Partial Generalized Bhaskar Rao Designs over Abelian Groups

William Douglas Palmer

The School of Mathematics and Statistics
and
The Institute of Education
The University of Sydney

Abstract. Let $G = \mathrm{EA}(g)$ of order g be the abelian group

$$Z_{p_1} \times Z_{p_1} \times \cdots \times Z_{p_1} \times \cdots \times Z_{p_n} \times Z_{p_n} \times \cdots \times Z_{p_n}$$

where Z_{p_i} occurs r_i times with $\prod_{i=1}^{n} p_i^{r_i}$ the prime decomposition of g.

It is shown that the necessary conditions

$$\lambda \equiv 0 \pmod{g}$$
$$v \geq 3n$$
$$v \equiv 0 \pmod{n}$$
$$\lambda(v-n) \equiv 0 \pmod{2}$$
$$\lambda v\,(v-n) \equiv \begin{cases} 0 \pmod{6} \text{ if } g \text{ is odd,} \\ 0 \pmod{24} \text{ if } g \text{ is even,} \end{cases}$$

are sufficient for the existence of a PGBRD(v, 3, λ, n; EA(g)).

1. Introduction

A *design* is a pair (X, $\boldsymbol{B}$) where X is a finite set (whose elements are called *points*) and $\boldsymbol{B}$ is a collection of (not necessarily distinct) subsets B_i (called *blocks*) of X. A point and a block are said to be *incident* if and only if the point belongs to the block. For a design (X, $\boldsymbol{B}$) with v points and b blocks, the incidence matrix N *is a* $v \times b$ matrix, $N = (n_{ij})$, such that

$$n_{ij} = \begin{cases} 1 \text{ if point } i \text{ belongs to block } j \\ 0 \text{ otherwise.} \end{cases}$$

A *balanced incomplete block design*, BIBD(v, b, r, k, λ), is a design (X, $\boldsymbol{B}$) with v points and b blocks such that:

(i) each element of X appears in exactly r blocks;
(ii) each block contains exactly k ($< v$) elements of X; and
(iii) each pair of distinct elements of X appear together in exactly λ blocks.

As $r(k-1) = \lambda(v-1)$ and $vr = bk$ are well-known necessary conditions for the existence of a BIBD(v, b, r, k, λ) we denote this design by BIBD(v, k, λ).

Let v and λ be positive integers and K a set of positive integers. A design (X, $\boldsymbol{B}$) with v points and b blocks is a *pairwise balanced design*, PBD(v; K; λ), if:

(i) X contains exactly v points;
(ii) if a block contains k points then k belongs to K ;
(iii) each pair of distinct points appear together in exactly λ blocks.

A pairwise balanced design PBD(v; $\{k\}$; λ), where $K = \{k\}$ consists of exactly one integer, is a BIBD(v, k, λ).

A *group divisible design*, GDD(v, b, r, λ_1, λ_2, m, n), is a triple (X, S, A) where:
(i) X is a set of v elements (called *points*);
(ii) S is a class of m subsets of X (called *groups*), each of size n, which partitions X ;
(iii) A is a class of b (not necessarily distinct) subsets of X (called *blocks*), each of size $k \geq 2$;
(iv) each point appears in exactly r blocks;
(v) each pair $\{x, y\}$ of points contained in a group is contained in exactly λ_1 blocks;
(vi) each pair $\{x, y\}$ of elements of X not contained in a group is contained in exactly λ_2 blocks.

We apply the term "group" here to describe elements of S and the reader is reminded not to confuse the use of this term with the word "group" used in the *algebraic* sense of the word.

A *transversal design*, TD, with k groups each of size n and index λ, denoted by TD(k, λ; n), is a GDD on kn points where:
(i) each block intersects each group in exactly one point;
(ii) each pair $\{x, y\}$ of points not contained in a group is contained in exactly λ blocks.

It is well-known (see, for example, Street and Street (1987)) that a TD(k, λ; n) is a GDD(kn, λn^2, λn, k, 0, λ, k, n).

