction. This completes the proof. □

Lemma 2.5: Let $G \in \mathcal{G}(1,n,k)$. If G has girth at least 5, then $G \in \mathcal{G}(1,n+\ell,k-\ell)$ for $1 \le \ell \le k-1$.

Proof: Let $u,v_1,v_2,\ldots,v_{n+\ell}$ be any $n + \ell + 1$ vertices of G. Let

$$N(u/v_{1+\ell},v_{2+\ell},\ldots,v_{n+\ell}) = \{x_1,x_2,\ldots,x_d\} .$$

Then $d \ge k$. Since G has girth at least 5, we have for each i, $v_i \in N(x_j/\ )\cup\{x_j\}$ for at most one j. Consequently $n(u/v_1,v_2,\ldots,v_{n+\ell}) \ge k-\ell$ and hence $G \in \mathcal{G}(1,n+\ell,k-\ell)$ as required. □

As a corollary we have :

Corollary: If $G \in \mathcal{G}(1,n,k)$ is (n+k)-regular, then $G \in \mathcal{G}(1,n+\ell,k-\ell)$ for $1 \le \ell \le k-1$. □

3. MAIN RESULTS

In this section we will establish some adjacency properties of the Paley graph G_p of prime order p defined in Section 1. We begin with some number theoretic results which we make use of in our proofs.

For odd prime p the **Legendre symbol** $\left(\frac{a}{p}\right)$ is defined as :

$$\left(\frac{a}{p}\right) = \begin{cases} 1\ , & \text{if a is a quadratic residue modulo p,} \\ 0\ , & \text{if } p|a, \\ -1\ , & \text{otherwise.} \end{cases}$$

It is well known (see [2]) that

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)\ , \text{ if } a \equiv b (\text{mod } p)\ , \tag{3.1}$$

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\ , \tag{3.2}$$

and

$$\sum_{x=0}^{p-1} \left(\frac{x}{p}\right) = 0\ . \tag{3.3}$$

It follows from (3.3) that

$$\sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right) = 0\ . \tag{3.4}$$

In our next two lemmas we make use of the following standard terminology. We write " $\sum_{x(\text{mod} p)}$ " whenever the summation is taken over a complete residue system modulo p. More specifically, if $x_1, x_2, \ldots, x_p$ is any complete residue system modulo p and $C_j = C_{x_i}$ whenever $j \equiv x_i$ (mod p), then

$$\sum_{j=0}^{p-1} C_j = \sum_{i=1}^{p} C_{x_i} = \sum_{x(\text{mod } p)} C_x\ .$$

Lemma 3.1: (Burgess [6]) Let p be an odd prime and let $a_1, a_2, \ldots, a_s$ be distinct residues modulo p. Then

$$\left| \sum_{x(\text{mod } p)} \left(\frac{(x-a_1)(x-a_2) \ \ldots \ (x-a_s)}{p} \right) \right| \le (s-1)\sqrt{p} \ . \qquad \square$$

Lemma 3.2: Let p be an odd prime and let $a_1, a_2, \ldots, a_s$ be distinct residues modulo p. Then for even s

$$\sum_{x=0}^{p-1} \left(\frac{(x-a_1)(x-a_2) \ \ldots \ (x-a_s)}{p} \right)$$

$$= -1 \pm \sum_{y(\text{mod } p)} \left(\frac{(y+b_1)(y+b_2) \ \ldots \ (y+b_{s-1})}{p} \right)$$

for some set $\{b_1, b_2, \ldots, b_{s-1}\}$ of distinct residues modulo p.

Proof: We write

$$\sum_{x=0}^{p-1} \left(\frac{(x-a_1)(x-a_2) \ \ldots \ (x-a_s)}{p} \right)$$

$$= \sum_{x(\text{mod } p)} \left(\frac{(x-a_1)(x-a_2) \ \ldots \ (x-a_s)}{p} \right)$$

$$= \sum_{x(\text{mod } p)} \left(\frac{x(x+a_1-a_2)(x+a_1-a_3)\ldots(x+a_1-a_s)}{p} \right). \qquad (3.5)$$

Note the latter equality is valid, since x and hence $x + a_1$ assume all values in a complete residue system modulo p. Now since $a_1, a_2, \ldots, a_s$ are distinct (mod p), then $\lambda_i = a_1 - a_{i+1} \not\equiv 0 (\text{mod } p)$ for $1 \le i \le s-1$.

