On the Classification of 2-extendable Cayley Graphs on Dihedral Groups

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Abstract. A graph X with at least two independent edges is 2-extendable if any two independent edges of X are contained in a perfect matching of X. In this paper, we prove that a connected Cayley graph of even order on a dihedral group is 2-extendable if and only if it is not isomorphic to any one of the following circulant graphs:

(I) $\mathbf{Z}_{2n}(1, 2n-1)$, $n \geq 3$;
(II) $\mathbf{Z}_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$;
(III) $\mathbf{Z}_{4n}(1, 4n-1, 2n)$, $n \geq 2$;
(IV) $\mathbf{Z}_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$; and
(V) $\mathbf{Z}_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$.

1. Introduction.

For a simple graph X, we use $V(X)$ and $E(X)$ to denote the vertex-set and the edge-set of X respectively. For any set $S \subseteq V(X)$, we denote by $X[S]$ the subgraph of X induced by S. The edge incident with vertices x and y is denoted by xy.

Let G be a group and S a subset of G such that the identity element $1 \notin S$ and $x^{-1} \in S$ for each $x \in S$. The *Cayley graph* $X(G; S)$ on the group G has the elements of G as its vertices and edges joining g and gs for all $g \in G$ and $s \in S$. We call S the *symbol set*, and say that the edge $g(gs)$ has the symbol s. It is well-known that every Cayley graph is vertex-transitive. For $S \subseteq G$, we denote by $\langle S \rangle$ the subgroup of G induced by S. When the group is cyclic, or $G \cong \mathbf{Z}_m$, the Cayley graph $X(G; S)$ is called a *circulant* and is denoted by $\mathbf{Z}_m(S)$.

The dihedral group D_n is a group which is generated by two elements ρ and τ, where $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$. We denote $\{x\tau \mid x \in \langle\rho\rangle\}$ by $\langle\rho\rangle\tau$. From the relations $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$, we can easily obtain $(\rho^i\tau)^2 = 1$ and $\rho^i\tau\rho^{-j} = \tau\rho^{-(i+j)} = \rho^{i+j}\tau$, which are useful later. It is easy to see that D_n has a cyclic subgroup $\langle\rho\rangle$ of index 2 which is isomorphic to $\mathbf{Z}_n$. Moreover, $D_n = \langle\rho\rangle \cup \langle\rho\rangle\tau$.

A *perfect matching* of a graph X is a set of independent edges which together cover all the vertices of X. For a positive integer k, if M is a set of k independent edges of X and M^* is a perfect matching of X such that $M \subseteq M^*$, we call M^* a *perfect matching extension* of M, or *M can be extended to M^**. A graph X is said to be *k-extendable* if it contains k independent edges and any k independent edges of X can be extended to a perfect matching of X.

The concept of k-extendability was introduced by Plummer [5] in 1980 and he (see [5], [6], [7]) studied the relationship between k-extendability and other graph parameters, e.g., degree, connectivity, genus, etc.. The motivation for studying k-extendable graph is to determine a greatest lower bound on the number of different perfect matchings in a graph (which has a perfect matching). See, for instance, [5] for more details. Little, Grant and Holton [3] gave a characterization of 1-extendable graphs and Yu [11] further obtained a characterization for k-extendable graphs. Schrag and Cammack [8] and Yu [10] classified the 2-extendable generalized Petersen graphs. Recently, Chan, Chen and Yu [2] classified the 2-extendable Cayley graphs on abelian groups. Their classification, as stated below, will be used in the proof later.

Theorem 1.1. (Chan, Chen and Yu [2]) *Let $X = X(G; S)$ be a Cayley graph on the abelian group G of even order. Then X is 2-extendable if and only if it is not isomorphic to any of the following graphs:*

(I) $\mathbf{Z}_{2n}(1, 2n-1)$, $n \geq 3$;

(II) $\mathbf{Z}_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$;

(III) $\mathbf{Z}_{4n}(1, 4n-1, 2n)$, $n \geq 2$;

(IV) $\mathbf{Z}_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$; *and*

(V) $\mathbf{Z}_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$.