Suppose that x and y are distinct points in a GDD. We say that x and y are *first associates* if $\{x, y\}$ is contained in a group. If $\{x, y\}$ is not contained in a group then x and y are said to be *second associates*. For a GDD(v, b, r, λ_1, λ_2, m, n) we define the *association matrices*

$$B_i = \left(b^i_{st}\right),\ 1 \leq i \leq 2, \text{ and } 1 \leq s, t \leq v$$

as $v \times v$ (0, 1) – matrices given by

$$b^i_{st} = \begin{cases} 1 \text{ if points s and } t \text{ are } i\text{th associates,} \\ 0 \text{ otherwise.} \end{cases}$$

It is well-known (see, for example, Street and Street (1987)) that, if N is the incidence matrix of a GDD(v, b, r, λ_1, λ_2, m, n) then

$$NN^{\mathrm{T}} = rI_v + \lambda_1 B_1 + \lambda_2 B_2,$$

where I_v is the identity matrix of order v. Let us suppose that the association scheme of a GDD(v, b, r, λ_1, λ_2, m, n) is such that the ith group consists of the n points

$$(i-1)n+1,\ (i-1)n+2,\ \ldots,\ in$$

for $i = 1, \ldots, m$. Then the matrix NN^{T} can be partitioned into m^2 square submatrices each of order n. The diagonal submatrices have all diagonal entries equal to r and all off-diagonal entries equal to λ_1, while all entries of the off-diagonal submatrices are equal to λ_2. Thus, in this case, NN^{T} can be written as

$$NN^{\mathrm{T}} = I_m \otimes [(r-\lambda_1)I_n + \lambda_1 J_n] + (J_m - I_m) \otimes \lambda_2 J_n$$

where we write $A \otimes B$ for the Kronecker product of the matrices A and B and J_n for the square matrix of order n whose entries are all 1's. When $\lambda_1 = 0$, $\lambda_2 = \lambda$ the expression for NN^{T} takes the form

$$NN^{\mathrm{T}} = rI_{mn} + (J_m - I_m) \otimes \lambda_2 J_n$$

In this paper we are concerned with the class of GDDs with $\lambda_1 = 0$ and $\lambda_2 = \lambda$; and a GDD in this class will be denoted by GDD(v, b, r, k, λ, n). When no confusion is likely, a GDD(v, b, r, k, λ, n) is denoted in terms of the independent parameters v, k, λ and n by GDD(v, k, λ, n). We note that all TDs belong to this class of GDDs.

Let $G = \{h_1 = e, h_2, \ldots, h_g\}$ be a finite group (with identity e) of order g. Form the matrix W,

$$W = h_1A_1 + \ldots + h_gA_g,$$

where $A_1, \ldots, A_g$ are $v \times b$ (0, 1) – matrices such that the Hadamard product $A_k{}^* A_j = 0$ for any $k \neq j$. Now let

$$W^+ = (h_1^{-1}A_1 + \ldots + h_g^{-1}A_g)^{\mathrm{T}},$$

and

$$N = A_1 + \cdots + A_g.$$

Then we say that W is a *partial generalized Bhaskar Rao design with two association classes over* G, PGBRD, denoted by PGBRD($v, b, r, k, \lambda, n; G$), or in abbreviated form PGBRD($v, k, \lambda, n; G$), if:
N is the incidence matrix of the GDD(v, b, r, k, λ, n), that is,

$$NN^{\mathrm{T}} = rI_v + \lambda B_2,$$

where B_2 is association matrix of the GDD(v, b, r, k, λ, n) corresponding to $\lambda_2 = \lambda$; and

$$WW^+ = reI_v + (\lambda/g)(h_1 + \cdots + h_g)B_2.$$

A partial generalized Bhaskar Rao design with *one* association class, denoted by GBRD($v, k, \lambda; G$), satisfies

$$NN^{\mathrm{T}} = (r-\lambda)I_v + \lambda J_v,$$

that is, if $k < v$, N is the incidence matrix of the BIBD(v, b, r, k, λ), and

$$WW^+ = reI_v + (\lambda/g)(h_1 + \cdots + h_g)(J_v - I_v).$$

For both a partial generalized Bhaskar Rao design with *two* association classes over G and a generalized Bhaskar Rao design with *one* association class over G, we say that the design W is *based* on the incidence matrix N.