If $x \not\equiv 0(\text{mod } p)$, then there exists an y such that $xy \equiv 1(\text{mod}$

p). Furthermore, $\left(\frac{y^s}{p}\right) = 1$, since s is even. If $x \equiv 0 \pmod p$, then $\left(\frac{x}{p}\right) = 0$. Thus we can write (3.5) as

$$\sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{x(x+\lambda_1)(x+\lambda_2)\ \ldots\ (x+\lambda_{s-1})}{p}\right)$$

$$= \sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{y^s}{p}\right)\left(\frac{x(x+\lambda_1)(x+\lambda_2)\ \ldots\ (x+\lambda_{s-1})}{p}\right)$$

$$= \sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{xy(xy+\lambda_1 y)(xy+\lambda_2 y)\ \ldots\ (xy+\lambda_{s-1} y)}{p}\right)$$

$$= \sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{(1+\lambda_1 y)(1+\lambda_2 y)\ \ldots\ (1+\lambda_{s-1} y)}{p}\right).$$

Since, for each i, $\lambda_i \not\equiv 0 \pmod p$ there exists λ'_i , such that $\lambda_i\lambda'_i = 1$. Furthermore,

$$\left(\frac{\lambda_1\lambda'_1\ \lambda_2\lambda'_2\ \ldots\ \lambda_{s-1}\ \lambda'_{s-1}}{p}\right) = 1.$$

Now using the same idea as above we can write :

$$\sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{(1+\lambda_1 y)(1+\lambda_2 y)\ \ldots\ (1+\lambda_{s-1} y)}{p}\right)$$

$$= \sum_{\substack{x \pmod p \\ x \not\equiv 0 \pmod p}} \left(\frac{(\lambda_1\lambda_2\ \ldots\ \lambda_{s-1})(\lambda'_1+y)(\lambda'_2+y)\ \ldots\ (\lambda'_{s-1}+y)}{p}\right). \tag{3.6}$$

Let $\lambda = \lambda_1\lambda_2 \ldots \lambda_{s-1}$ and $\lambda' = \lambda'_1\lambda'_2 \ldots \lambda'_{s-1}$. Since $\lambda_i \not\equiv 0 \pmod p$ for each i, we have $\lambda \not\equiv 0 \pmod p$ and so $\left(\frac{\lambda}{p}\right) = \pm 1$. As x assumes all values in a reduced residue system modulo p, so does y. Hence we can write (3.6) as :

$$\sum_{\substack{y \pmod p \\ y \not\equiv 0 \pmod p}} \left(\frac{\lambda}{p}\right)\left(\frac{(y+\lambda'_1)(y+\lambda'_2) \ldots (y+\lambda'_{s-1})}{p}\right)$$

$$= \sum_{y \pmod p} \left(\frac{\lambda}{p}\right)\left(\frac{(y+\lambda'_1)(y+\lambda'_2) \ldots (y+\lambda'_{s-1})}{p}\right) - \left(\frac{\lambda}{p}\right)\left(\frac{\lambda'}{p}\right)$$

$$= \left(\frac{\lambda}{p}\right) \sum_{y \pmod p} \left(\frac{(y+\lambda'_1)(y+\lambda'_2) \ldots (y+\lambda'_{s-1})}{p}\right) - 1$$

$$= -1 \pm \sum_{y \pmod p} \left(\frac{(y+\lambda'_1)(y+\lambda'_2) \ldots (y+\lambda'_{s-1})}{p}\right).$$

This completes the proof of the lemma. □

Using (3.4) and Lemma 3.1 we have the following corollaries to Lemma 3.2.

Corollary 1: If p is an odd prime, then for $a \not\equiv b \pmod p$

$$\sum_{x=0}^{p-1} \left(\frac{(x-a)(x-b)}{p}\right) = -1.$$ □

Corollary 2: Let p be an odd prime and let $a_1, a_2, \ldots, a_s$ be distinct residues modulo p. Then for even s

$$\left| \sum_{x=0}^{p-1} \left(\frac{(x-a_1)(x-a_2)\ \dots\ (x-a_s)}{p} \right) \right| \le 1 + (s-2)\sqrt{p} \ . \qquad \square$$