Stong [9] has proved that any Cayley graph on a dihedral group is 1-factorizable. His result implies that $X(D_n; S)$ is 1-extendable. In this paper, we shall give a classification for 2-extendable Cayley graphs on dihedral groups by showing that, except for the five classes of graphs in Theorem 1.1, $X(D_n; S)$ is 2-extendable.

From now on, we shall assume that $X = X(D_n; S)$ is connected, that is, S is a generating set of D_n, or $\langle S \rangle = D_n$. For convenience, we let $S' = S \cap \langle \rho \rangle$ and $S'' = S \cap (\langle \rho \rangle \tau)$. Then clearly, $S'' \neq \emptyset$ as $X(D_n; S)$ is connected. Also, without loss of generality, we may always assume $\tau \in S''$. Let E_s be a set of edges which has the symbol s for any $s \in S$. Then, for $s \in S''$, E_s is a perfect matching of $X(D_n; S)$.

We introduce a class of graphs, denoted by $C[2q, s, t]$ (where $s+t \equiv 0 \pmod 2$), which are defined as follows. The vertex-set is $\{(i,j) \mid 0 \leq i \leq 2q-1, 0 \leq j \leq s-1\}$, which is the cartesian product of $\mathbf{Z}_{2q}$ and $\mathbf{Z}_s$. The edge-set consists of three types of pairs as given below:

(1) $(i,j)(i+1,j)$ and $(2q-1,j)(0,j)$, where $i = 0,1,2,...,2q-2$ and $j = 0,1,2,...,s-1$;

(2) $(i,j)(i,j+1)$, where $i+j \equiv 0 \pmod 2$, $i = 0,1,2,...,2q-1$ and $j = 0,1,2,...,s-1$; and

(3) $(2i+1,0)(2i+1+t,s-1)$, where $i = 0,1,...,q-1$ and the first coordinates are computed modulo $2q$.

Clearly, $C[2q,s,t]$ is a 3-regular graph. Alspach and Zhang [1] introduced the brick-product of C_{2q} with P_s which is a $C[2q,s,t]$ without edges of type (3). It was proven in [1] that $C[2q,s,t]$ is a Cayley graph on a dihedral group. As an example, The graph $C[6,5,1]$ is given in Figure 1.1.

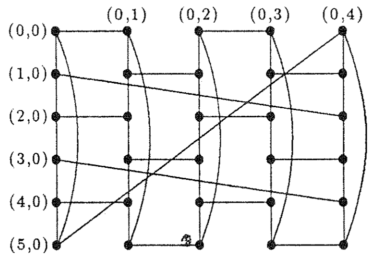

Figure 1.1. The graph $C[6,5,1]$

To conclude this section, we make the following observation which sketches the structure of Cayley graphs on dihedral groups.

Observation 1.2. A Cayley graph $X = X(D_n;S)$ on a dihedral group D_n can be decomposed into two subgraphs on $\langle\rho\rangle$ and $\langle\rho\rangle\tau$ together with a class of perfect matchings joining them. The two subgraphs on $\langle\rho\rangle$ and $\langle\rho\rangle\tau$ are isomorphic and both are isomorphic to a circulant H on $\mathbf{Z}_n$. Furthermore, if $|S''| = 1$, then X is isomorphic to $H \times K_2$.

Proof. Let $X[\langle\rho\rangle]$ and $X[\langle\rho\rangle\tau]$ be the induced subgraphs on $\langle\rho\rangle$ and $\langle\rho\rangle\tau$, respectively. Then $X[\langle\rho\rangle] = X(\langle\rho\rangle;S') \cong \mathbf{Z}_n(S^*)$, where $S^* = \{i \mid \rho^i \in S'\}$, which is a circulant and $\phi : X[\langle\rho\rangle] \to X[\langle\rho\rangle\tau]$ defined by $\phi(\rho^i) = \rho^i\tau$ is an isomorphism (note that $X[\langle\rho\rangle]$ may be edgeless).