We shall reserve the name *generalized Bhaskar Rao design*, GBRD, for a partial generalized Bhaskar Rao design with *one* association class.

A GBRD in which $v = b$ is a *symmetric* GBRD or a *generalized weighing matrix*. A generalized weighing matrix which contains no zero entries is also known as a *generalized Hadamard matrix*. Generalized Hadamard matrices have been studied by Brock (1988), Dawson (1985), de Launey (1984,1986, 1989A, 1989B), Jungnickel (1979), Seberry (1979), and Street (1979).

GBRDs over elementary abelian groups other than Z_2 have been studied recently by Lam and Seberry (1984) and Seberry (1985). de Launey, Sarvate and Seberry (1985) considered GBRDs over Z_4 which is an abelian (but not elementary) group. Some GBRDs over various groups (abelian and non-abelian) have been studied by Gibbons and Mathon (1987A, 1987B). Palmer and Seberry (1988) have shown that the necessary conditions are sufficient for the existence of GBRDs over the non-abelian groups Q, S_3, D_4, D_6 and over the abelian group $Z_2 \times Z_4$. GBRDs over cyclic groups of even order have been considered recently by Bowler, Quinn and Seberry (199).

Recently Curran and Vanstone (1989) have used GBRDs to construct doubly resolvable BIBDs. Sarvate and Seberry (199) have used GBRDs in the construction of balanced ternary designs. Generalized Bhaskar Rao designs and generalized Hadamard matrices have been used by Mackenzie and Seberry (1988) to obtain q – ary codes.

Our aim in this paper is to establish the existence of the designs PGBRD(v, 3, λ, n ; EA(g)). For each integer g, EA(g) is the abelian group

$$Z_{p_1} \times Z_{p_1} \times \cdots \times Z_{p_1} \times \cdots \times Z_{p_n} \times Z_{p_n} \times \cdots \times Z_{p_n}$$

where $g = p_1 \ldots p_n$ and each p_i is a prime. In an earlier paper (Palmer (1990) it was shown that the necessary conditions are sufficient for the existence of a PGBRD(v, 3, λ, 2; EA(g)).

2. Constructions

The constructions which will be used extensively in this paper are contained in the following five theorems.

Theorem 2.1 (Palmer (1990)) *Suppose that a* GDD(v, k, λ, n) *and a* GBRD(k, k, μ; G) *exists. Then a* PGBRD(v, k, $\lambda\mu$, n; G) *exists.*

Theorem 2.2 (Palmer (1990)) *Suppose that a* PBD(v; H; λ) *exists and that for each h belonging to H a* PGBRD(nh, k, μ, n; G) *exists. Then a* PGBRD(nv, k, $\lambda\mu$, n; G) *exists.*

Theorem 2.3 (Palmer (1990)) *Suppose that a* BIBD(v, k, λ) *and a* PGBRD(nk, j, μ, n; G) *exists. Then there exists a* PGBRD(nv, j, $\lambda\mu$, n; G).

The next theorem is a generalization of Theorem 2.4 found in Palmer (1990).

Theorem 2.4 *Let G and H be groups of orders g and h respectively. Suppose that a* GBRD(v, k, λ; $G \times H$) *exists, then a* PGBRD(hv, k, λ/h, h; G) *exists.*

Proof: Let A = GBRD(v, k, λ; $G \times H$) and suppose that (α, β), where α and β belong to G and H respectively, is any non-zero entry in A. We form the matrix B by replacing the zero entries of A by square zero matrices of size h and by replacing every non-zero entry (α, β) by the matrix αP_β where P_β corresponds to β in the right regular representation of H . We claim that B is a PGBRD(hv, k, λ/h, h ; G).