Recall that for prime $p \equiv 1 (\mathrm{mod}\ 4)$, G_p denotes the Paley graph of order p, that is the graph with $V(G_p) = \{0,1,\dots,p-1\}$ and $E(G_p) = \{(i,j) : i-j \equiv y^2 (\mathrm{mod}\ p)$ for some $y \in GF(p)\}$. Observe that if $a,b \in V(G_p)$, then

$$\left(\frac{a-b}{p} \right) = \begin{cases} 1 & , \text{ if a is adjacent to b,} \\ 0 & , \text{ if } a = b, \\ -1 & , \text{ otherwise.} \end{cases}$$

Further, since $p \equiv 1(\mathrm{mop}\ 4)$ then -1 is a quadratic residue modulo p. Consequently

$$\left(\frac{a-b}{p} \right) = \left(\frac{b-a}{p} \right).$$

We now illustrate the application of Lemma 3.2 by proving a result that was proved, using the theory of strongly regular graphs, by Exoo [10].

Theorem 3.1: Let $p = 4t + 1$ be a prime. Then $G_p \in \mathcal{G}(1,1,k)$ for every $k \le t$.

Proof: Let a and b be any two distinct vertices of G_p. Then $n(a/b) \ge k$ if any only if

$$f = \sum_{\substack{x=0 \\ x\neq a,b}}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 - \left(\frac{x-b}{p}\right)\right) \geq 4k .$$

We now show that $f \geq 4k$ for $t \geq k$. We can write

$$g = \sum_{x=0}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 - \left(\frac{x-b}{p}\right)\right)$$

$$= \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right) - \sum_{x=0}^{p-1} \left(\frac{x-b}{p}\right) - \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)$$

$$= p - \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right) \qquad \text{(by (3.4))}$$

$$= p + 1 . \qquad \text{(by Corollary 1 of Lemma 3.2)}$$

Hence $f = g - 2 = p - 1 = 4t \geq 4k$ for $t \geq k$ as required. □

Remark 1: When $t < k$ the above proof yields $f < 4k$, and hence $G_p \notin \mathcal{G}(1,1,k)$.

We noted in the introduction that Exoo and Harary [9] proved that the Petersen graph is the smallest member of $\mathcal{G}(1,2,1)$. In [10] Exoo proved that if $G \in \mathcal{G}(1,2,1) \cap \mathcal{G}(2,1,1)$, then $\nu(G) \geq 17$ and furthermore $G_{17} \in \mathcal{G}(1,2,1) \cap \mathcal{G}(2,1,1)$. Our next result concerns the classes $\mathcal{G}(1,2,k)$ and $\mathcal{G}(2,1,k)$.

Theorem 3.2: Let $p \equiv 1 \pmod 4$ be a prime and k a positive integer. If $p > (1 + 2\sqrt{2k})^2$, then $G_p \in \mathcal{G}(1,2,k) \cap \mathcal{G}(2,1,k)$.

Proof: Since G_p is a self-complementary graph it is sufficient to prove that $G_p \in \mathcal{G}(1,2,k)$. Let $S = \{a,b,c\}$ be any set of distinct vertices of G_p. Then $n(a/b,c) \geq k$ if and only if

$$f = \sum_{\substack{x=0 \\ x \notin S}}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 - \left(\frac{x-b}{p}\right)\right)\left(1 - \left(\frac{x-c}{p}\right)\right)$$

$$\geq 8k \ .$$

To show that $f \geq 8k$ it is clearly sufficient to establish that $f > 8(k-1)$.

We can write

$$g = \sum_{x=0}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 - \left(\frac{x-b}{p}\right)\right)\left(1 - \left(\frac{x-c}{p}\right)\right)$$

$$= \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \left\{\left(\frac{x-a}{p}\right) - \left(\frac{x-b}{p}\right) - \left(\frac{x-c}{p}\right)\right\}$$

$$- \sum_{x=0}^{p-1} \left\{\left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right) + \left(\frac{x-a}{p}\right)\left(\frac{x-c}{p}\right) - \left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right)\right\}$$

$$+ \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right)$$

$$= p + 1 + \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right) .$$

(by (3.4) and Corollary 1 of Lemma 3.2)

