Note on distance-regular graphs with a pair of equal subdegrees

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Abstract:

We bound the diameter of a distance-regular graph in which two subdegrees are equal.

Introduction:

Let G be a regular graph of diameter d with vertex set VG and edge set EG . We say that G is *distance-regular* if for vertices u, v Œ VG distance i apart, v is adjacent to c_i vertices in VG distance i-1 from u, a_i vertices in VG distance i from u and b_i vertices in VG distance i+1 from u and the numbers c_i, a_i, and b_i depend only upon the value of i and not upon the choice of u and v. These numbers are called *intersection numbers*. The values of c_0 and b_d are officially undefined although no inconsistency arises if they are considered to be zero.

If G is distance-regular there is associated with it an *intersection array*

$i(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$.

Since $b_i + a_i + c_i = b_0$ for all i = 1, 2,..., d-1 the intersection array gives us all intersection numbers associated with G.

It is clear that a distance-regular graph G, as above, has a constant number of vertices, k_i, distance i from v, for all v Œ VG, and for each i, $1 \leq i \leq d$. We call k_i the i^{th} *subdegree* of G. We also note that $k_1 = b_0$ is just the degree of regularity of G and is usually referred to as k.

In [3] it is shown that there are only finitely many distance-transitive graphs of any given degree greater than two. The analogous result for distance-regular graphs has been widely conjectured, but still remains to be proved. Progress toward establishing such a result has been made from several approaches. One of these is to bound the diameter of a distance-regular graph in terms of some specified property.

Ivanov [4] has bounded the diameter of a distance-regular graph, with degree greater than two, in terms of the smallest incontractible cycle other than a triangle. This was achieved with the following results.

Theorem 1 (Ivanov [4]): If in the intersection array of the distance-regular graph G the relations, $(c_{s-1}, b_{s-1}) \neq (c_s, b_s) = (c_{s+t}, b_{s+t})$ hold for $s > 1$, then $t < s$.//

Corollary 1.1 (Ivanov [4]): If G is a distance-regular graph with an incontractible cycle of length $g > 3$, then $d \geq g\, 2^{k-1}$.//

Taylor and Levingston [6] have shown that if we consider only those distance-regular graphs with $k_1 = k_2$, then regardless of the degree of the graph (except of course in the case of degree two', i.e. cycles) the maximum possible diameter of the graph is four.

Theorem 2 (Taylor & Levingston [6]): Let G be a distance-regular graph of diameter at least four, then G is a cycle if and only if $k_1 = k_2$.//

This can be "generalised" to the following result which has been obtained by the author.

Theorem 3: Let G be a distance-regular graph of degree $k \geq 3$, with $k_i = k_j$, $i < j$. Then the diameter of G is less than $\max\{3(i+1), 2j\}$.

As can be readily seen, the bound when $i = 1$ and $j = 2$ is not as good as the Taylor and Levingston result, but the cases covered beyond this are worthy of note.

The proof of the Theorem requires the Ivanov result as well as the following:

Proposition 4 (Smith [5]): Let G be a distance-regular graph of diameter d. If $i + j \leq d$, then $c_i \leq b_j$.//

Proposition 5 (Taylor & Levingston [6]): Let G be a distance-regular graph of diameter d.

(i) If $i + j \leq d$ and $i < j$, then $k_i \leq k_j$. Moreover, if $k_i = k_j$, then $k_{i+1} = k_{j-1}$ and so on.

(ii) For some integer $h \geq d/2$ we have $k_1 \leq k_2 \leq \ldots \leq k_h$ and $k_h \geq k_{h+1} \geq \ldots \geq k_d$.//

Proof of Theorem 3: Let us assume that G is as in the statement of the lemma and that the diameter of G is at least equal to max{3(i+1), 2j}. By Propositon 5 we have that $k_i \leq k_{i+1} \leq ... \leq k_{j-i} \leq k_j$ (since $d \geq 2j$) and from our assumption that $k_i = k_j$ we may conclude that equality holds throughout. (Notice, if i = 1, then this gives us $d \leq 4$ by Theorem 2)

It is shown in [5] that the b_m and c_m intersection numbers form, respectively descending and ascending sequences and that the subdegrees k_m and k_{m+1} obey the relation $k_m b_m = k_{m+1} c_{m+1}$. Thus we see that $b_m = c_{m+1}$ for $i \leq m \leq j - 1$. But, by Proposition 4, $b_{j-1} \geq c_j \geq ... \geq c_{i+1} = b_i \geq b_{j-1}$ which establishes the equality of all of these intersection numbers. In fact we can go further. Since the diameter is at least 3(i + 1) we know that $b_i \geq c_{2i+3}$ and $c_{i+1} \leq b_{2i+2}$ which in turn establishes the relations
$b_m = c_{m+1}, i \leq m \leq 2(i+1)$, and $b_m = b_{m+1}, i \leq m \leq 2i + 1$. In particular this tells us that $(c_{i+1}, b_{i+1}) = (c_{2(i+1)}, b_{2(i+1)})$ which contradicts Ivanov's lemma. Thus the diameter of G must be less than max{3(i+1), 2j}.//

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