Theorem 2.5 *Suppose that a* PGBRD(v, k, λ, n; G) *and a* TD(k, 1; s) *exists. Then a* PGBRD(sv, k, λ, sn; G) *exists.*

Proof: Let A be a PGBRD(v, k, λ, n; G). Let B be the incidence matrix of a TD(k, 1; s). We write B as

$$\begin{bmatrix} B_1 \\ \cdots \\ \vdots \\ \cdots \\ B_k \end{bmatrix}$$

where each B_i, $i = 1, 2, \ldots, k$, is a matrix of size $s \times s^2$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the non-zero entries of the first column of A. We now replace α_1 by $\alpha_1 B_1$, α_2 by $\alpha_2 B_2, \ldots, \alpha_k$ by $\alpha_k B_k$ and the zero entries by zero matrices of size $s \times s^2$. This process is repeated for the remaining columns of A. The new matrix thus formed is a PGBRD(sv, k, λ, sn; G).

It is well-known (see, for example, Street and Street (1987)), that a T(k, 1; s) exists if and only if there exist $k-2$ mutually orthogonal latin squares of order s. So a T(3, 1; s) exists when s is a positive integer. Thus we have the

Corollary 2.6 *Let s be a positive integer. Suppose that a* PGBRD(v, 3, λ, n; G) *exists. Then a* PGBRD(sv, 3, λ, sn; G) *exists.*

3. Necessary conditions

Hanani (1975) has shown that a GDD(v, 3, λ, n) exists if and only if

$$v \equiv 0 \pmod{n} \tag{3.1}$$
$$v \geq 3n \tag{3.2}$$
$$\lambda(v-n) \equiv 0 \pmod{2} \tag{3.3}$$
$$\lambda v(v-n) \equiv 0 \pmod{6} \tag{3.4}$$

For the existence of a PGBRD(v, k, λ, n; G) we also require

$$\lambda \equiv 0 \pmod{g} \tag{3.5}$$

where g is the order of the group G. In view of Theorems 2.4 and 3.1 (Palmer (1990)) we have the extra necessary condition,

$$\lambda v(v-n) \equiv 0 \pmod{24}, \tag{3.6}$$

for the existence of a PGBRD(v, k, λ, n; EA(g)) when g is even. Hence, we obtain

Theorem 3.1 *Necessary conditions for the existence of a* PGBRD(v, k, λ, n; EA(g)) *are*:

$$\lambda \equiv 0 \pmod{g} \tag{3.7}$$
$$v \equiv 0 \pmod{n} \tag{3.8}$$
$$v \geq 3n \tag{3.9}$$
$$\lambda(v-n) \equiv 0 \pmod{2} \tag{3.10}$$
$$\lambda v(v-n) \equiv 0 \pmod{6}, \textit{ if } g \textit{ is odd} \tag{3.11}$$
$$\lambda v(v-n) \equiv 0 \pmod{24}, \textit{ if } g \textit{ is even.} \tag{3.12}$$

In the remaining sections of the paper we will show that these necessary conditions are sufficient for the existence of a PGBRD(v, 3, λ, n; EA(g)).

4. PGBRD(v, 3, λ, n; EA(g)), $n \equiv 1$ or 5 (mod 6)

Theorem 4.1 *Suppose $n \equiv 1$ or 5 (mod 6). Then the necessary conditions* (3.7), (3.8), (3.9), (3.10), (3.11) *and* (3.12) *are sufficient for the existence of a* PGBRD(v, 3, λ, n; EA(g)).

Proof: Let p be a non-negative integer. By Theorem 3.1, a PGBRD($(6p+1)m$, 3, λ, $6p+1$; EA(g)) can exist only if

$$\lambda \equiv 0 \pmod{g} \quad (4.1)$$
$$m \geq 3 \quad (4.2)$$
$$\lambda(m-1) \equiv 0 \pmod{2} \quad (4.3)$$

$$\lambda m v(m-1) \equiv \begin{cases} 0 \pmod{6} \text{ if } g \text{ is odd,} \\ 0 \pmod{24} \text{ if } g \text{ is even.} \end{cases} \quad (4.4)$$

The conditions (4.1), (4.2),(4.3) and (4.4) are necessary and sufficient conditions for the existence of a PGBRD(m ,3, λ, 1; EA(g)) (Seberry (1985)). Thus, using Corollary 2.6, we can construct the design PGBRD($v = (6p+1)m$, 3, λ, $n = 6p+1$; EA(g)) from the design PGBRD(m , 3, λ, 1; EA(g)) whenever v, λ, n, and g satisfy the necessary conditions given in Theorem 3.1.