Thus

$$|g - p - 1| = \left| \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right) \right|$$

$$\leq 2\sqrt{p} \ . \qquad \text{(by Lemma 3.1)} \qquad (3.7)$$

Hence

$$g - f = \left(1 - \left(\frac{a-b}{p}\right)\right)\left(1 - \left(\frac{a-c}{p}\right)\right) + \left(1 + \left(\frac{b-a}{p}\right)\right)\left(1 - \left(\frac{b-c}{p}\right)\right)$$

$$+ \left(1 + \left(\frac{c-a}{p}\right)\right)\left(1 - \left(\frac{c-b}{p}\right)\right)$$

$$\leq 8 \ ,$$

since either $ab \in E(G_p)$ or $ab \notin E(G_p)$. Consequently

$$f \geq g - 8$$

$$\geq p + 1 - 2\sqrt{p} - 8 \ .$$

Hence $f > 8(k-1)$ for $p > (1 + 2\sqrt{2k}\ )^2$ as required. As S is arbitrary this completes the proof. □

Remark 2: We have verified, by computer, that if $p \equiv 1 (\mathrm{mod}\ 4)$ is a prime number less that or equal to 1009 and k is a positive integer with $p \leq (1 + 2\sqrt{2k})^2$, then $G_p \notin \mathcal{G}(1,2,k)$. We conjecture that this is true for all p. We can choose a, b and c in the proof of Theorem 3.2 so that $g - f = 8$ and hence

$$f = g - 8$$

$$\leq p + 2\sqrt{p} + 1 - 8. \qquad \text{(by (3.7))}$$

Consequently $f < 8k$ for $p < (-1 + 2\sqrt{2(k+1)})^2$. So the problem is to look at $(-1 + 2\sqrt{2(k+1)})^2 \le p \le (1 + 2\sqrt{2k})^2$.

We now turn our attention to the class $\mathcal{G}(2,2,k)$. This class has been studied for k = 1 by Blass et al [4] and Caccetta et al [8].

Theorem 3.3: Let $p \equiv 1 (\mathrm{mod}\ 4)$ be a prime and k a positive integer. If $p > (5 + 2\sqrt{4k+6})^2$, then $G_p \in \mathcal{G}(2,2,k)$.

Proof: The method of proof is similar to that of Theorem 3.2. Here we take $S = \{a,b,c,d\}$ to be any set of four distinct vertices of G_p and observe that $n(a,b/c,d) \ge k$ if and only if

$$f = \sum_{\substack{x=0 \\ x\notin S}}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 + \left(\frac{x-b}{p}\right)\right)\left(1 - \left(\frac{x-c}{p}\right)\right)\left(1 - \left(\frac{x-d}{p}\right)\right)$$

$$> 16(k-1) .$$

Simple algebra together with (3.4) and Corollary 1 of Lemma 3.2 yields :

$$g = \sum_{x=0}^{p-1} \left(1 + \left(\frac{x-a}{p}\right)\right)\left(1 + \left(\frac{x-b}{p}\right)\right)\left(1 - \left(\frac{x-c}{p}\right)\right)\left(1 - \left(\frac{x-d}{p}\right)\right)$$

$$= p + 2 + \sum_{x=0}^{p-1} \left\{\left(\frac{x-a}{p}\right)\left(\frac{x-c}{p}\right)\left(\frac{x-d}{p}\right) + \left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right)\left(\frac{x-d}{p}\right)\right.$$

$$\left. - \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right) - \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-d}{p}\right)\right\}$$

$$+ \sum_{x=0}^{p-1} \left(\frac{x-a}{p}\right)\left(\frac{x-b}{p}\right)\left(\frac{x-c}{p}\right)\left(\frac{x-d}{p}\right) .$$

Now by Lemma 3.1 and Corollary 2 of Lemma 3.2 we have

$$|g - (p + 2)| \le 10\sqrt{p} + 1,$$

and hence

$$g \geq p + 1 - 10\sqrt{p}\ .$$

Now

$$g - f = \left(1 + \left(\frac{a-b}{p}\right)\right)\left(1 - \left(\frac{a-c}{p}\right)\right)\left(1 - \left(\frac{a-d}{p}\right)\right)$$
$$+ \left(1 + \left(\frac{d-a}{p}\right)\right)\left(1 + \left(\frac{d-b}{p}\right)\right)\left(1 - \left(\frac{d-c}{p}\right)\right)$$
$$+ \left(1 + \left(\frac{c-a}{p}\right)\right)\left(1 + \left(\frac{c-b}{p}\right)\right)\left(1 - \left(\frac{c-d}{p}\right)\right)$$
$$+ \left(1 + \left(\frac{b-a}{p}\right)\right)\left(1 - \left(\frac{b-c}{p}\right)\right)\left(1 - \left(\frac{b-d}{p}\right)\right).$$