The class of perfect matchings is $\{E_s \mid s \in S''\}$. Moreover, if $S'' = \{\tau\}$, then, since the isomorphism ϕ carries each ρ^i to $\rho^i\tau$ which is adjacent to ρ^i, we have: $X \cong H \times K_2$. ∎

We set $E_1 = E(X[\langle\rho\rangle])$, $E_2 = E(X[\langle\rho\rangle\tau])$ and $E_3 = E(X(D_n; S''))$. Then $E(X) = E_1 \cup E_2 \cup E_3$.

2. Basic Lemmas.

We need the following lemmas in the proof of the main theorem.

Lemma 2.1. *If n is odd, then $\mathbf{Z}_n(S) \times K_2 \cong \mathbf{Z}_{2n}(2S \cup \{n\})$.*

Proof. Define a mapping f from $\mathbf{Z}_n(S) \times K_2$ to $\mathbf{Z}_{2n}(2S \cup \{n\})$ by

$$f(x, y) = \begin{cases} 2x \pmod{2n} & \text{if } y = 0, \\ 2x + n \pmod{2n} & \text{if } y = 1. \end{cases}$$

Then, it is easy to see that f is the required isomorphism. ∎

Lemma 2.2. *Let $X = X(D_n; \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\})$ be connected.*

(1) If $X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is connected, then X is a 3- or 4-regular circulant.

(2) If $X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then X has $C_{2m} \times P_h$ as a spanning subgraph for some $m \geq 2$ and $h \geq 2$.

Proof. (1) Let $X_1 = X(D_n; \{\rho^i\tau, \rho^j\tau\})$. Since $\rho^i\tau$ and $\rho^j\tau$ are of order 2, X_1 is a 2-regular graph. If it is connected, then it is a $2n$-cycle $1(\rho^i\tau)(\rho^{i-j})(\rho^{2i-j}\tau)(\rho^{2(i-j)})\cdots(\rho^{(n-1)(i-j)})(\rho^{ni-(n-1)j}\tau)1$. We use $\{0, 1, 2, \ldots, 2n-1\}$ to relabel this cycle so that $\rho^{ti-(t-1)j}\tau \leftrightarrow 2t-1$ and $\rho^{t(i-j)} \leftrightarrow 2t$. Then the cycle becomes $012\cdots(2n-1)0$ after the relabelling.

Let $\rho^k = \rho^{h(i-j)}$. Then edges of X with symbol ρ^k (resp., ρ^{-k}) become edges with symbol $2h$ (resp., $-2h$) after relabelling. Therefore, $X = X(D_n; \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\}) \cong \mathbf{Z}_{2n}(\{1, 2n-1, \pm 2h\})$. If $h = \frac{n}{2}$, then X is 3-regular. Otherwise, it is 4-regular.

(2) If $X_1 = X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then it is a union of h disjoint even cycles C_{2m}, for some $m > 1, h > 1$. We can arrange the vertices of each cycle in a column such that the first column begins with 1, the second column begins with ρ^k (note that ρ^k does not belong to the first column, for otherwise X will be disconnected), the third column begins with ρ^{2k}, and so on. We thus obtain a $2m \times h$ array in which each row forms an h-path whose edges have the same symbol ρ^k or ρ^{-k} (an example with $X = X(D_{12}; \{\tau, \rho^4\tau, \rho^5, \rho^{-5}\})$ is illustrated in Figure 2.1). Therefore, X has a spanning subgraph $C_{2m} \times P_h$. ∎

We quote the following result from [1], which is implied in the proof of Theorem 3.1 of [1].

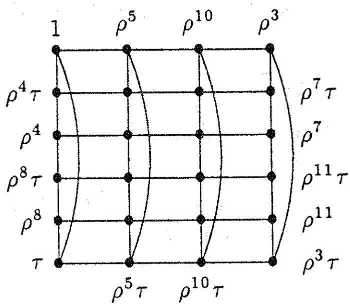

Figure 2.1.