Also, by repeating the argument of the previous paragraph for the case where $n = 6p+5$, it can be shown that a PGBRD(v , 3, λ, n; EA(g)) exists if and only if v, λ, n and g satisfy the necessary conditions given in Theorem 3.1.

5. PGBRD(v, 3, λ, n; EA(g)), $n \equiv 2$ or 4 (mod 6)

Theorem 5.1 *Suppose $n \equiv 2$ or 4 (mod 6). The necessary conditions* (3.7), (3.8), (3.9), (3.10), (3.11) *and* (3.12) *are sufficient for the existence of a* PGBRD(v, 3, λ, n; EA(g)).

Proof: Let p be a non-negative integer. By Theorem 3.1, the design PGBRD($(6p+2)m$, 3, λ, $6p+2$; EA(g)) can exist only if

$$\lambda \equiv 0 \pmod{g} \quad (5.1)$$
$$m \geq 3 \quad (5.2)$$
$$\lambda m(m-1) \equiv 0 \pmod{3}. \quad (5.3)$$

We note that Palmer (1990) has shown that (5.1), (5.2) and (5.3) are necessary and sufficient conditions for the existence of the design PGBRD($v = 2m$, 3, λ, 2; EA(g)). Thus, for $n = 6p+2 \equiv 2$ (mod 6) and v and λ satisfying the necessary conditions (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12), we can apply Corollary 2.6 to construct the design PGBRD($v = (6p+2)m$, 3, λ, $6p+2$; EA(g)) from a PGBRD($v = 2m$, 3, λ, 2; EA(g)).

By similar reasoning, we can show that the necessary conditions are sufficient for the existence of the design PGBRD($v = (6p+4)m$, 3, λ, $6p+4$; EA(g)).

6. PGBRD(v, 3, λ, n; EA(g)), $n \equiv 0 \pmod 6$

Lemma 6.1 *There exists a* PGBRD($v = 6m$, 3, λ, 6; EA(g)) *if and only if*

$$m \geq 3$$
$$\lambda \equiv 0 \pmod g$$

Proof: When $g \equiv 0, 1, 3 \pmod 4$ a GBRD(3,3, g; EA(g)) exists (Seberry (1985)). Also when $m \geq 3$, a GDD($6m$, 3,1,6) exists (Hanani (1975)). Hence, on application of Theorem 2.1 (Palmer (1990)), we can construct a PGBRD($v = 6m$, 3, g, 6; EA(g)), $g \equiv 0, 1, 3 \pmod 4$, $m \geq 3$.

A GBRD(m, 3, $12h$; $Z_2 \times EA(h) \times Z_2 \times Z_3$), h odd, exists if and only if $m \geq 3$ (Seberry (1985)). Thus, by applying Theorem 2.4, we can construct the design PGBRD($v = 6m$, 3, $2h$, 6; $Z_2 \times EA(h)$) if and only if $g = 2h \equiv 2 \pmod 4$ and $m \geq 3$. A PGBRD($v = 6m$, 3, $\lambda = gt$, 6; EA(g)) can be constructed by taking t copies of a PGBRD($v = 6m$, 3, g, 6; EA(g)).

Theorem 6.2 *The necessary conditions* (3.7), (3.8), (3.9), (3.10), (3.11) *and* (3.12) *are sufficient for the existence of the design* PGBRD(v, 3, λ, n; EA(g)), $n \equiv 0$ (*mod* 6).

Proof: The necessary conditions for the existence of the design PGBRD($v = 6mp$, 3, λ, $n = 6p$; EA(g)), p a positive integer, are now:

$$\lambda \equiv 0 \pmod g$$
$$m \geq 3.$$

By application of Corollary 2.6 and Lemma 6.1, we see that these conditions are sufficient for the existence of the design PGBRD(v, 3, λ, n ; EA(g)), $n \equiv 0$ (*mod* 6).