Observing that at least one of the first two terms and at least one of the last two terms on the right hand side of the above expression is zero, we conclude that $g - f \leq 16$. Consequently

$$f \geq g - 16$$
$$\geq p + 1 - 10\sqrt{p} - 16\ .$$

Hence $f > 16(k-1)$ for $p > (5 + 2\sqrt{4k + 6})^2$ as required. Since S is arbitrary this completes the proof. □

Remark 3: Blass et al [4] proved that $G_p \in \mathcal{G}(n,n,1)$ for $p \equiv 1 (\mathrm{mod}\ 4)$ and $p > n^2 2^{4n}$. For the particular case $n = 2$, this result asserts that $G_p \in \mathcal{G}(2,2,1)$ for prime $p \geq 1033$. When $k = 1$ Theorem 3.3 asserts that $G_p \in \mathcal{G}(2,2,1)$ for all prime $p \geq 137$. We have verified, using the computer that $G_p \in \mathcal{G}(2,2,1)$, only for prime $p \geq 61$. Thus Theorem 3.3 is not sharp. In fact, computer analysis shows that the bound on p given in Theorem 3.3 is fairly close to best possible. Table 3.1 gives the maximum k for which $G_p \in \mathcal{G}(2,2,k)$; we give only some of the computational results.

Maximum k	Order p
0	≤ 53
1	61, 73
2	89, 97, 101, 109, 113
3	137
4	149, 157, 173
5	181
6	193, 197, 233
7	229
8	241, 257
9	269, 277, 281
10	293, 313, 317
11	337
12	349, 353
14	373
15	389, 397, 401
16	409, 421, 433
17	449
18	457, 461
20	521
21	509
⋮	⋮
46	997

Table 3.1 : Maximum k For Which $G_p \in \mathcal{G}(2,2,k)$.

Our next result concerns the class $\mathcal{G}(n,n,1)$.

Theorem 3.4: Let $p \equiv 1 \pmod 4$ be a prime. If $p > ((2n-3)2^{2n-1}+4)^2$, then $G_p \in \mathcal{G}(n,n,1)$.

Proof: Let $S = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ be any set of 2n distinct vertices of G_p. Then $n(a_1, a_2, \ldots, a_n / b_1, b_2, \ldots, b_n) \geq 1$ if any only if

$$f = \sum_{\substack{x=0 \\ x \notin S}}^{p-1} \prod_{i=1}^{n} \left(1 + \left(\frac{x-a_i}{p}\right)\right)\left(1 - \left(\frac{x-b_i}{p}\right)\right) > 0 .$$

Now

$$g = \sum_{x=0}^{p-1} \prod_{i=1}^{n} \left(1 + \left(\frac{x-a_i}{p}\right)\right)\left(1 - \left(\frac{x-b_i}{p}\right)\right)$$

$$= \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \sum_{i=1}^{n} \left\{\left(\frac{x-a_i}{p}\right) - \left(\frac{x-b_i}{p}\right)\right\}$$

$$+ \sum_{x=0}^{p-1} \left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\frac{x-a_i}{p}\right)\left(\frac{x-a_j}{p}\right)\right.$$

$$\left. - \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x-a_i}{p}\right)\left(\frac{x-b_j}{p}\right) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\frac{x-b_i}{p}\right)\left(\frac{x-b_j}{p}\right)\right]$$

$$+ \ldots + \sum_{x=0}^{p-1} \prod_{i=1}^{n} \left(\frac{x-a_i}{p}\right)\left(\frac{x-b_i}{p}\right) .$$

Observe that the first term in the above expression is equal to p and the second term is 0.