Lemma 2.3. (Alspach and Zhang [1]) *Let $X = X(D_n; \{\rho^i\tau, \rho^j\tau, \rho^k\tau\})$ be connected. If $X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then X is isomorphic to $C[2q, s, t]$ for some $q \geq 2, s \geq 2$ and $t \geq 1$.*

We also need the following result in [2].

Lemma 2.4. (Chan, Chen and Yu [2]) *$C_{2m} \times P_h$ $(m \geq 2, h \geq 2)$ is 2-extendable.*

3. The Main Theorem.

In this section, we shall prove the following result which is a characterization of 2-extendable Cayley graphs on dihedral groups.

Theorem 3.1. *Let $X = X(D_n; S)$ be connected, $n \geq 2$. Then X is 2-extendable if and only if X is not isomorphic to any of the following graphs.*

(I) $\mathbf{Z}_{2n}(1, 2n-1)$, $n \geq 3$;

(II) $\mathbf{Z}_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$;

(III) $\mathbf{Z}_{4n}(1, 4n-1, 2n)$, $n \geq 2$;

(IV) $\mathbf{Z}_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$; *and*

(V) $\mathbf{Z}_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$.

Proof. It is not hard to see that each class of graphs in (I) - (V) can be realized by Cayley graphs on dihedral groups. If X is isomorphic to any graph in these classes, then X is not 2-extendable, by Theorem 1.1.

Let $X = X(D_n; S)$. We shall show that if X is not isomorphic to any of the graphs in the five classes, then X is 2-extendable.

If $n = 2$, then $X = X(D_2; S)$ is either C_4 or K_4. In any case, X is 2-extendable. So we may assume that $n \geq 3$. Choose arbitrarily two independent edges e_1 and

e_2 of X. Recall that $E_1 = E(X[\langle\rho\rangle])$, $E_2 = E(X[\langle\rho\rangle\tau])$, $E_3 = E(X(D_n; S''))$ and $\tau \in S$.

Case 1. $M = \{e_1, e_2\} \subseteq E_1$ or E_2.

Since $X[\langle\rho\rangle] \cong X[\langle\rho\rangle\tau]$, we may assume that $M \subseteq E_1$. Suppose $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k)(\rho^h)$. Then i, j, k and h are all distinct. Let $M^* = (E_\tau \cup \{e_1, e_2, (\rho^i\tau)(\rho^j\tau), (\rho^k\tau)(\rho^h\tau)\}) - \{(\rho^i)(\rho^i\tau), (\rho^j)(\rho^j\tau), (\rho^k)(\rho^k\tau), (\rho^h)(\rho^h\tau)\}$. Then M^* is a perfect matching containing M.

Case 2. $M \cap E_3 \neq \emptyset$ and $M \cap (E_1 \cup E_2) \neq \emptyset$.

Without loss of generality, assume $e_1 = (\rho^i)(\rho^j) \in E_1$ and $e_2 = (\rho^k)(\rho^{k+h}\tau) \in E_3$, where k, i and j are all distinct and $\rho^h\tau \in S''$. Then $(E_{\rho^h\tau} \cup \{e_1, (\rho^{i+h}\tau)(\rho^{j+h}\tau)\}) - \{(\rho^i)(\rho^{i+h}\tau), (\rho^j)(\rho^{j+h}\tau)\}$ is a perfect matching of X which contains M.

Case 3. $e_1 \in E_1$, $e_2 \in E_2$.

Let $G_1, G_2, ..., G_r$ be the components of $X[\langle\rho\rangle]$. Then $G_i \cong G_j$ for $1 \leq i, j \leq r$. Let G'_i be the subgraph of $X[\langle\rho\rangle\tau]$ induced by $\{x\tau \mid x \in V(G_i)\}$. Then $G'_i \cong G_i$ $(1 \leq i \leq r)$.

Then, we have the following subcases to consider.

Case 3.1. e_1 and e_2 lie in G_i and G'_j, respectively, and $i \neq j$.