7. PGBRD(v , 3, λ, n; EA(g)), $n \equiv 3 \pmod 6$

Lemma 7.1 *The necessary conditions are sufficient for the existence of a* PGBRD($v = 3m$, 3, $\lambda = gt$, 3; EA(g)).

Proof: By Theorem 3.1, the necessary conditions for the existence of the design PGBRD($v = 3m$, 3, $\lambda = gt$, 3; EA(g)) give rise to the following cases:

(a) g odd, t odd, $m \equiv 1 \pmod 2$ and $m \geq 3$;
(b) g odd, t even, $m \geq 3$;
(c) $g \equiv 0 \pmod 4$, $m \geq 3$;
(d) $g \equiv 2 \pmod 4$, t odd, $m \equiv 0$ or $1 \pmod 4$and $m \geq 3$;
(e) $g \equiv 2 \pmod 4$, t even, $m \equiv 0$ or $1 \pmod 4$ and $m \geq 3$.

<u>Cases (a) and (b)</u>: Here a GBRD(3, 3, g; EA(g)) exists (Seberry (1985)) Hanani (1975) has shown that a GDD($3m$, 3, 1, 3) exists if and only if $m \equiv 1 \pmod 2$ and $m \geq 3$, and a GDD($3m$,3, 2, 3) exists if and only if $m \geq 3$. Hence, by Theorem 2.1, we see that a PGBRD($3m$, 3, g; EA(g)) exists for $m \equiv 1 \pmod 2$ and $m \geq 3$, and a PGBRD($3m$,3, $2g$; EA(g)) exists for $m \geq 3$. The designs PGBRD($3m$, 3, gt; EA(g)) and PGBRD($3m$,3, $2gt$; EA(g)) can be obtained by taking t multiples of the designs PGBRD($3m$, 3, g; EA(g)) and PGBRD($3m$, 3, $2g$; EA(g)) respectively.

Case (c): As in (a), for all odd $m \geq 3$, a PGBRD($3m$, 3, g, 3; EA(g)) exists. By Seberry (1985) a GBRD($2p$, 3, $12q$; EA($12q$)), $p \geq 2$, $q \geq 1$, exists. Thus, by Theorem 2.4, a PGBRD($3(2p)$,3, g, $4q$); EA(g)) exists for all even $m = 2p \geq 2$. The design PGBRD($3m$, 3, $\lambda = tg$, 3; EA(g)) is obtained by taking a t multiple of a PGBRD($3m$,3, g, 3; EA(g)).

Case (d): By Seberry (1985), a GBRD(m, 3, $(4q+2)3$; EA($(4q+2)3$)) exists if and only if $m \equiv 0$ or 1 (mod 4) and $m \geq 3$. Thus, by Theorem 2.4, a PGBRD($3m$, 3, $(4q+2)$, 3; EA($(4q+2)$)) exists for $m \equiv 0$ or 1 (mod 4) and $m \geq 3$. We can produce a PGBRD($3m$, 3, $(4q+2)t$, 3; EA($(4q+2)$)) by taking t copies of a PGBRD($3m$, 3, $(4q+2)$, 3; EA($(4q+2)$)).

Case (e): The design GBRD(m, 3, $(4q+2)6$; EA($(4q+2)3$) exists if and only if $m \geq 3$. Theorem 2.4 shows that a PGBRD($3m$, 3, $(4q+2)2$, 3; EA($(4q+2)$)) exists if and only if $m \geq 3$. For all $m \geq 3$, we can construct the design PGBRD($3m$, 3, $(4q+2)2t$, 3; EA($(4q+2)$)) by taking t copies of the design PGBRD($3m$, 3, $(4q+2)2$, 3; EA($(4q+2)$)).

Theorem 7.2 *Suppose that* $n \equiv 3$ (*mod* 6). *Then the necessary conditions* (3.7), (3.8), (3.9), (3.10), (3.11)*and* (3.12) *are sufficient for the existence of the design* PGBRD(v, 3, λ, n; EA(g)).