Using Corollary 1 of Lemma 3.2 the third term of the above expression is equal to $n^2 - \binom{n}{2} - \binom{n}{2} = n$. Hence

$$|g - p - n| \le \left| \sum_{x=0}^{p-1} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{k=j+1}^{2n} \left(\frac{x-c_i}{p}\right)\left(\frac{x-c_j}{p}\right)\left(\frac{x-c_k}{p}\right) \right|$$

$$+ \left| \sum_{x=0}^{p-1} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{k=j+1}^{2n-1} \sum_{\ell=k+1}^{2n} \left(\frac{x-c_i}{p}\right)\left(\frac{x-c_j}{p}\right) \left(\frac{x-c_k}{p}\right)\left(\frac{x-c_\ell}{p}\right) \right| + \cdots + \left| \sum_{x=0}^{p-1} \prod_{i=1}^{2n} \left(\frac{x-c_i}{p}\right) \right|, \tag{3.8}$$

where $\{c_1, c_2, \ldots, c_{2n}\} = S$. Now Lemma 3.1 and Corollary 2 of Lemma 3.2 together imply

$$\left| \sum_{x=0}^{p-1} \sum_{i_1<i_2<\ldots<i_s} \left(\frac{x-c_{i_1}}{p}\right)\left(\frac{x-c_{i_2}}{p}\right) \cdots \left(\frac{x-c_{i_s}}{p}\right) \right|$$

$$\le \begin{cases} \binom{2n}{s}(s-1)\sqrt{p} & \text{, if } s \text{ is odd,} \\ \binom{2n}{s}(1 + (s-2)\sqrt{p}) & \text{, otherwise.} \end{cases} \tag{3.9}$$

Making use of (3.9) we get from (3.8)

$$|g - p - n| \le \sum_{t=1}^{n-1} \left[\left(\binom{2n}{2t+1}(2t)\sqrt{p} \right) + \binom{2n}{2t+2}\left(1 + 2t\sqrt{p}\right) \right]$$

$$= \sqrt{p}\left[\sum_{i=3}^{2n} i\binom{2n}{i} - \sum_{t=1}^{n-1} \left\{ \binom{2n}{2t+1} + 2\binom{2n}{2t+2} \right\} \right]$$

$$+ \sum_{t=1}^{n-1} \binom{2n}{2t+2}$$

$$= \sqrt{p}\{(2n-3)2^{2n-1}+2\} + 2^{2n-1} - 2n^2 + n - 1 .$$

Hence

$$g \geq p + n - 2^{2n-1} + 2n^2 - n + 1 - \sqrt{p}\{(2n-3)2^{2n-1}+2\}$$

$$= p - 2^{2n-1} + 2n^2 + 1 - \sqrt{p}\{(2n-3)2^{2n-1}+2\}. \qquad (3.10)$$

Now

$$g - f = \sum_{x \in S} \prod_{i=1}^{n} \left(1 + \left(\frac{x-a_i}{p}\right)\right)\left(1 - \left(\frac{x-b_i}{p}\right)\right) . \qquad (3.11)$$

If $g - f \neq 0$, then for some x_j the product

$$\prod_{i=1}^{n} \left(1 + \left(\frac{x_j-a_i}{p}\right)\right)\left(1 - \left(\frac{x_j-b_i}{p}\right)\right) \neq 0 . \qquad (3.12)$$

Without any loss of generality suppose $x_j = a_k$. For (3.12) to hold we must have $\left(\frac{a_k - b_i}{p}\right) = -1$ for all i. Hence the term in (3.11) with $x = b_i$ contributes zero to the sum. Hence we can write (3.11) as

$$g - f = \sum_{x=a_1}^{a_n} \prod_{i=1}^{n} \left(1 + \left(\frac{x-a_i}{p}\right)\right)\left(1 - \left(\frac{x-b_i}{p}\right)\right)$$

$$\leq n2^{2n-1} ,$$

since

$$\prod_{i=1}^{n} \left(1 + \left(\frac{x-a_i}{p}\right)\right)\left(1 - \left(\frac{x-b_i}{p}\right)\right) \leq 2^{2n-1}$$

for each x; note that each factor is at most 2 and at least one factor is 1. Hence

$$f \geq g - n2^{2n-1}$$

$$\geq p - (n+1)2^{2n-1} + 2n^2 + 1 - \{(2n-3)2^{2n-1} + 2\}\sqrt{p} .$$

(using (3.10))

So if $p > ((2n - 3)2^{2n-1} + 4)^2$, then $f > 0$ as required. Since S is arbitrary, this completes the proof of the theorem. □

ACKNOWLEDGEMENTS

The authors express their thanks to Mr Peter Caccetta for his assistance with the computational work. This work has been supported by an Australian Research Council Grant A48932119.

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(Received 23/1/92)