Let $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k\tau)(\rho^h\tau)$. Then

$$E_\tau \cup \{e_1, e_2, (\rho^i\tau)(\rho^j\tau), (\rho^k)(\rho^h)\}) - \{(\rho^i)(\rho^i\tau), (\rho^j)(\rho^j\tau), (\rho^k)(\rho^k\tau), (\rho^h)(\rho^h\tau)\}$$

is a perfect matching containing e_1 and e_2.

Case 3.2. e_1 and e_2 lie in G_i and G'_i, respectively, and $|V(G_i)| = |V(G'_i)|$ is even.

It is easy to see that every connected circulant of even order is 1-extendable and each component of $X[\langle\rho\rangle]$ is a circulant. Hence e_1 can be extended to a perfect matching M_1 in $X[\langle\rho\rangle]$ and e_2 can be extended to a perfect matching M_2 in $X[\langle\rho\rangle\tau]$. Then $M_1 \cup M_2$ is a perfect matching of X as required.

Case 3.3. $e_1 \in E(G_i)$, $e_2 \in E(G'_i)$ for some i and $|V(G_i)| = |V(G'_i)|$ is odd.

Let $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k\tau)(\rho^h\tau)$.

(a) If $X[\langle\rho\rangle]$ is disconnected, then so is $X(D_n; S' \cup \{\tau\})$. Since X is connected, there exists $\rho^m\tau \in S''$ so that $\rho^i \cdot \rho^m\tau = \rho^{i+m}\tau \neq V(G'_i)$. Therefore, $\{x \cdot (\rho^m\tau) \mid x \in V(G_i)\} \cap V(G'_i) = \emptyset$. In this case,

$$\begin{aligned}&(E_{\rho^m\tau} \cup \{e_1, e_2, (\rho^{i+m}\tau)(\rho^{j+m}\tau), (\rho^{k-m})(\rho^{h-m})\})\\ &-\{(\rho^i)(\rho^{i+m}\tau), (\rho^j)(\rho^{j+m}\tau), (\rho^{k-m})(\rho^k\tau), (\rho^{h-m})(\rho^h\tau)\}\end{aligned}$$

is a perfect matching which contains e_1 and e_2.

(b) If $X[\langle\rho\rangle]$ is connected, then n is odd. Let $n = 2k+1$.

If $|S'| \geq 4$, then, by Observation 1.2 and Lemma 2.1, $X' = X(D_n; S' \cup \{\tau\}) \cong \mathbf{Z}_n(S^*) \times K_2 \cong \mathbf{Z}_{2n}(\{n\} \cup 2S^*)$ where $S^* = \{i \mid \rho^i \in S'\}$ (by Lemma 2.1). Hence X' is a circulant of degree at least 5 and is 2-extendable by Theorem 1.1. But X' is a spanning subgraph of X which contains e_1 and e_2. Hence $\{e_1, e_2\}$ can be extended to a perfect matching of X.

Suppose now $S' = \{\rho^{\pm i}\}$. Then e_1 and e_2 have the same symbol. If $S'' = \{\tau\}$, then X is 3-regular and $X \cong \mathbf{Z}_{4k+2}(2k+1, 2, 4k)$, which is a graph belonging to class (IV). Hence we must have $|S''| \geq 2$. When $|S''| = 2$ and $X(D_n; S'')$ is disconnected, $X(D_n; S'' \cup S')$ has $C_{2m} \times P_h$ as a spanning subgraph by Lemma 2.2, where $h \geq 2$. Since h is odd, so we have $h \geq 3$. Therefore, we can rearrange the column in the proof of Lemma 2.2, such that $e_1, e_2 \in E(C_{2m} \times P_h)$. But $C_{2m} \times P_h$ is 2-extendable (by Lemma 2.4). Hence e_1 and e_2 can be extended to a perfect matching of X.

When $S'' = 2$ and $X(D_n; S'')$ is connected, $X(D_n; S'' \cup S')$ is a 4-regular circulant by Lemma 2.2 again. If $X = X(D_n; S) = X(D_{2k+1}; S) \cong \mathbf{Z}_{4k+2}(1, 4k+1, 2k, 2k+2)$, then X is a graph of class (V), which is not 2-extendable. (For instance, $X(D_5; \{\tau, \rho\tau, \rho^2, \rho^3\}) \cong \mathbf{Z}_{10}(1, 4, 6, 9)$ is such a graph.) In any other cases, $X(D_n; S)$ is 2-extendable by Theorem 1.1.