Proof: Let p be a positive integer. A PGBRD($v = (6p+3)m$, 3, λ, $6p+3$; EA(g)) can be constructed from a PGBRD($v = 3m$, 3, λ, 3; EA(g)) on application of Corollary 2.6. However, by Theorem 3.1, a PGBRD($v = (6p+3)m$, 3, λ, $6p+3$; EA(g)) exists only if a PGBRD($v = 3m$, 3, λ, 3; EA(g)) exists. We know, by Lemma 7.1, that the necessary conditions (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) are sufficient for the existence of the design PGBRD($v = 3m$, 3, λ, 3; EA(g)). Hence the necessary conditions (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) are also sufficient for the existence of the design PGBRD($v = (6p+3)m$, 3, λ, $6p+3$; EA(g)).

8. Main Result and Applications

By virtue of Theorems 4.1, 5.1, 6.2 and 7.2 we have the

Theorem 8.1 *The necessary conditions* (3.7), (3.8), (3.9), (3.10), (3.11) *and* (3.12) *are sufficient for the existence of the design* PGBRD(v, 3, λ, n ; EA(g)).

Let EA(g) = $\{e = h_1, \ldots, h_g\}$ where EA(g) is the elementary abelian group defined in section 1. Suppose that EA(g) is represented by the $g \times g$ permutation matrices $P_1, \ldots, P_g$ so that the element h_i corresponds to the matrix P_i, $1 \leq i \leq g$. As in Street and Rodger (1980), Seberry (1982) and Palmer (1990), we construct, by replacing each group element entry of a PGBRD(v, 3, λ, n ; EA(g)) by its corresponding $g \times g$ permutation matrix, the incidence matrix of a group divisible design with v/n groups each of size ng and with the parameters

$$v^* = vg,\ b^* = bg,\ r^* = r,\ k^* = 3,\ \lambda_1^* = 0,\ \lambda_2^* = \lambda/g.$$

Hence we have part of Hanani's Theorem 6.2 (Hanani (1975)) but by a different approach:

Theorem 8.2 *The conditions*

$$\lambda \equiv 0 \pmod{g}$$
$$v \geq 3n$$
$$v \equiv 0 \pmod{n}$$
$$\lambda(v - n) \equiv 0 \pmod{2}$$

$$\lambda v(v - n) \equiv \begin{cases} 0 \pmod{6} \text{ if } g \text{ is odd,} \\ 0 \pmod{24} \text{ if } g \text{ is even,} \end{cases}$$

are sufficient for the existence of a group divisible design with v/n *groups each of size ng and the parameters :*

$$v^* = vg,\ b^* = b\ ,\ r^* = r,\ k^* = 3,\ \lambda_1^* = 0,\ \lambda_2^* = \lambda/g.$$

9. Acknowlegements

The author wishes to thank Professor J. Seberry and Dr. P.B. Kirkpatrick for some helpful suggestions in the preparation of this paper.

References

Beth, T., Jungnickel, D. and Lenz, H. (1986). *Design Theory*, Cambridge University Press, Cambridge.

Bhaskar Rao, M. (1966). Group divisible family of PBIBD designs. *J. Indian Statist.Assoc.* **4**, 14–28.

Bhaskar Rao, M. (1970). Balanced orthogonal designs and their application in the construction of some BIB and group divisible designs. *Sankhya Ser.* **A 32**, 439–448.

Bowler, A., Quinn, K. and Seberry, J. (199). Generalised Bhaskar Rao designs with elements from cyclic groups of even order. *Australas. J. Combin.* (to appear).

Brock, B.W. (1988). Hermitian congruence and the existence and completion of generalized Hadamard matrices, relative difference sets and maximal matrices. *Journal of Combinatorial Theory, Ser.* **A 49**, 233–261.

Clatworthy, W.H. (1973). *Tables of two-associate-class Partially Balanced Designs*, NBS Applied Math. Ser. No. (63).

Curran, D.J. and Vanstone, S.A. (1989). Doubly resolvable designs from generalized Bhaskar Rao designs. *Discrete Math.* **73**, 49–63.