Now assume $|S''| > 2$, we shall show that e_1 and e_2 can be extended to a perfect matching of X. Note again that e_1 and e_2 have the same symbol (as $S' = \{\rho^{\pm i}\}$). Without loss of generality, we assume that $e_1 = 1(\rho^i)$, $e_2 = (\rho^i\tau)(\rho^{2i}\tau)$. If $\rho^i\tau \in S''$, then $(E_{\rho^i\tau} \cup \{e_1, e_2\}) - \{1(\rho^i\tau), (\rho^i)(\rho^{2i}\tau)\}$ is a perfect matching containing e_1 and e_2. If $\rho^i\tau \notin S''$, then there is a $\rho^j\tau \in S''$ such that $j \neq 0$, $j \neq 2i$ as $|S''| \geq 3$. Let

$$\begin{aligned} M^* = &(E_{\rho^j\tau} \cup \{e_1, e_2, (\rho^j\tau)(\rho^{i+j}\tau), (\rho^{i-j})(\rho^{2i-j})\}) - \{1(\rho^j\tau), (\rho^i)(\rho^{i+j}\tau)), \\ &(\rho^{i-j})(\rho^i\tau), (\rho^{2i-j})(\rho^{2i}\tau)\}. \end{aligned}$$

Then M^* is a perfect matching of X which extends e_1 and e_2.

Case 4. $\{e_1, e_2\} \subseteq E_3$.

If e_1 and e_2 have the same symbol $\rho^i\tau$. Then $E_{\rho^i\tau}$ is a perfect matching of X which contains e_1 and e_2. So we assume that e_1 has symbol $\rho^i\tau$ and e_2 has symbol $\rho^j\tau$, $i > j$.

Case 4.1. If $X_1 = X(D_n; \{\rho^i\tau, \rho^j\tau)\})$ is disconnected. Then X_1 is a disjoint union of some even cycles. If e_1, e_2 belong to different cycles, then we can easily extend e_1 and e_2 to a perfect matching of X. So suppose that e_1 and e_2 belong to the same cycle and no perfect matching of this cycle contains both e_1 and e_2. Let $G_1, G_2, ..., G_h$ be disjoint cycles of X_1, where $G_i \cong C_{2m}$ $(1 \leq i \leq h)$ and $e_1, e_2 \in E(G_1)$. Since X is vertex-transitive, we may assume $e_1 = 1(\rho^i\tau)$. Thus G_1 is a $2m$-cycle $1(\rho^i\tau)(\rho^{i-j})(\rho^{2i-j}\tau)(\rho^{2(i-j)})\cdots(\rho^{(m-1)(i-j)})(\rho^{mi-(m-1)j}\tau)1$ (where $m(i-j) \equiv 0 \pmod{n}$).

(a) Suppose $S' - \{\rho^{i-j}, \rho^{2(i-j)}, ..., \rho^{(m-1)(i-j)}\}$ is not empty, say containing ρ^k. Since $\rho^k \notin V(G_1)$, we may assume that $\rho^k \in V(G_2)$. Then, by Lemma 2.2, the subgraph of $X(D_n; \{\rho^i\tau, \rho^j\tau, \rho^k, \rho^{-k}\})$ induced by $V(G_1) \cup V(G_2)$ contains a spanning subgraph which is isomorphic to $C_{2m} \times K_2$ and contains e_1, e_2. By Lemma 2.4, $C_{2m} \times K_2$ is 2-extendable. Thus there is a perfect matching M' of $C_{2m} \times K_2$ containing e_1 and e_2. For other G_i, $i \geq 3$, simply choose a perfect matching M_i of G_i. Then $M' \cup (\bigcup_{j=3}^{h} M_i)$ is a perfect matching of X containing e_1 and e_2.

(b) If $S' - \{\rho^{i-j}, \rho^{2(i-j)}, ..., \rho^{(m-1)(i-j)}\} = \emptyset$, then $X(D_n; S' \cup \{\rho^i\tau, \rho^j\tau\})$ is disconnected. Since X is connected, there is a $\rho^r\tau \in S''$ such that the edges with symbol $\rho^r\tau$ join G_1 and another G_i. Let $X' = X(D_n; \{\rho^i\tau, \rho^j\tau, \rho^r\tau\})$. Then each component of X' is also a Cayley graph on a dihedral group D_b for some b. So, without loss of generality, we assume that X' is connected. By Lemma 2.3, X' is isomorphic to $C[2q, s, t]$ for some $q \geq 2$, $s \geq 2$ and $t \geq 1$.

In this case, we may assume that $e_1 = (0,0)(1,0)$ and $e_2 = (2p+1,0)(2p+2,0)$. If s is even, let

$$\begin{aligned} M = &\{(0,j)(1,j) \mid j = 0,1,2,...,s-2\} \cup \{(2,i)(2,i+1) \mid i = 0,2,4,...,s-2\} \cup \\ &\{(i,j)(i+1,j) \mid i = 3,5,...,2q-3; j = 0,1,2,...,s-2\} \cup \\ &\{(2q-1,0)(2q-1+t,s-1),(2q-1,1)(2q-1,2,),...,(2q-1,s-3)(2q-1,s-2)\} \\ &\cup B \end{aligned}$$

where B is a perfect matching of $(C_{2q} \times \{s-1\}) - \{(2,s-1),(2q-1+t,s-1)\}$ which is a union of paths of odd length (since $2q-1+t-2 = 2q-3+t$ is odd). Then M is a perfect matching of X which contains e_1 and e_2.

If s is odd, let

$$\begin{aligned} M = &\{(0,j)(1,j) \mid j = 0,1,2,...,s-2\} \cup \{(2,i)(2,i+1) \mid i = 0,2,4,...,s-3\} \cup \\ &\{(i,j)(i+1,j) \mid i = 3,5,...,2q-3; j = 0,1,2,...,s-2\} \cup \\ &\{(2q-1,0)(2q-1+t,s-1),(2q-1,1)(2q-1,2,),...,(2q-1,s-2)(2q-1,s-1)\} \\ &\cup B \end{aligned}$$

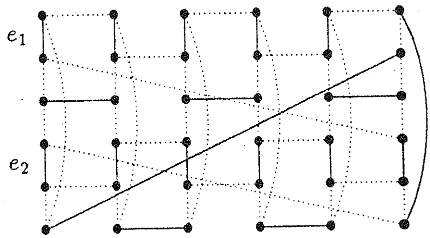

Figure 3.1.

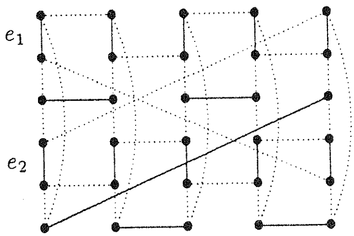

Figure 3.2.

where B is a perfect matching of $(C_{2q} \times \{s-1\}) - \{(2q-1, s-1), (2q-1+t, s-1)\}$ which is a union of paths of odd length (since $2q-1+t-(2q-1) = t$ is odd). Then M is a perfect matching of X which contains e_1 and e_2. (We illustrated the above patterns with $C[6,6,2]$ and $C[6,5,3]$ in Figures 3.1 and 3.2, respectively.)

Case 4.2. $X_1 = X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is connected. Then $X_1 \cong C_{2n}$.

(a) If $S = \{\rho^i\tau, \rho^j\tau\}$, then $X \cong C_{2n} = \mathbf{Z}_{2n}(1, 2n-1)$, $(n \geq 3)$, which is in class (I).

(b) If $S = \{\rho^i\tau, \rho^j\tau, \rho^{n/2}\}$, then n is even, say $n = 2m$. By the proof of Lemma 2.2, $X(D_n; S)$ is a 3-regular circulant and $X(D_n; S) \cong \mathbf{Z}_{2n}(1, 2n-1, n) = \mathbf{Z}_{4m}(1, 4m-1, 2m)$. This is a graph of class (III).