Dawson, J.E. (1985). A construction for the generalized Hadamard matrices GH($4q$, EA(q)). *J. Statist. Plann. and Inference* **11**, 103–110.

de Launey, W. (1984). On the non-existence of generalized Hadamard matrices. *J. Statist. Plann. and Inference* **10**, 385–396.

de Launey, W. (1986). A survey of generalized Hadamard matrices and difference matrices D(k, λ; G) with large k. *Utilitas Math.* **38**, 5–29.

de Launey, W. (1989A). Square GBRDs over non-abelian groups. *Ars Combinatoria* **27**,40–49.

de Launey, W. (1989B). Some new constructions for difference matrices, generalized Hadamard matrices and balanced generalized weighing matrices. *Graphs and Combinatorics* **5**, 125–135.

de Launey, W., Sarvate, D.G., and Seberry, J. (1985). Generalized Bhaskar Rao designs with blocks size 3 over Z_4.
Gibbons, P.B. and Mathon, R. (1987A). Construction methods for Bhaskar Rao and related designs. *J. Australian Math. Soc.* A **42**,5–30.
Gibbons, P.B. and Mathon, R. (1987B). Group signings of symmetic balanced incomplete block designs. *Ars Combinatoria.* **23A**, 123–134.
Hanani, H. (1975). Balanced incomplete block designs and related designs. *Discrete Math.* **11**, 255–369.
Jungnickel, D. (1979). On difference matrices, resolvable TD's and generalized Hadamard matrices. *Math. Z.* **167**, 49–60.
Lam, C. and Seberry, J. (1984). Generalized Bhaskar Rao designs. *J. Statist. Plann. and Inference* **10**, 83–95.
Mackenzie, C. and Seberry, J. (1988). Maximal q–ary codes and Plotkin's bound. *Ars Combinatoria* **26B**, 37–50.
Palmer, W.D. (1990). Generalized Bhaskar Rao designs with two association classes, *Australas. J. Combin.* **1**, 161–180.
Palmer, W.D. and Seberry, J. (1988). Bhaskar Rao designs over small groups. *Ars Combinatoria* **26A**, 125–148.
Raghavarao, D. (1971). *Construction and combinatorial problems in design of experiments*, Wiley, New York.
Sarvate, D.G., and Seberry, J. (199). Constructions of balanced ternary designs based on generalized Bkaskar Rao designs. *J. Statist. Plann. and Inference* (submitted).
Seberry, J. (1979). Some remarks on generalized Hadamard matrices and theorems of Rajkundlia on SBIBDs. *Combinatorial Mathematics IV, Lecture Notes in Math.*, **748**, Springer, Berlin, 154–164.
Seberry, J. (1982). Some families of partially balanced incomplete block designs. In: *Combinatorial Mathematics IX, Lecture Notes in Mathematics No. 952*, Springer - Verlag, Berlin-Heidelberg- New York, 378–386.
Seberry, J. (1984). Regular group divisible designs and Bhaskar Rao designs with block size 3. *J. Statist. Plann. and Inference* **10**,69–82.
Seberry, J. (1985). Generalized Bhaskar Rao designs of block size three. *J. Statist. Plann. and Inference* **11**, 373–379.
Singh, S.J. (1982). Some Bhaskar Rao designs and applications for $k = 3, \lambda = 2$. *University of Indore J. of Science* **7**, 8–15.
Street, D.J. (1979). Generalized Hadamard matrices, orthogonal arrays and F–squares. *Ars Combinatoria* **8**, 131–141.
Street, D.J. and Rodger, C.A. (1980). Some results on Bhaskar Rao designs. *Combinatorial Mathematics VII, Lecture Notes in Mathematics No. 829,* Springer - Verlag, Berlin-Heidelberg- New York, 238–245.
Street, A. P. and Street, D. J. (1987) *Combinatorics of Experiment Design,* Oxford University Press, Oxford.
Wilson, R.M. (1974). A few more squares. *Proceedings of the fifth Southeastern Conference on Combinatorics and Computing, Congressus Numerantium X.* Utilitas math., Winnipeg, 675–680.
Wilson, R.M. (1975). Construction and uses of pairwise balanced designs, *combinatorics* Edited by M. Hall Jr. and J. H. van Lint, (Mathematisch Centrum, Amsterdam), 19–42.

(Received 1/2/1991)