(c) If $S = \{\rho^i\tau, \rho^j\tau, \rho^k, \rho^{-k}\}$, $(k \neq \frac{n}{2})$, then $X(D_n; S)$ is a 4-regular circulant by the proof of Lemma 2.2. Since a circulant is a Cayley graph on abelian group, by Theorem 1.1, $X(D_n; S)$ is 2-extendable if it is either not isomorphic to $\mathbf{Z}_{4k+2}(1, 4k+1, 2k, 2k+2)$, (which belongs to class (V)), or to $\mathbf{Z}_{2n}(1, 2, 2n-1, 2n-2)$, (which is a graph in class (II)).

(d) If $|S'| \geq 3$, then $X(D_n; S' \cup \{\rho^i\tau, \rho^j\tau\})$ is a circulant of degree at least 5, by the proof of Lemma 2.2. By Theorem 1.1, $X(D_n; S \cup \{\rho^i\tau, \rho^j\tau\})$ is 2-extendable. Hence $\{e_1, e_2\}$ can be extended to a perfect matching of X.

(e) If $|S'| = 0$, then $|S''| \geq 3$. We have $\rho^k\tau \in S''$ for some k distinct from i and j. We shall show that, for some $\rho^k\tau \in S''$, $X' = X(D_n; S^*)$ has a perfect matching containing $\{e_1, e_2\}$, where $S^* = \{\rho^i\tau, \rho^j\tau, \rho^k\tau\}$.

If $X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then we can choose $\rho^k\tau \in S''$ (because X is connected) such that the edges with symbol $\rho^k\tau$ join two cycles produced by $\rho^i\tau$ and $\rho^j\tau$. This case was dealt in Case 4.1(b) and we know the e_1 and e_2 can be extended to a perfect matching of $X(D_n; S^*)$ and then a perfect matching of X.

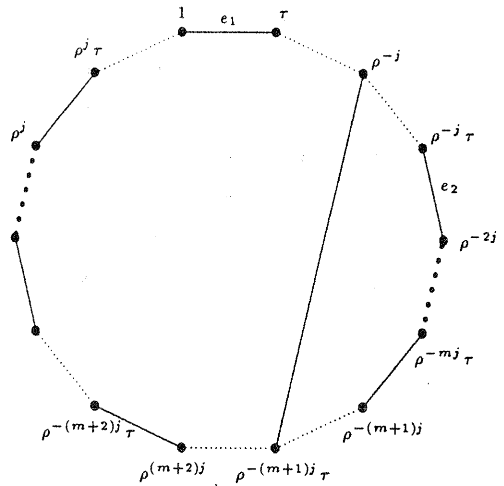

Figure 3.3.

Assume now that $X(D_n; \{\rho^i\tau, \rho^j\tau\})$ is connected. Then it is isomorphic to C_{2n}. For convenience, we can assume that $\rho^i\tau = \tau$. Then $C_{2n} = 1(\tau)(\rho^{-j})(\rho^{-j}\tau)(\rho^{-2j})(\rho^{-2j}\tau)\cdots(\rho^j\tau)1$. Also assume that $e_1 = 1(\tau)$, $e_2 = (\rho^{-qj}\tau)(\rho^{-(q+1)j})$. Let $\rho^k = \rho^{-mj}$. We can assume that $m > q+1$, (or else consider ρ^{-k}). Let $e_3 = (\rho^{-j})(\rho^{-(m+1)j}\tau)$. Then $C_{2n} - \{1, \tau, \rho^{-qj}\tau, \rho^{-(q+1)j}, \rho^{-j}, \rho^{-(m+1)j}\}$ is a union of paths of even order and so contains a perfect matching M. Then $M \cup \{e_1, e_2, e_3\}$ is a perfect matching of X' which contains e_1 and e_2 (see the illustration in Figure 3.3, for the case $q = 1$). ∎

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